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Theta functions and the metaplectic group

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1 Introduction

Over the course of the 20th century (see e.g. [Tat1967, BJ1979]), the classical theory of modular forms (see e.g. [Miy2006]) was recast in adelic terms to give the modern theory of "automorphic forms on adelic groups." For a number field F and a reductive algebraic group G/F, automorphic forms on $G(\mathbf{A}_F)$ are functions on $G(\mathbf{A}_F)$ which are left-invariant under $G(F) \subset G(\mathbf{A}_F)$ and satisfy some extra regularity and decay conditions. For example, the classical modular forms and Maass forms may be interpreted as automorphic forms on $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$. The adelic perspective on automorphic forms has been extremely fruitful in modern algebraic number theory, for example because it clarifies the role of representation theory at the $\mathfrak{p}$-adic places in the local Euler factors of L-functions and has therefore allowed very general local and global (Arthur–)Langlands conjectures (see e.g. [CL2019, §6.4]) to be stated and studied.

Along with the modular forms of integral weight, the classical theory also deals with modular forms of half-integral weight, the chief example of which being the *theta functions* that were originally studied by Jacobi for the purposes of constructing elliptic functions and later used by Riemann to prove the analytic continuation and functional equation for his zeta function (the key ingredient of Riemann's proof was the modularity/automorphy property of Jacobi's theta function). Theta functions provide an important explicit example of a function that *turns out* to be automorphic as a result of non-trivial input. This leads to a reasonable question:

Question 1.1. Can the classical theta functions be viewed as $\mathrm{SL}_2(\mathbf{Q})$-invariant functions on some group containing $\mathrm{SL}_2(\mathbf{Q})$ that is somehow related to $\mathrm{SL}_2(\mathbf{A}_\mathbf{Q})$?

Answering Question 1.1 is most of the way towards the most basic form of the *theta correspondence* (see e.g. [Gan2022]), which is a key tool in the Langlands program and one of the most useful methods of explicit construction of automorphic forms for the purposes of realizing the Langlands functoriality conjecture. Some specific examples of the importance of the theta correspondence within and outside the Langlands program proper include:

- Modularity of theta functions of lattices explains, for example, why even unimodular lattices must have dimension divisible by 8 (see [Elk2019b, CL2019] for more).
- One special case of the theta correspondence is the modular forms that Hecke attached to Dirichlet characters for a real quadratic field (see [Pra1993, 8.3.2]). Among other things, this provides a proof of existence of the modular form of weight 1 that proves the modularity of the Galois representations with dihedral projective image.
- A construction related to the Jacquet–Langlands correspondence can be obtained as a special case of the theta correspondence (see [Pra1993, 8.3.4]).

In a landmark paper [Wei1964], André Weil drew inspiration from physics in order to provide a satisfactory answer to Question 1.1. The purpose of this talk was to explain how Weil might have come up with his answer by first closely examining the full detail of the most classical and concrete example (as we like to do at the PROMYS program). Weil's theory provides a unified perspective on theta functions that allows one to prove the automorphy of all the theta functions mentioned above in one go.

2 Classical theta functions

Jacobi considered the following holomorphic function of two complex variables $z \in \mathbf{C}$ and $\tau \in \mathbf{H} := \{x+iy \in \mathbf{C} : y > 0\}$:

$$\vartheta(z;\tau) := \sum_{n\in\mathbf{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}.$$

The reason for constructing it is that it has nice transformation properties under the natural addition action of the lattice $\Lambda_\tau = \mathbf{Z} + \tau\mathbf{Z}$ on the z-variable. Of course, it is NOT periodic with respect to this lattice (otherwise it would define a holomorphic function on the compact Riemann surface $\mathbf{C}/\Lambda_\tau$, and would therefore be constant), but it is nice enough that Jacobi was able to use them to construct meromorphic functions that are (this was a massive achievement in complex analysis and algebraic geometry; the classic reference on this and its generalizations is [Mum2007, Mum1984]).

Less obvious is the transformation property that $\vartheta(z;-)$ satisfies under the usual action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}$, which is a perfectly natural question to ask (it is analogous to looking at the Eisenstein series as coefficients of the differential equation satisfied by the Weierstrass $\wp$-function and asking if they have a modularity property, which they famously do). One such property is as follows:

Theorem 2.1. Let $z \in \mathbf{C}$ and $\tau \in \mathbf{H}$. We have (after choosing the right branch of $(-)^{1/2}$ on $\mathbf{H}$)

$$\vartheta\left(\frac{z}{\tau}; \frac{-1}{\tau}\right) = (-i\tau)^{1/2} e^{\frac{\pi}{\tau} i z^2} \vartheta(z;\tau).$$

Corollary 2.2. $\vartheta(0;-1/\tau) = (-i\tau)^{1/2}\vartheta(0,\tau)$.

Corollary 2.2 is very important: it is saying that $\vartheta(0;-)$ provides an explicit construction of a *weight*-$1/2$ *modular form.* After taking a Mellin transform (as in [Elk2019a, `zeta1.pdf`]), it is equivalent to the analytic continuation and functional equation of the Riemann zeta function. This is a very special case of the grand promises that were made in §1 about applications to the Langlands program. Typically, it is written that Poisson summation (whether in the archimedean or in the adelic context as in Tate's thesis) is the key tool in the proof of Corollary 2.2 and Theorem 2.1. But for the maximum insight to be drawn for the purposes of motivating Weil's construction of theta functions, we write down the proof by plugging in the relevant information to the typical proof of Poisson summation.

Proof of Theorem 2.1. Let $\Phi \in \mathcal{S}(\mathbf{R})$, and let $\chi : \mathbf{R} \to S^1$ be the character given by $x \mapsto e^{2\pi i x}$. Consider the function

$$\vartheta(x* | \Phi) := \sum_{n\in\mathbf{Z}} \Phi(n)\chi(nx^*)$$

of a real variable x^*, which coincides[1] with $\vartheta(t;x^*)$ when $\Phi(x) = \Phi_\tau(x) = e^{-\pi i \tau x^2}$ (which is Schwartz at least when $\tau \in i\mathbf{R}$). This is NOT periodic in x^*. However, if we add an extra variable $x \in \mathbf{R}$, we can obtain a function

$$\Omega(x, x^*|\Phi) := \sum_{n\in\mathbf{Z}} \Phi(x+n)\chi((x+n)x^*).$$

On one hand, rearranging yields

$$\Omega(x, x^*|\Phi) := \chi(xx^*) \sum_{n\in\mathbf{Z}} \Phi(x+n)\chi(nx*). \tag{1}$$

On the other hand, treating $\Omega(x, x^*|\Phi)$ like a function on $\mathbf{R}/\mathbf{Z}$ and using the fact that Φ is Schwartz, we may expand it in a Fourier series and computing (using Fubini/Tonelli):

$$\Omega(x, x^*|\Phi) = \sum_{m\in\mathbf{Z}} \left(\int_0^1 \Omega(y, x^*|\Phi)\chi(-my)\,dy\right)\chi(mx) \tag{2}$$

[1] up to negating z

$$= \sum_{m\in\mathbf{Z}} \left(\sum_{n\in\mathbf{Z}} \int_0^1 \Phi(y+n)\chi((y+n)x^*)\chi(-my)\,dy \right) \chi(mx) \tag{3}$$

$$= \sum_{m\in\mathbf{Z}} \left(\int_{\mathbf{R}} \Phi(y)\chi(y(x^*-m))\,dy \right) \chi(mx) \tag{4}$$

$$= \sum_{m\in\mathbf{Z}} \widehat{\Phi}(m-x^*)\chi(mx). \tag{5}$$

Combining (5) with (1) and defining the slightly modified function

$$\Theta(x, x^*|\Phi) := \sum_{n\in\mathbf{Z}} \Phi(x+n)\chi(nx*),$$

we obtain

$$\chi(xx^*)\Theta(x, x^*|\Phi) = \Theta(-x^*, x|\widehat{\Phi}),$$

which is equivalent to saying

$$\chi(-xx^*)\Theta(x^*, -x|\Phi) = \Theta(x, x^*|\widehat{\Phi}). \tag{6}$$

Expanding the definitions of everything and using the standard computation of the Fourier transform of the Gaussian, one obtains exactly the claimed result. □

Remark In (6), the point is as follows:

- On the left hand side, Θ is being transformed by applying the Weyl element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and multiplying by the exponential of the quadratic form $-xx^*$. This should be the action of some group on some function space.
- On the right hand side, the only thing going on is the Fourier transform.

So it stands to reason that if we are looking for the group that answers Question 1.1, then we should find a group whose elements consist of an element of SL_2 together with a quadratic form, and which has a natural representation on an L^2 space for which the action of the Weyl element is by the Fourier transform. This is what is done in the next section.

3 The metaplectic group and the Weil representation

In this section, we construct the metaplectic group and its representation that satisfies the desiderata set forth in the previous section and which will therefore allow us to answer Question 1.1 in the next section. The construction itself first came from physics. I do not know whether Weil came up with it himself independently or learned about it from the physicists and then realized that it had the right properties.

Let V be a free module of finite rank over a local field K of characteristic not equal to 2, or over $\mathbf{A}_F$ where F is a number field or a function field of characteristic not equal to 2. Let A denote the base ring (so A is $\mathbf{A}_F$ or K as the case may be). Fix a nontrivial continuous character $\chi : K \to S^1$ or $\chi : \mathbf{A}_F/F \to S^1$ so that $\mathrm{Hom}(V, A)$ is identified with the Pontryagin dual V^*. For $v \in V$ and $v \in V^*$, we denote by $\langle v, v^* \rangle$ the value in A of v^* applied to v (so that $\chi(\langle v, v^* \rangle)$ is the value of v^* applied to v when considered as being in the Pontryagin dual rather than the vector space dual).

There are three very reasonable operations on functions $\Phi \in L^2(V)$:

- For $v \in V$, there is the translation operation $(v \cdot \Phi)(x) = \Phi(x+v)$.
- For $v^* \in V^*$, there is the multiplication-by-phase operation $(v^* \cdot \Phi)(x) = \chi(\langle x, v^* \rangle)\Phi(x)$.

- For $t \in K$, there is the scaling operation $(t \cdot \Phi)(x) = \chi(t)\Phi(x)$.

For $z = (v, v^*, t) \in V \times V^* \times K$, we therefore set the operator $U(z) \in \mathrm{U}(L^2(V))$ to be defined by

$$(U(z)\Phi)(x) := \chi(t)\chi(\langle x, v^* \rangle)\Phi(x + v).$$

We have

$$\begin{aligned}(U(v_1, v_1^*, t_1)U(v_2, v_2^*, t)\Phi)(x) &= \chi(t_1)\chi(\langle x, v_1^* \rangle)(U(v_2, v_2^*, t)\Phi)(x + v_1) \\ &= \chi(t_1 + t_2)\chi(\langle x, v_1^* + v_2^* \rangle)\chi(v_1, v_2^*)\Phi(x + v_1 + v_2) \\ &= U(v_1 + v_2, v_1^* + v_2^*, t_1 + t_2 + \langle v_1, v_2^* \rangle).\end{aligned}$$

So we define

Definition 3.1. The *Heisenberg group* $\mathrm{Heis}(V)$ is defined to be the set $V \times V^* \times K$ with the group operation

$$(v_1, v_1^*, t_1) \cdot (v_2, v_2^*, t_2) = (v_1 + v_2, v_1^* + v_2^*, t_1 + t_2 + \langle v_1, v_2^* \rangle).$$

It is equipped with a representation $U = U_{\chi,V} : \mathrm{Heis}(V) \to \mathrm{U}(L^2(V))$ as defined above.

The following theorem from physics / operator algebras underpins the entire theory:

Theorem 3.2 (Stone–Von Neumann)**.** For any choice of χ and V, the representation $U_{\chi,V}$ is (up to isomorphism) the unique infinite-dimensional irreducible unitary Hilbert space representation of $\mathrm{Heis}(V)$ with the property that the central character is $U_{\chi,V}|_K = \chi$.

Proof. See [Mac1949]. □

Note that Weil got around the lack of access to Theorem 3.2 by simply constructing the isomorphism in the cases where he needed it, through some clever functional analysis [Wei1964, §I].

In any event, the application of Theorem 3.2 to our situation is in combination with Schur's lemma and the action of any other group on $\mathrm{Heis}(V)$. This other group is the one that generalizes the role of SL_2. Let us figure out what it can be:

Lemma 3.3. Let s be an automorphism of $\mathrm{Heis}(V)$ with the property that $s|_K = 1$. Then s is of the form $s = (\sigma, f)$, where $\sigma \in \mathrm{Sp}(V \oplus V^*)$ (the module $V \oplus V^*$ being given the obvious nondegenerate alternating form $\langle (v_1, v_1^*), (v_2, v_2^*) \rangle := \langle v_1, v_2^* \rangle - \langle v_2, v_1^* \rangle$), where f is a function on $V \oplus V^*$, and where $(\sigma, f) \cdot (v, v^*, t) = (\sigma(v, v^*), f(v, v^*) + t)$. A pair (σ, f) defines a valid such automorphism if and only if

$$f(w_1 + w_2) - f(w_1) - f(w_2) = F(\sigma w_1, \sigma w_2) - F(w_1, w_2) \tag{7}$$

for all $w_1, w_2 \in V \oplus V^*$, where $F(w_1, w_2) := \langle v_1, v_2^* \rangle$ (where $w_i = (v_i, v_i^*)$).

Proof. Left as a (straightforward) exercise to the reader. □

Given the right hand side of (7) is a bilinear form on $V \oplus V^*$, f must be a polynomial function of degree ≤ 2. It makes sense to restrict to the case where f is a quadratic form, in which case (7) tells us that f is uniquely determined by σ (as the right hand side of (7) tells us what the symmetric bilinear form associated to f is, and A does not have characteristic 2 so this is the same information as f). In particular, we have

$$f_\sigma(w) = \frac{f(2w) - 2f(w)}{2} = \frac{F(\sigma w, \sigma w) - F(w, w)}{2}. \tag{8}$$

In particular, it is natural to consider the action of $\mathrm{Sp}(V \oplus V^*)$ on $\mathrm{Heis}(V)$ where σ acts via (σ, f_σ).

Proposition 3.4. There is a unique projective representation $\omega_{\chi,V} : \mathrm{Sp}(V \oplus V^*) \to \mathrm{PGL}(L^2(V))$ with the property that for all $\sigma \in \mathrm{Sp}(V \oplus V^*)$ and all $z \in \mathrm{Heis}(V)$,

$$\omega_{\chi,V}(\sigma) \circ U(\sigma \cdot z) = U(z) \circ \omega_{\chi,V}(\sigma).$$

Proof. Let $\sigma \in \mathrm{Sp}(V \otimes V^*)$. By Theorem 3.2, the representations $z \mapsto U(z)$ and $z \mapsto U(\sigma \cdot z)$ are isomorphic, so there exists an invertible operator $\widetilde{\omega}_{\chi,V}$ on $L^2(V)$ such that the diagram

$$\begin{tikzcd} L^2(V) \arrow[r, "\widetilde{\omega}_{\chi,V}(\sigma)"] \arrow[d, "U(\sigma\cdot z)"] & L^2(V) \arrow[d, "U(z)"] \\ L^2(V) \arrow[r, "\widetilde{\omega}_{\chi,V}(\sigma)"] & L^2(V) \end{tikzcd}$$

commutes. By Schur's lemma, the operator $\widetilde{\omega}_{\chi,V}$ is unique up to scalars, which ensures that the map $\sigma \mapsto \omega_{\chi,V}(\sigma) := \pi(\widetilde{\omega}_{\chi,V}(\sigma))$ is a homomorphism, as desired. Here $\pi : \mathrm{GL}(L^2(V)) \to \mathrm{PGL}(L^2(V))$. $\square$

Example 1. Let $\gamma : V^* \to V$ be an isomorphism, and consider the "Weyl element"

$$\sigma = \begin{pmatrix} 0 & -\gamma^{*,-1} \\ \gamma & 0 \end{pmatrix} \in \mathrm{Sp}(V \oplus V^*).$$

The corresponding f_σ is given by $f_\sigma(u, u^*) = -\langle u, u^* \rangle$, according to (8) For all $z = (v, v^*, 0) \in \mathrm{Heis}(V)$, the operator $\widetilde{\omega}_{\chi,V}(\sigma)$ must make the diagram

$$\begin{tikzcd} L^2(V) \arrow[r, "\widetilde{\omega}_{\chi,V}(\sigma)"] \arrow[d, "U(\sigma\cdot z)"] & L^2(V) \arrow[d, "U(z)"] \\ L^2(V) \arrow[r, "\widetilde{\omega}_{\chi,V}(\sigma)"] & L^2(V) \end{tikzcd}$$

commute. In other words, for all $\Phi \in L^2(V)$, we need (for almost all $x \in V$)

$$(U(v,0,0)\widetilde{\omega}_{\chi,V}(\sigma)\Phi)(x) = (\widetilde{\omega}_{\chi,V}(\sigma)U(0, -\gamma^{*,-1}v, 0))(x)$$

(Weil's convention is that matrices act on the right and I am too lazy to change this). In English, $\widetilde{\omega}_{\chi,V}(\sigma)$ should switch translation by v with multiplication by the phase $\chi(\langle -, -\gamma^{*,-1}v\rangle)$. This is exactly what the Fourier transform does ! In particular,

$$\int_V \Phi(y+v)\chi(\langle y, -\gamma^{*,-1}x\rangle)\,dy = \int_V \Phi(y)\chi(\langle y-v, -\gamma^{*,-1}x\rangle)\,dy = \chi(\langle x, \gamma^{*,-1}v\rangle)\int_V \Phi(y)\chi(\langle y, -\gamma^{*,-1}x).$$

So we must have $\widetilde{\omega}_{\chi,V}(\Phi) = \widehat{\Phi}$ up to scalars, for the appropriate Fourier transform convention (where V and V^* are identified via γ). The reader is encouraged to figure out what that convention ought to be from the above equation.

There is no natural choice of $\widetilde{\omega}_{\chi,V}$ that works for all σ, i.e. no lifting

$$\begin{tikzcd} \mathrm{Sp}(V \oplus V^*) \arrow[r, dotted] \arrow[dr, "\omega_{\chi,V}"'] & \mathrm{GL}(L^2(V)) \arrow[d, "\pi"] \\ & \mathrm{PGL}(L^2(V)) \end{tikzcd}$$

For this reason, we define

Definition 3.5. The *metaplectic group* $\mathrm{Mp}(V)$ is defined to be the fibered product

$$\{(\sigma, \widetilde{\omega}) \in \mathrm{Sp}(V \oplus V^*) \times \mathrm{GL}(L^2(V)) : \pi(\widetilde{\omega}) = \omega_{\chi,V}(\sigma)\}.$$

It is equipped with the bona fide (not projective) representation $\widetilde{\omega}_{\chi,V}$, which is just projection to the second coordinate and is called the *Weil representation.*

For a subgroup $\Omega \subset \mathrm{Sp}(V \oplus V^*)$, the data of a lifting of $\omega_{\chi,V}$ to a genuine representation of Ω is the same as a section of the projection $\mathrm{Mp}(V) \to \mathrm{Sp}(V \oplus V^*)$ over Ω.

4 Theta functions as automorphic forms on the metaplectic group

Given that the Weil representation acts on $L^2(V)$ by the Fourier transform, we expect/hope (6) (which is for $V = \mathbf{R}$ and σ equal to the Weyl element from Example 1) to generalize to the existence of a subgroup $\Omega \subset \mathrm{Sp}(V \oplus V^*)$ containing a Weyl element and a lifting $\omega_{\chi,V}$ of the Weil representation to Ω such that

$$\chi(f_\sigma(x, x^*))\Theta(\sigma(x, x^*)|\Phi) = \Theta(x, x^*|\omega_{\chi,V}(s)\Phi)$$

for all $s \in \Omega$, where

$$\Theta(x, x^*|\Phi) := \int_\Gamma \Phi(x + \xi)\chi(\xi, x^*)\, d\xi$$

and $\Gamma \subset V$ is a lattice in the usual sense that is useful for Fourier analysis[2].

In general, this Ω is the "$B(G, \Gamma)$" of [Wei1964, §1]. In order to make things very slightly simpler, we will assume that $A = \mathbf{A}_F$ for a global field F of characteristic not equal to 2, in which case Ω can be chosen to be the obvious discrete subgroup $\mathrm{Sp}_{2r}(F) \subset \mathrm{Sp}(V \oplus V^*)$, where r is the rank of V over A.

Theorem 4.1. Let $A = \mathbf{A}_F$ and $V = V_F \otimes \mathbf{A}_F$ for some finite dimensional F-vector space B_F. There is a lifting over $\mathrm{Sp}_{2r}(F) \subset \mathrm{Sp}(V \oplus V^*)$ of the Weil representation to a genuine representation $\mathbf{r}_{\chi,V} : \mathrm{Sp}_{2r}(F) \to \mathrm{GL}(L^2(V))$ satisfying

$$\chi(f_\sigma(x, x^*))\Theta(\sigma(x, x^*)|\Phi) = \Theta(x, x^*|\mathbf{r}_{\chi,V}(s)\Phi)$$

for all $s \in \mathrm{Sp}_{2r}(F)$.

Proof. The required transformation law for Θ under the action of the lifting of the Weil representation gives us all we need to define it. The reason for this is that the function taking Φ to Θ is in fact a bijection. Rigorously, note that $\Theta(x, x^*|\Phi)$ satisfies the quasiperiodicity condition

$$\Theta(x + \xi, x^* + \xi^*|\Phi) = \Theta(x, x^*|\Phi)\chi(\langle \xi, -x^*\rangle) \tag{9}$$

for all $\xi \in \Gamma$ and $\xi^* \in \Gamma_* := \{\xi^* \in V^* : \langle \xi, \xi^*\rangle = 1 \text{ for all } \xi \in \Gamma\}$. This makes it Γ_*-periodic in $x^* \in V^*$. The function $|\Theta(x, x^*|\Phi)|$ on $V \times V^*$ is $(\Gamma \times \Gamma_*)$-periodic, since χ is valued in S^1. So we may define an L^2-norm on all measurable functions on $V \times V^*$ that satisfy (9) by

$$\|\Theta\|_{L^2} := \int_{(V \times V^*)/(\Gamma \times \Gamma_*)} |\Theta|^2\, d\overline{x} d\overline{x}^*.$$

Define the Hilbert space $H(V, \Gamma)$ to consist of measurable functions $\Theta : V \oplus V^* \to \mathbf{C}$ satisfying (9) and $\|\Theta\|_{L^2} < \infty$ and which are locally integrable on $V \oplus V^*$. We first claim that the function $Z : L^2(V) \to H(V, \Gamma)$ given by $\Phi \mapsto \Theta(-, -|\Phi)$ is an isomorphism. This fact is actually at the heart of the proof. Indeed, this is where we copy the same key step as in the proof of the special case which is Theorem 2.1, namely Fourier expansion of Θ as a periodic function in one of the variables (the only difference is that because we are expanding Θ and not "Ω" in the notation of that proof, the variable in which it is periodic is x^* and not x, though we could have set this proof up differently to match with the conventions of the previous one). By Fourier inversion between V^*/Γ_* and its Pontryagin dual Γ (which is just Fourier series expansion when $V = \mathbf{R}$ and $\Gamma = \mathbf{Z}$),

$$\begin{aligned}\Theta(x, x^*|\Phi) &= \int_\Gamma \left(\int_{V^*/\Gamma_*} \Theta(x, y^*|\Phi)\chi(\langle -\xi, y^*\rangle)\, d\overline{y}^* \right) \chi(\langle \xi, x^*\rangle)\, d\xi \\ &= \int_\Gamma \left(\int_{V^*/\Gamma_*} \Theta(x + \xi, y^*|\Phi)\, d\overline{y}^* \right) \chi(\langle \xi, x^*\rangle)\, d\xi\end{aligned}$$

[2] If $A = \mathbf{R}$ or $\mathbf{C}$, Γ is just a $\mathbf{Z}$-lattice. If A is a nonarchimedean local field, Γ is an $\mathcal{O}_K$-lattice. If $A = \mathbf{A}_F$ and $V = V_F \otimes \mathbf{A}_F$ for a finite-dimensional F-vector space V_F, then $\Gamma = V_F \subset V$.

which implies that Z is an isomorphism with inverse given by

$$\Phi(x) = \int_{V^*/\Gamma_*} \Theta(x, x^*)\, d\overline{x}^*.$$

One checks explicitly (using the fact that $\mathrm{Sp}_{2r}(F)$ by definition restricts to isomorphisms on $\Gamma \times \Gamma_*$ and that $\chi(f_\sigma|_{\Gamma\times\Gamma_*}) = 1$ for all $\sigma \in \mathrm{Sp}_{2r}(F)$ since χ is trivial on F) that there is a genuine representation of $\mathrm{Sp}_{2r}(F)$ valued in $H(V, \Gamma)$ given by

$$(\mathbf{r}_F(\sigma)\Theta)(x, x^*) = \chi(f_\sigma(x, x^*))\Theta(\sigma(x, x^*)).$$

Using that $Z : \Phi \mapsto \Theta(-, -|\Phi)$ is an isomorphism from $L^2(V)$ to $H(V, \Gamma)$, this provides a representation $\mathbf{r}_{\chi,V} : \mathrm{Sp}_{2r}(F) \to \mathrm{GL}(L^2(V))$ with the property that

$$\Theta(x, x^*|\mathbf{r}_{\chi,V}(\sigma)\Phi) = \chi(f_\sigma(x, x^*))\Theta(\sigma(x, x^*)).$$

It remains to check that this is a lift of $\omega_{\chi,V}$, i.e. that

$$\mathbf{r}_{\chi,V}(\sigma)^{-1}U(v, v^*, t)\mathbf{r}_{\chi,V}(\sigma) = U(\sigma(v, v^*), t)\chi(f_\sigma(x, x^*))U(\sigma(x, x^*), t),$$

which is readily verified from the definitions of Z and Z^{-1}. □

Recall from §1 that $\Theta(0, 0|\Phi)$ as Φ varies is the function that we expect to be some kind of automorphic form in Φ (which is what parametrized the variable τ before). This is exactly what happens:

Corollary 4.2. For all $\Phi \in \mathcal{S}(V)$, the function $\Theta(\Phi) : \mathrm{Mp}(V) \to \mathbf{C}$ given by

$$\Theta(\Phi)(\widetilde{\sigma}) = \Theta(0, 0|\widetilde{\omega}_{\chi,V}(\widetilde{\sigma})\Phi) = \sum_{\xi\in F} (\widetilde{\omega}_{\chi,V}(\widetilde{\sigma})\Phi)(\xi)$$

is invariant under $\mathrm{Sp}_{2r}(F) \subset \mathrm{Mp}(V)$ (embedded via the section of Theorem 4.1).

Proof. Let $\sigma \in \mathrm{Sp}_{2r}(F)$ and $\tilde{s} \in \mathrm{Mp}(V)$. Then we compute directly using Theorem 4.1:

$$\begin{aligned}
\Theta(\Phi)(\sigma\widetilde{s}) &= \Theta(0, 0|\widetilde{\omega}_{\chi,V}(\sigma\widetilde{s}) \\
&= \Theta(0, 0|\widetilde{\omega}_{\chi,V}(\sigma)\widetilde{\omega}_{\chi,V}(\widetilde{s})) \\
&= \Theta(0, 0|\mathbf{r}_{\chi,V}(\sigma)\widetilde{\omega}_{\chi,V}(\sigma)) \\
&= \Theta(0, 0|\widetilde{\omega}_{\chi,V}(\sigma)) \\
&= \Theta(\Phi)(\widetilde{s}),
\end{aligned}$$

as desired. □

In particular, $\Theta(\Phi)$ deserves to be called an automorphic form on $\mathrm{Mp}(V)$. One should note that $\mathrm{Mp}(V)$ is NOT an algebraic group, so the theory of automorphic forms needs to be redeveloped from scratch for the metaplectic group (see for example [Gel1976]). To further convince ourselves that this deserves to be called a theta function, let us compute the usual evaluation on elements corresponding to points on the upper half plane:

Example 2. As in example 1, there are other explicit examples of elements of $\mathrm{Sp}(V \oplus V^*)$ for which the projective Weil representation may be computed. From this it is not hard to fiddle with the constant factors to get a bona fide lift of $\omega_{\chi,V}$ to a genuine representation (but only on those special elements). For example, if f is a quadratic form on V corresponding to the symmetric morphism $\rho : V \to V^*$, then we have the set of elements of $\mathrm{Sp}(V \oplus V^*)$ of the form

$$\sigma = \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}$$

on which we can (for example by ad hoc considerations similar to those of Example 1) define a lift of the projective Weil representation, namely

$$\mathbf{r}_{\chi,V}\left(\begin{pmatrix}1 & \rho\\ 0 & 1\end{pmatrix}\right)\Phi(x) := \Phi(x)\chi(f(x)).$$

Similarly, for elements of the form

$$\sigma = \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{*,-1}\end{pmatrix}$$

with $\alpha \in \mathrm{Aut}(V)$, we can define a lift given by

$$\mathbf{r}_{\chi,V}\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{*,-1}\end{pmatrix}\right)\Phi(x) := |\alpha|^{1/2}\Phi(\alpha x).$$

Putting these two lifts together, we obtain a lift over the Siegel parabolic

$$\begin{pmatrix}* & *\\ 0 & *\end{pmatrix},$$

which is great because these are the matrices we need to describe the values of an automorphic form on the upper half-plane model. Recall that this is because

$$\begin{pmatrix}y^{1/2} & xy^{-1/2}\\ 0 & y^{-1/2}\end{pmatrix} i = x + iy$$

for all $x \in \mathbf{R}$, $y > 0$.

Now let us follow [Gel1976] in recovering the classical theta function $\vartheta(0;\tau)$ as a special case of $\Theta(\Phi)$. Once we have done this, the work of Theorem 4.1 replaces the work of Theorem 2.1 and allows us to immediately conclude the half-integral weight modularity property of ϑ. Let $x + iy \in \mathbf{H}$, and let us denote by $\begin{pmatrix}y^{1/2} & xy^{-1/2}\\ 0 & y^{-1/2}\end{pmatrix}$ the element of $\mathrm{SL}_2(\mathbf{A_Q})$ that is equal to this matrix at the real place and equal to 1 everywhere else.

Let $V = \mathbf{A_Q}$ so that $\mathrm{Sp}(V \oplus V*) = \mathrm{SL}_2(\mathbf{A_Q})$. Let $\Phi = \prod_v \Phi_v$, where $\Phi_\infty(x) = e^{-2\pi x^2} \in \mathcal{S}(\mathbf{R})$ and $\Phi_p(x) = \mathbf{1}_{\mathbf{Z}_p}$ for all rational primes $p < \infty$. Let $\chi = \prod_v \chi_v$, where $\chi_\infty(x) = e^{2\pi i x}$. For $x + iy \in \mathbf{H}$, we may compute (using the explicit sections of the metaplectic covering over the Siegel parabolic described above)

$$\begin{aligned}\Theta(\Phi)\left(\begin{pmatrix}y^{1/2} & xy^{-1/2}\\ 0 & y^{-1/2}\end{pmatrix}\right) &= \sum_{\xi\in\mathbf{Q}}\left(\left(\mathbf{r}_{\chi,V}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\mathbf{r}_{\chi,V}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\right)\Phi\right)(\xi)\\ &= \sum_{\xi\in\mathbf{Q}} e^{2\pi i\xi^2 x}\left(\mathbf{r}_{\chi,V}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\Phi\right)(\xi)\\ &= y^{1/4}\sum_{\xi\in\mathbf{Q}} e^{2\pi i\xi^2 x}\Phi(y^{1/2}\xi)\end{aligned}$$

The term $\Phi(y^{1/2}\xi)$ is zero unless $\xi_p \in \mathbf{Z}_p$ for all $p < \infty$, i.e. unless $\xi \in \mathbf{Z}$, by the explicit form of Φ_p and because the adele y is 1 everywhere except ∞. Using the explicit form of Φ_∞, we are left with

$$y^{1/4}\sum_{\xi\in\mathbf{Z}} e^{2\pi i\xi^2 x}e^{-2\pi y\xi^2} = y^{1/4}\sum_{\xi\in\mathbf{Z}} e^{2\pi i\xi^2(x+iy)} = y^{1/4}\vartheta(0; x+iy),$$

confirming that this automorphic form corresponds in the usual way to what is essentially the classical theta function.

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