Talk #2: Artin L-functions

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Let K be a number field, and $\overline{K}$ its algebraic closure.

1 Galois representations with complex coefficients have finite image

Lemma 1.1 (No small subgroups property)**.** For all Lie groups G, there exists an open neighborhood $U \subset G$ of $1 \in G$ such that U contains no nontrivial subgroups of G.

Proof. The point is to use the exponential map to deduce this fact from the special case of $G = (\mathbf{R}^n, +)$. Let $\mathfrak{g}$ be the Lie algebra of G, and let $1 \in U' \subset G$ be an open neighborhood of the identity small enough that the exponential map $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism $\Omega \xrightarrow{\sim} V$, where Ω is an open neighborhood of 0 in $\mathfrak{g}$ and V is an open neighborhood of 1 in G. The manifold structure on $\mathfrak{g}$ here is the standard one on $\mathbf{R}^{\dim G} \cong \mathfrak{g}$, and can therefore be given by any of the various equivalent norms; choose one of those norms and call it $|\cdot| : \mathfrak{g} \to \mathbf{R}$. Since Ω is an open neighborhood of 0, it contains an open ball around 0 of radius $r > 0$. Let

$$\Omega' := B_{|\cdot|}\left(0, \frac{r}{2}\right) \subset \Omega$$

be the open ball around 0 of radius $r/2$. It has the property that $2 \cdot \Omega' \subset \Omega$ as well.

Since $\exp|_\Omega$ is a diffeomorphism onto its image, which is the open neighborhood $1 \in U' \subset G$, we know that $U := \exp(\Omega')$ is an open neighborhood of 1 in G. We claim that U satisfies the property claimed in the statement of the lemma. Indeed, let $H \subset U$ be a subgroup of G, suppose for the sake of contradiction that there exists a nontrivial $h \in H$, and (as made possible by $\exp^{-1}(h) \neq 0$ since $h \neq 1$) let $n \geq 1$ be such that

$$\frac{r}{2^{n+1}} \leq |\exp^{-1}(h)| < \frac{r}{2^n}.$$

Since $H \subset U$ is a subgroup of G, we have

$$h^{2^n} \in H \subset U.$$

Moreover, $2^n \cdot \exp^{-1}(h) \in \mathfrak{g}$ has the property that

$$|2^n \cdot \exp^{-1}(h)| = 2^n |\exp^{-1}(h)| \in \left[\frac{r}{2}, r\right),$$

i.e.,

$$2^n \cdot \exp^{-1}(h) \in \Omega \setminus \Omega'.$$

But $\exp(2^n \cdot \exp^{-1}(h)) = h^{2^n}$ by the general properties of the exponential map for Lie groups, and since $\exp|_\Omega : \Omega \to U'$ is a bijection, this means $2^n \cdot \exp^{-1}(h)$ is the unique preimage in Ω of h^{2^n} under exp. On the other hand, $h^{2^n} \in U = \exp(\Omega')$, so this preimage must be in $\Omega' \subset \Omega$, which contradicts the fact that $2^n \cdot \exp^{-1}(h) \in \Omega \setminus \Omega'$. Therefore, the original assumption that H contained a nontrivial element was false, and U has the desired property. $\square$

The reason for proving Lemma 1.1 is to apply it to the Lie group $G = \mathrm{GL}_n(\mathbf{C})$ in order to show the following fact about finite-dimensional complex Galois representations.

Proposition 1.2. Let $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbf{C})$ be a continuous representation. Then ρ has finite image [and hence factors through $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(M/K)$ for some finite Galois extension M/K contained inside $\overline{K}$].

Proof. Since $\mathrm{GL}_n(\mathbf{C})$ is a Lie group, by Lemma 1.1, there is an open neighborhood $U \subset \mathrm{GL}_n(\mathbf{C})$ of $1 \in \mathrm{GL}_n(\mathbf{C})$ such that U contains no nontrivial subgroup of $\mathrm{GL}_n(\mathbf{C})$.

Since ρ is continuous, $\rho^{-1}(U) \subset \mathrm{Gal}(\overline{K}/K)$ is an open neighborhood of $1 \in \mathrm{Gal}(\overline{K}/K)$. Since the topology of $\mathrm{Gal}(\overline{K}/K)$ is generated by the system of open subgroups given by $\mathrm{Gal}(\overline{K}/M)$ where M ranges over the finite Galois extensions of K contained in $\overline{K}$, the fact that $\rho^{-1}(U)$ is an open neighborhood of 1 means that

$$\mathrm{Gal}(\overline{K}/M) \subset \rho^{-1}(U)$$

for some finite Galois extension M/K contained in $\overline{K}$. In other words, $\rho(\mathrm{Gal}(\overline{K}/M)) \subset U$. Since $\rho(\mathrm{Gal}(\overline{K}/M))$ is a subgroup of $\mathrm{GL}_n(\mathbf{C})$, this implies it is the trivial subgroup by definition of U. Therefore, ρ factors through the finite quotient $\mathrm{Gal}(M/K)$, and has finite image, as desired. $\square$

Thanks to Proposition 1.2, instead of discussing complex continuous representations of the infinite group $\mathrm{Gal}(\overline{K}/K)$, it will suffice to consider complex representations of groups of the form $\mathrm{Gal}(M/K)$, where M is a finite Galois extension of K.

2 Abelian Artin L-functions, class field theory, and the Cebotarev density theorem

In this section, we consider the case where $n = 1$. In this case, we are looking at characters $\mathrm{Gal}(M/K) \to \mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^\times$. In fact, we can assume that M/K is abelian, since all such characters factor through the abelianization (as $\mathbf{C}^\times$ is abelian).

For any finite prime $\mathfrak{P}$ of M lying over the prime $\mathfrak{p}$ of K, the decomposition group $D(\mathfrak{P}|\mathfrak{p})$ and its normal subgroup the inertia group $I(\mathfrak{P}|\mathfrak{p})$ are useful invariants that tell us about the splitting behavior at $\mathfrak{P}$ of $\mathfrak{p}$ in M. Indeed, recall that $|I(\mathfrak{P}|\mathfrak{p})| = e(\mathfrak{P}|\mathfrak{p})$, and $|D(\mathfrak{P}|\mathfrak{p})| = e(\mathfrak{P}|\mathfrak{p})f(\mathfrak{P}|\mathfrak{p})$. Since $\mathrm{Gal}(M/K)$ transitively permutes the primes lying over $\mathfrak{p}$, the ramification and inertial degrees $e(\mathfrak{P}|\mathfrak{p})$ and $f(\mathfrak{P}|\mathfrak{p})$ do not depend on the choice of $\mathfrak{P}$ lying over $\mathfrak{p}$. Moreover, since we can assume that $\mathrm{Gal}(M/K)$ is abelian, we have

$$D(\sigma\mathfrak{P}|\mathfrak{p}) = \sigma D(\mathfrak{P}|\mathfrak{p})\sigma^{-1} = D(\mathfrak{P}|\mathfrak{p}),$$

and similarly for inertia groups, which implies that in this situation the groups $D(\mathfrak{P}|\mathfrak{p})$ and $I(\mathfrak{P}|\mathfrak{p})$ depend only on $\mathfrak{p}$ and not on the choice of $\mathfrak{P}$ lying over $\mathfrak{p}$.

When $\mathfrak{p}$ is unramified in M (true for all but finitely many primes $\mathfrak{p}$ of K), we have $I(\mathfrak{P}|\mathfrak{p}) = 1$ for all $\mathfrak{P}|\mathfrak{p}$, and therefore we have an isomorphism

$$D(\mathfrak{P}|\mathfrak{p}) \cong \mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p})) = \langle \mathrm{Frob} \rangle.$$

Pulling back $\mathrm{Frob} = [x \mapsto x^{|\kappa(\mathfrak{p})|}]$ via this isomorphism defines the Frobenius element $\mathrm{Frob}_\mathfrak{p}$, which generates the cyclic group $D(\mathfrak{P}|\mathfrak{p})$ and doesn't depend on the choice of $\mathfrak{P}$ since M/K is abelian. The element $\mathrm{Frob}_\mathfrak{p}$, which in this case ($M/K$ abelian and $\mathfrak{p}$ unramified) is a well-defined generator of the cyclic group $D(\mathfrak{P}|\mathfrak{p}) \subset \mathrm{Gal}(M/K)$ that depends only on $\mathfrak{p}$. This single element of $\mathrm{Gal}(M/K)$ therefore tells us the full detail of how an unramified prime $\mathfrak{p}$ splits in M, and is very important to understanding the arithmetic of the extension M/K. For example, if $K = \mathbf{Q}$ and $M = \mathbf{Q}(\sqrt{d})$, d a squarefree integer, then (say, when $p \neq 2, d$ to guarantee it is not ramified in M) $\mathrm{Frob}_p \in \mathbf{Z}/2\mathbf{Z} \cong \mathrm{Gal}(M/K)$ is the Legendre symbol of d mod p. As such, it is interesting to ask about the statistics of the element $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(M/K)$ as $\mathfrak{p}$ varies over the primes of K unramified in M. If we order the $\mathfrak{p}$ by their norm, for a fixed $g \in \mathrm{Gal}(M/K)$, what is the probability that $\mathrm{Frob}_\mathfrak{p} = g$?

By copying the proof of Dirichlet's theorem on primes in arithmetic progression, one might expect to answer this question by studying the following L-function.

Definition 2.1. Let $\chi : \mathrm{Gal}(M/K) \to \mathbf{C}^\times$ be a character. The associated *Artin L-function* is defined to be

$$L(s, \chi) := \prod_{\substack{\mathfrak{p} \\ \chi(I(\mathfrak{P}|\mathfrak{p}))=1}} \left(\frac{1}{1 - \chi(\mathrm{Frob}_\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s}} \right),$$

for all $s \in \mathbf{C}$ such that this converges.

Example 1. Let $M/K = \mathbf{Q}(\sqrt{-1})/\mathbf{Q}$, and $\chi : \mathrm{Gal}(M/K) = \langle\sigma\rangle \to \mathbf{C}^\times$ the unique nontrivial character, which is given by $\chi(\sigma) = -1$. Since an odd rational prime p is split in $\mathbf{Q}(\sqrt{-1})$ if and only if -1 is a quadratic residue modulo p, and 2 is the only ramified prime, we have

$$L(s,\chi) = \prod_{p\neq 2} \frac{1}{1 - \left(\frac{-1}{p}\right) p^{-s}}.$$

Since

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$$

by quadratic reciprocity / Euler's criterion, this is exactly the same as the Dirichlet L-function

$$\begin{aligned}\left(\frac{1}{1+3^{-s}}\right)\left(\frac{1}{1-5^{-s}}\right)\left(\frac{1}{1+7^{-s}}\right)\left(\frac{1}{1+11^{-s}}\right)\cdots &= 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \frac{1}{13^s} + \cdots \\ &= L^{\mathrm{Hecke}}(s,\widetilde{\chi}),\end{aligned}$$

where $\widetilde{\chi} : (\mathbf{Z}/4\mathbf{Z})^\times \to \mathbf{C}^\times$ is the unique nontrivial Dirichlet character, that is, the one that takes $(-1 \mod 4)$ to -1.

Example 2. More generally, let $M/K = \mathbf{Q}(\sqrt{d})/\mathbf{Q}$, where $d \in \mathbf{Z}$ is squarefree, and again let $\chi : \mathrm{Gal}(M/K) = \langle\sigma\rangle \to \mathbf{C}^\times$ be the unique nontrivial character, namely the one sending $\sigma \mapsto -1$. Let Δ_d be the discriminant of M/K. It is d when $d \equiv 1 \pmod 4$ and $4d$ when $d \equiv 2, 3 \pmod 4$. Then we have

$$L(s,\chi). = \prod_{p\nmid\Delta_d} \frac{1}{1 - \left(\frac{d}{p}\right) p^{-s}} = \sum_{(n,\Delta_d)=1} \left(\frac{d}{n}\right) n^{-s} = \sum_{n=1}^{\infty} \left(\frac{\Delta_d}{n}\right) n^{-s} = L^{\mathrm{Dirichlet}}(s,\widetilde{\chi}),$$

where $\left(\frac{\cdot}{n}\right)$ denotes the Kronecker symbol and $\widetilde{\chi}$ is just another name for the Dirichlet character that is $n \mapsto \left(\frac{\Delta_d}{n}\right)$ (the fact that it is a Dirichlet character, and of conductor $|\Delta_d|$, is a consequence of quadratic reciprocity).

Example 3. Via class field theory (i.e. Kronecker–Weber) for $\mathbf{Q}$ or Gauss sums, the previous two examples are in turn consequences of the example $M/K = \mathbf{Q}(\zeta_N)/\mathbf{Q}$, where $N \geq 1$. In this situation, we have $\mathrm{Gal}(M/K) \overset{\varphi}{\cong} (\mathbf{Z}/N\mathbf{Z})^\times$, and we can consider an arbitrary primitive[1] character $\widetilde{\chi} : (\mathbf{Z}/N\mathbf{Z})^\times$, setting $\chi : \mathrm{Gal}(M/K) \to \mathbf{C}^\times$ to be

$$\chi := \widetilde{\chi} \circ \varphi.$$

Then by the usual fact that $\varphi(\mathrm{Frob}_p) = (p \mod N)$, we have

$$L(s,\chi) = \prod_{p\nmid N} \frac{1}{1 - \widetilde{\chi}(p \mod N)p^{-s}} = L^{\mathrm{Dirichlet}}(s,\widetilde{\chi}).$$

As usual, we can at least verify that this product converges absolutely and uniformly to a holomorphic function on horizontal half-planes to the right of $\mathrm{Re}(s) = 1$:

Lemma 2.2. The Artin L-series $L(s,\chi)$ converges to a holomorphic function on all horizontal half-planes of the form

$$\mathrm{Re}(s) \geq 1 + \varepsilon,$$

for $\varepsilon > 0$.

[1] Just so we don't need to do the completely doable exercise of figuring out inertia groups in cyclotomic extensions of $\mathbf{Q}$. In general the two L-functions will differ by some Euler factors at the primes dividing N, as governed by what the inertia groups are. This inertia group calculation is done explicitly in my notes [Gun2019, example 3.23].

Proof. It suffices to to prove the statement for the product

$$\prod_{\mathfrak{p}\nmid\Delta_{M/K}} \left(1-\chi(\mathrm{Frob}_{\mathfrak{p}})\mathrm{N}\mathfrak{p}^{-s}\right).$$

Letting $c_{\mathfrak{p}} = |\mathrm{N}\mathfrak{p}^{-s}|$, which is a positive number with the property that

$$c_{\mathfrak{p}} = \mathrm{N}\mathfrak{p}^{-\Re(s)} \leq \mathrm{N}\mathfrak{p}^{-(1+\varepsilon)},$$

and using the fact that there are at most $[K:\mathbf{Q}]$ distinct primes $\mathfrak{p}$ of K lying over a given rational prime p, we have

$$\sum_{\mathfrak{p}} c_{\mathfrak{p}} \leq [K:\mathbf{Q}]\sum_{p} p^{-(1+\varepsilon)},$$

where the second sum is over all rational primes p under a prime involved in the first sum. This sum converges when $\varepsilon > 0$ (it is bounded above by $\zeta_{\mathbf{Q}}(1+\varepsilon)$ for example), so since $|\chi(\mathrm{Frob}_{\mathfrak{p}})| = 1$ for all $\mathfrak{p}$ unramified in M, we conclude the desired uniform convergence to a holomorphic function by [SS2003, Proposition 5.3.2]. □

For Hecke L-functions, we automatically get convergence in a region containing 1 (I think this will have been talked about in the talk of Fernando Trejos Suarez and I know it from [Lan1994, Ch. VIII]) and meromorphic continuation holomorphic everywhere except possibly 1 and functional equation. For our Artin L-functions, this will not be obvious unless we connect them to Hecke L-functions. Indeed, one proof that Hecke L-series for finite-order Hecke characters of K converge to a holomorphic function on the region $\mathrm{Re}(s) > 1-1/[K:\mathbf{Q}]$, in a direct generalization of the same proof for Dirichlet L-functions, uses the fact that

$$\sum_{\mathfrak{c}}\sum_{\substack{\mathfrak{a}\in\mathfrak{c}\\ \mathrm{N}\mathfrak{a}\leq n}} \chi(\mathfrak{a}) = O(n^{1-1/[K:\mathbf{Q}]}),$$

where $\mathfrak{c}$ runs over the appropriate (ray) classes of ideals in K. The point is to get the savings $1/[K:\mathbf{Q}]$ by grouping together one ideal in each class, and proving through independent means (in this case by Davenport-type results on lattice points in expanding domains[2]) that the number of integral ideals $\mathfrak{a}\in\mathfrak{c}$ with $\mathrm{N}\mathfrak{a}\leq n$ doesn't depend on $\mathfrak{c}$ up to an $O(n^{1-1/[K:\mathbf{Q}]})$ error. In the case of Galois groups, there is *a priori* no general way to do the counting of integral ideals of bounded norm whose Frobenius (extended linearly to all ideals supported away from the discriminant) equals a fixed element of $\mathrm{Gal}(M/K)$, which is why Lemma 2.2 is as far as we will go without relating our Artin L-functions to Hecke L-functions, by hook or by crook.

We really want to do be able to extend/have convergence at least a little bit to the left of 1, since (again by analogy to the proof of Dirichlet's theorem) the proof of Cebotarev's density theorem will require understanding the Artin L-functions near $s=1$. In this section, where M/K is abelian, the point will be to do this by relating to Hecke L-functions via technique of class field theory.

Definition 2.3. We keep the following notation for the basic objects on Hecke side of class field theory.

- A *modulus* of K is a $\mathbf{Z}_{\geq 0}$-linear combination of (possibly infinite) primes of K.
- For a modulus $\mathfrak{m}$ of K, define $I_K(\mathfrak{m})$ to be the group of fractional ideals of K supported away from the finite part of $\mathfrak{m}$.
- For a modulus $\mathfrak{m} = \prod_{v|\infty} v^{a_v}\prod_{\mathfrak{p}<\infty}\mathfrak{p}^{a_{\mathfrak{p}}}$, define the subgroup $P_{\mathfrak{m}}\subset I_K(\mathfrak{m})$ to be the set of principal fractional ideals (α) such that $\alpha\in K^\times$ has the property that for all real $v|\infty$ with $a_v\geq 1$,

  $$v(\alpha) > 0$$

  and that for all finite $\mathfrak{p}$ with $a_{\mathfrak{p}}\geq 1$,

  $$v_{\mathfrak{p}}(1-\alpha)\geq a_{\mathfrak{p}}.$$

[2] This geometry of numbers argument also gives you the exact residue of the pole at $s=1$ when it exists, i.e. the analytic class number formula.

- The group $I_K(\mathfrak{m})/P_{\mathfrak{m}}$ is called the *ray class group of K modulo* $\mathfrak{m}$, and it is a finite abelian group (follows from finiteness of the class group and some dévissage).
- The subgroup $\mathfrak{N}(\mathfrak{m}) \subset I_K(\mathfrak{m})$ is the one consisting of all norms of elements of $I_K(\mathfrak{m})$.

Theorem 2.4 (Class field theory)**.** Let M/K be a finite abelian extension. There is a modulus $\mathfrak{f} = \mathfrak{f}(M/K)$ divisible exactly by all the primes (finite and infinite) of K that are ramified in M such that the map $\mathfrak{p} \mapsto \mathrm{Frob}_{\mathfrak{p}}$ (extended linearly to $I_K(\mathfrak{f})$) defines an isomorphism

$$\mathrm{Art} : I_K(\mathfrak{f})/P_{\mathfrak{f}}\mathfrak{N}(\mathfrak{f}) \xrightarrow{\sim} \mathrm{Gal}(M/K).$$

Corollary 2.5. Let M/K be a finite abelian extension, and $\chi : \mathrm{Gal}(M/K) \to \mathbf{C}^\times$ a character. Then on the region $\mathrm{Re}(s) > 1$, we have

$$L^{\mathrm{Artin}}(s, \chi) = L^{\mathrm{Hecke}}(s, \chi \circ \mathrm{Art})$$

up to a finite number of Euler factors, which is an exact equality in the case that χ is injective.

Proof. By Lemma 2.2, and its analog for the Hecke L-functions (also follows from the Euler product for those), it suffices to prove that the Euler factors at each $\mathfrak{p}$ coincide. Both the Hecke and Artin L-functions here have trivial Euler factors at the finite primes dividing $\mathfrak{f}$ (since those primes are exactly those that ramify in M). For a prime $\mathfrak{p}$ not dividing $\mathfrak{f}$, the Euler factor at $\mathfrak{p}$ of $L^{\mathrm{Hecke}}(s, \chi \circ \mathrm{Art})$ is (according to the definition)

$$\frac{1}{1 - \chi(\mathrm{Art}([\mathfrak{p}]))\mathrm{N}\mathfrak{p}^{-s}}.$$

But the definition of the Artin reciprocity map was that it took classes of primes to Frobenii, so this is just

$$\frac{1}{1 - \chi(\mathrm{Frob}_{\mathfrak{p}})\mathrm{N}\mathfrak{p}^{-s}},$$

which is exactly the Euler factor at $\mathfrak{p}$ of $L^{\mathrm{Artin}}(s, \chi)$.

When χ is injective, the primes $\mathfrak{p}$ of K such that $\chi(I(\mathfrak{P}|\mathfrak{p})) \neq 1$ are exactly the primes which are ramified in L, i.e., those dividing $\mathfrak{f}(L/K)$, so the Euler factors at primes dividing $\mathfrak{f}$ for both the Dirichlet and Artin L-functions here are all trivial. □

Corollary 2.6. Let M/K be a finite abelian extension and $\chi : \mathrm{Gal}(M/K) \to \mathbf{C}^\times$ be a character. Then the L-series $L^{\mathrm{Artin}}(s, \chi)$ converges (except where it has poles) to a meromorphic function on $\mathrm{Re}(s) > 1 - 1/[K : \mathbf{Q}]$. Furthermore, it admits a meromorphic continuation to the entire complex plane. If $\chi = 1$, then the only pole of $L^{\mathrm{Artin}}(s, \chi)$ is a simple pole at $s = 1$. If $\chi \neq 1$, then it is holomorphic.

Proof. Follows from the same properties of Hecke L-functions (hopefully proved in Fernando's talk). In particular, $\chi \circ \mathrm{Art}$ can be viewed as a character of $I_K(\mathfrak{f})/P_{\mathfrak{f}}$ that happens to also vanish on $\mathfrak{N}(\mathfrak{f})$, so the L-functions and the theory of [Lan1994, Ch. VIII] can be used to prove the claim about convergence and about the possible pole at $s = 1$. For the same reason, Tate's thesis [Tat1967] or Hecke's work (hopefully also done in Fernando's talk) also applies, and shows the meromorphic continuation to all of $\mathbf{C}$ as well as the claimed information about the poles. It also gives us a functional equation for $L^{\mathrm{Artin}}(s, \chi)$, which will depend on the ε and γ -factors for the Hecke character $\chi \circ \mathrm{Art}$. Note that we can assume in this whole argument that χ is injective and therefore not worry about the possible extra Euler factors in Corollary 2.6 by replacing χ with the character $\mathrm{Gal}(M'/K) \to \mathbf{C}^\times$ it factors through (where $\mathrm{Gal}(M/M')$ is the kernel of χ); this makes no difference to $L^{\mathrm{Artin}}(s, \chi)$, as is easy to check (we will check it in greater generality in Lemma 3.4 anyway). □

To prove the Cebotarev density theorem, now that we have extended $L^{\mathrm{Artin}}(s, \chi)$ slightly to the left of $s = 1$ (though of course we really have it to the entire complex plane except possible a pole at $s = 1$ by the general meromorphic continuation for Hecke L-functions, though this will not be necess), we will need the nonvanishing of $L^{\mathrm{Artin}}(1, \chi)$ for nontrivial Galois characters χ. This is a consequence of the following

Proposition 2.7. Let M/K be a finite abelian extension. Then

$$\zeta_M(s) = \prod_{\chi \in \widehat{\mathrm{Gal}(M/K)}} L^{\mathrm{Artin}}(s, \chi)$$

as meromorphic functions on $\mathbf{C}$ up to a finite product of holomorphic functions not vanishing at $s = 1$.

Proof. By Corollary 2.6, it suffices to verify an equality of Euler factors, where for $\zeta_M(s)$ they are grouped by the prime of K that they lie over. In other words, we need to prove that for all finite primes $\mathfrak{p}$ of K,

$$\prod_{\mathfrak{P}|\mathfrak{p}} \frac{1}{1 - \mathrm{N}\mathfrak{P}^{-s}} = \prod_{\chi \in \widehat{\mathrm{Gal}(M/K)}} \frac{1}{1 - \chi(\mathrm{Frob}_{\mathfrak{p}})\mathrm{N}\mathfrak{p}^{-s}}.$$

What is sufficient (by plugging in $X = \mathrm{N}\mathfrak{p}^{-s}$), we will prove that

$$\prod_{\mathfrak{P}|\mathfrak{p}} (1 - X^{f(\mathfrak{P}|\mathfrak{p})}) = \prod_{\chi \in \widehat{\mathrm{Gal}(M/K)}} (1 - \chi(\mathrm{Frob}_{\mathfrak{p}})X).$$

Since M/K is Galois, the quantities $f(\mathfrak{P}|\mathfrak{p})$ do not depend on $\mathfrak{P}$, so call this common value $f_{\mathfrak{p}}$ and let $r_{\mathfrak{p}}$ be the number of $\mathfrak{P}|\mathfrak{p}$, so that the left hand side becomes

$$(1 - X^{f_{\mathfrak{p}}})^{r_{\mathfrak{p}}} = \prod_{i=0}^{f_{\mathfrak{p}}-1} (1 - \zeta_{f_{\mathfrak{p}}}^i)^{r_{\mathfrak{p}}}.$$

This means that we just need to show that the values $\chi(\mathrm{Frob}_{\mathfrak{p}})$, as χ runs over $\widehat{\mathrm{Gal}(M/K)}$, run over all the $f_{\mathfrak{p}}$-th roots of unity, repeating each possibility exactly $r_{\mathfrak{p}}$ times. Since $\langle \mathrm{Frob}_{\mathfrak{p}} \rangle = D(\mathfrak{P}|\mathfrak{p})$, which has order $f_{\mathfrak{p}}$, the value of $\chi(\mathrm{Frob}_{\mathfrak{p}})$ is always an $f_{\mathfrak{p}}$-th root of unity. Moreover, $[\mathrm{Gal}(M/K) : D(\mathfrak{P}|\mathfrak{p})] = r_{\mathfrak{p}}$, so the characters $\chi : \mathrm{Gal}(M/K) \to \mathbf{C}^\times$ can be all be written in the form $\chi_i \psi_j$, where the χ_i are arbitrary extensions to $\mathrm{Gal}(M/K)$ of the $f_{\mathfrak{p}}$ characters of the cyclic group $D(\mathfrak{P}|\mathfrak{p}) \subset \mathrm{Gal}(M/K)$, and the ψ_j are the $r_{\mathfrak{p}}$ characters of $\mathrm{Gal}(M/K)$ that vanish on $D(\mathfrak{P}|\mathfrak{p})$. The characters χ_i are determined by which $f_{\mathfrak{p}}$-th root of unity $\zeta_{f_{\mathfrak{p}}}^i$ they send $\mathrm{Frob}_{\mathfrak{p}}$ to, and by definition multiplying by ψ_j does not affect the value at $\mathrm{Frob}_{\mathfrak{p}} \in D(\mathfrak{P}|\mathfrak{p})$, so this is exactly what we get. $\square$

Corollary 2.8. Let M/K be a finite abelian extension. For any nontrivial $\chi : \mathrm{Gal}(M/K) \to \mathbf{C}^\times$,

$$L^{\mathrm{Artin}}(1, \chi) \neq 0$$

(we already know it is a well-defined complex number by Corollary 2.6)

Proof. We use Proposition 2.7: for s near 1, we have

$$\zeta_M(s) = \zeta_K(s) \left(\prod_{\chi \neq 1} L^{\mathrm{Artin}}(s, \chi) \right) \varphi(s)$$

where $\varphi(s)$ is a meromorphic function holomorphic and nonzero at $s = 1$. The left hand side has a simple pole at $s = 1$, so the same must be true of the right hand side. The $\varphi(s)$-term contributes no zeros or poles. The $\zeta_K(s)$-term contributes a simple pole. Therefore,

$$\mathrm{ord}_{s=1} \left(\prod_{\chi \neq 1} L^{\mathrm{Artin}}(s, \chi) \right) = 0.$$

But by Corollary 2.6, none of the terms in this product can have a pole at $s = 1$, so (since a sum of nonnegative integers being zero means all the integers were zero to begin with) none of those terms can have a zero at $s = 1$. $\square$

As promised, Corollary 2.8 and Corollary 2.6 directly imply the Dirichlet density version of the Cebotarev density theorem:

Theorem 2.9 (Cebotarev density theorem)**.** Let M/K be a finite abelian extension. Then for any $\sigma \in \mathrm{Gal}(M/K)$, the set

$$\mathcal{P}_\sigma := \{\text{primes } \mathfrak{p} \nmid \Delta_{M/K} \text{ such that } \mathrm{Frob}_\mathfrak{p} = \sigma\}$$

has the property that

$$\lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in \mathcal{P}_\sigma} \frac{1}{\mathrm{N}\mathfrak{p}^s}}{\sum_{\mathfrak{p} \nmid \Delta_{K/M}} \frac{1}{\mathrm{N}\mathfrak{p}^s}} = \frac{1}{[M:K]}.$$

Proof. The point is to copy the proof of Dirichlet's theorem on primes in arithmetic progressions. For $\sigma \in \mathrm{Gal}(M/K)$, and $x > 0$, let $\pi(x;\sigma)$ be the number of primes $\mathfrak{p} \in \mathcal{P}_\sigma$ there are such that $\mathrm{N}\mathfrak{p} \leq x$. The key idea is to consider $\pi(x;-)$ as a function on the finite abelian group $\mathrm{Gal}(M/K)$, and use the fact from Fourier analysis on finite abelian groups to decompose it as a linear combination of characters of $\mathrm{Gal}(M/K)$. In particular, for all $\sigma \in \mathrm{Gal}(M/K)$ and $y > 0$, we have

$$\pi(y;\sigma) = \sum_{\chi \in \widehat{\mathrm{Gal}(M/K)}} \langle \chi, \pi(y;-)\rangle \chi(\sigma),$$

where

$$\langle \chi, \pi(y;-)\rangle := \frac{1}{[M:K]} \sum_{\tau \in \mathrm{Gal}(M/K)} \overline{\chi}(\tau)\pi(y;\tau).$$

We apply summation by parts and apply this Fourier decomposition of $\pi(y;-)$ once the $\pi(y;\tau)$ is inside the integral: for real $s > 1$,

$$\begin{aligned}
\sum_{\mathfrak{p} \in \mathcal{P}_\sigma} \frac{1}{\mathrm{N}\mathfrak{p}^s} &= \int_1^\infty \frac{1}{y^s} d(\pi(y;\sigma)) \\
&= -s \int_1^\infty \pi(y;\sigma) y^{-s} \frac{dy}{y} \\
&= -s \int_1^\infty \left(\sum_{\chi \in \widehat{\mathrm{Gal}(M/K)}} \langle \chi, \pi(y;-)\rangle \chi(\sigma) \right) y^{-s} \frac{dy}{y} \\
&= -\frac{s}{[M:K]} \int_1^\infty \left(\sum_{\chi \in \widehat{\mathrm{Gal}(M/K)}} \sum_{\tau \in \mathrm{Gal}(M/K)} \overline{\chi}(\tau)\pi(y;\tau)\chi(\sigma) \right) y^{-s} \frac{dy}{y} \\
&= \frac{1}{[M:K]} \sum_{\chi \in \widehat{\mathrm{Gal}(M/K)}} \chi(\sigma) \cdot \left(-s \int_1^\infty \sum_{\tau \in \mathrm{Gal}(M/K)} \overline{\chi}(\tau)\pi(y;\tau) y^{-s} \frac{dy}{y} \right) \\
&= \frac{1}{[M:K]} \sum_{\chi \in \widehat{\mathrm{Gal}(M/K)}} \chi(\sigma) \cdot \sum_{\mathfrak{p} \nmid \Delta_{M/K}} \frac{\overline{\chi}(\mathrm{Frob}_\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{-s}} \\
&= \frac{1}{[M:K]} \sum_{\chi \in \widehat{\mathrm{Gal}(M/K)}} \overline{\chi}(\sigma) \cdot \sum_{\mathfrak{p} \nmid \Delta_{M/K}} \frac{\chi(\mathrm{Frob}_\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{-s}}
\end{aligned}$$

Choose a branch of the complex logarithm near 1. The sum on the inside is very similar to $\log L^{\mathrm{Artin}}(s,\chi)$, which is

$$\log \prod_{\mathfrak{p} \nmid \Delta_{M/K}} \frac{1}{1 - \chi(\mathrm{Frob}_\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s}} = \sum_{\mathfrak{p} \nmid \Delta_{M/K}} -\log(1 - \chi(\mathrm{Frob}_\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})$$

$$
\begin{aligned}
&= \sum_{\mathfrak{p}\nmid\Delta_{M/K}} \sum_{m\geq 1} \frac{\chi(\mathrm{Frob}_\mathfrak{p})^m}{m\mathrm{N}\mathfrak{p}^{ms}} \\
&= \sum_{\mathfrak{p}\nmid\Delta_{M/K}} \frac{\chi(\mathrm{Frob}_\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{s}} + \sum_{m\geq 2} \sum_{\mathfrak{p}\nmid\Delta_{M/K}} \frac{\chi(\mathrm{Frob}_\mathfrak{p})^m}{m\mathrm{N}\mathfrak{p}^{ms}}.
\end{aligned}
$$

And the $m \geq 2$ term is actually $O(1)$ for $s \to 1^+$, because in absolute value it is at most

$$
\sum_{\mathfrak{p}\nmid\Delta_{M/K}} \sum_{m\geq 2} \mathrm{N}\mathfrak{p}^{-ms} = \sum_{\mathfrak{p}\nmid\Delta_{M/K}} \frac{\mathrm{N}\mathfrak{p}^{-2}}{1-\mathrm{N}\mathfrak{p}^{-s}} \leq 2\zeta_K(2) < \infty.
$$

Continuing our calculation, we conclude that

$$
\begin{aligned}
\sum_{\mathfrak{p}\in\mathcal{P}_\sigma} \frac{1}{\mathrm{N}\mathfrak{p}^s} &= \frac{1}{[M:K]} \sum_{\chi\in\widehat{\mathrm{Gal}(M/K)}} \overline{\chi}(\sigma)\cdot\left(\log L^{\mathrm{Artin}}(s,\chi) + O_{s\to 1^+}(1)\right) \\
&= \frac{1}{[M:K]} \sum_{\chi\in\widehat{\mathrm{Gal}(M/K)}} \overline{\chi}(\sigma)\log L^{\mathrm{Artin}}(s,\chi) + O_{s\to 1^+}(1).
\end{aligned}
$$

By Corollary 2.8, the quantities $\log L^{\mathrm{Artin}}(s,\chi)$ are bounded as $s \to 1^+$, so the only term that isn't absorbed into the $O_{s\to 1^+}(1)$ error is the $\chi = 1$ term. That term is

$$
\log L^{\mathrm{Artin}}(s,1) = \sum_{\mathfrak{p}\nmid\Delta_{M/K}} \frac{1}{\mathrm{N}\mathfrak{p}^s} + O_{s\to 1^+}(1),
$$

so we deduce after substituting all of this in that

$$
\sum_{\mathfrak{p}\in\mathcal{P}_\sigma} \frac{1}{\mathrm{N}\mathfrak{p}^s} = \frac{1}{[M:K]} \sum_{\mathfrak{p}\nmid\Delta_{M/K}} \frac{1}{\mathrm{N}\mathfrak{p}^s} + O_{s\to 1^+}(1),
$$

and hence that

$$
\lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in\mathcal{P}_\sigma} \frac{1}{\mathrm{N}\mathfrak{p}^s}}{\sum_{\mathfrak{p}\nmid\Delta_{M/K}} \frac{1}{\mathrm{N}\mathfrak{p}^s}} = \frac{1}{[M:K]},
$$

as desired. □

Remark Cebotarev actually proved the non-abelian generalization of Theorem 2.9, where $\mathrm{Frob}_\mathfrak{p}$ is considered as a conjugacy class. That generalization says that the Dirichlet density of the set of unramified $\mathfrak{p}$ with $\mathrm{Frob}_\mathfrak{p}$ equal to a given conjugacy class $C \subset \mathrm{Gal}(M/K)$ is equal to $|C|/[M:K]$

Remark By class field theory, Theorem 2.9 also implies the generalization of Dirichlet's theorem on primes in arithmetic progression for primes in ray classes.

Remark Although we only stated and proved Theorem 2.9 for Dirichlet density, one can run the usual machine with zero-free regions and logarithmic derivatives to prove the natural density version of it. This was done by Serre–Odlyzko [Ser1981].

3 Non-abelian Artin L-functions, Brauer's theorem, and meromorphic continuation

Now let M/K be a finite (possibly non-abelian) Galois extension. For an arbitrary representation $\rho : \mathrm{Gal}(M/K) \to \mathrm{GL}(V)$ where V is an n-dimensional $\mathbf{C}$-vector space, we can define an Artin L-function for ρ analogously to Definition 2.1 as follows.

Definition 3.1. The Artin L-series attached to ρ is

$$L^{\mathrm{Artin}}(s, \rho) := \prod_{\mathfrak{p}} \frac{1}{\det((1 - \mathrm{N}\mathfrak{p}^{-s}\rho(\mathrm{Frob}_{\mathfrak{P}}))|_{V^{I(\mathfrak{P}|\mathfrak{p})}})}.$$

Note that since $\mathrm{Gal}(M/K)$ is not necessarily abelian, making this definition required a choice of $\mathfrak{P}|\mathfrak{p}$, but that it does not depend on this choice because $I(\mathfrak{P}|\mathfrak{p})$ is normal in $D(\mathfrak{P}|\mathfrak{p})$. When $\mathfrak{p}$ is ramified in M, the Euler factor at $\mathfrak{p}$ is well-defined (it doesn't depend on the choice of $\mathrm{Frob}_{\mathfrak{P}}$, which is only well-defined up to $I(\mathfrak{P}|\mathfrak{p})$) because the operator involved in the denominator is acting only on the fixed points of inertia.

Remark In Artin's original paper [Art1924], he defined the L-functions to have trivial L-factors at the ramified primes. In a later paper [Art1931], he changed the definition and verified that it still satisfied the properties we will prove here (in particular independence on induction of characters), since those factors had to appear in the functional equation that we will prove in this section.

Example 4. Consider the polynomial $f(X) = X^5 - X - 1 \in \mathbf{Z}[X]$. This polynomial is irreducible, as it is an Artin–Schreier polynomial modulo 5. Let us first recall the proof that f is irreducible modulo 5, following the general proof for Artin–Schreier polynomials. $f \mod 5$ is separable over $\mathbf{F}_5$, since its derivative is the nonzero constant -1 and therefore shares no roots with f in $\overline{\mathbf{F}}_5$. The Frobenius $x \mapsto x^5$ sends a root $x \in \overline{\mathbf{F}}_5$ of f to $x+1$ (by definition of f), so the roots of f are $x, x+1, x+2, x+3, x+4$, which are all distinct, and we see that $\mathrm{Gal}(\overline{\mathbf{F}}_5/\mathbf{F}_5)$ acts transitively on the roots of f, i.e. that f is irreducible over $\mathbf{F}_5$.

Let $K = \mathbf{Q}[X]/(X^5 - X - 1) = \mathbf{Q}(\alpha)$, where WLOG $\alpha = \alpha_5$ is a choice of roots $\alpha_1, \dots, \alpha_5$ of f in $\overline{F}_5$. By PARI/GP computation, the prime factorization of the discriminant of $X^5 - X - 1$ is

$$\mathrm{disc}(X^5 - X - 1) = 19 \cdot 151.$$

In particular, this is squarefree, which implies that

$$\mathcal{O}_K = \mathbf{Z}[\alpha].$$

Therefore, by the Dedekind–Kummer theorem, for *every* rational prime p, the splitting type of p in K can be read off of the splitting type of f modulo p: the irreducible factors of $f \mod p$ are in bijection with the primes of K lying over p, and this bijection takes degrees of irreducible factors to inertial degrees and multiplicities to ramification indices. As a reminder of how this works, e.g. from my notes [Kis2019, Lemma 11.3]: if $\overline{f} := f \mod p$ has prime factorization

$$\overline{f} = \prod_{i=1}^{r} \overline{f}_i^{e_i} \in \mathbf{F}_p[X],$$

then the ideal $\mathfrak{p}_i := (p, f_i(\alpha)) \subset \mathcal{O}_K$ is prime (its quotient is $\mathbf{F}_p[X]/(\overline{f_i})$), and $I = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i} \subset (p)$, since modulo p all the generators of $\mathfrak{p}_i$ are zero because they have a factor of p in their definition, except maybe the last one $\prod_{i=1}^{r} f_i(\alpha)^{e_i}$, which is still zero mod p because $f(\alpha) = 0$ coincides with this product modulo p. For the opposite inclusion, write $(p) = \prod_i \mathfrak{p}_i^{e_i'}$, where $e_i' \le e_i$ by what we just proved. Taking this modulo (p), we must get zero, but the ideal on the right hand side is generated modulo p by

$$\prod_{i=1}^{r} \overline{f}_i^{e_i'}(\overline{\alpha}) \in \mathcal{O}_K/(p),$$

which we conclude is zero as well, since $\mathcal{O}_K/(p) = \mathbf{F}_p[X]/(\overline{f}(X))$ where X corresponds to $\overline{\alpha}$.

It is a very interesting question to try and understand the splitting type of $X^5 - X - 1$ modulo p as p varies. This is the question that is answered for polynomials with abelian Galois group by class field theory (for example in the abelian examples of the previous section we did it for quadratic polynomials and cyclotomic polynomials, which happened in our analysis of how the Artin L-functions we were looking at in that case were actually Hecke L-functions). So it makes sense to build an L-function that encodes the splitting information of rational primes in K. On the other hand, Artin L-functions take in the Frobenii of the primes as input, which are a priori defined to be in some Galois group. $K/\mathbf{Q}$ is NOT Galois, so we will need to do the exercise of figuring out splitting types of rational primes p in K based on their Frobenii and other Galois decomposition data in the Galois closure M of $K/\mathbf{Q}$. We assemble the answer to this exercise from [Woo2011], [Gun2019, §4], [Neu1999, §II.9]. Let

$$G = \mathrm{Gal}(M/\mathbf{Q}) = \mathrm{Gal}(\mathbf{Q}(\alpha_1, \ldots, \alpha_5)/\mathbf{Q}).$$

It is a subgroup of S_5 according to how it permutes these five roots considered in the order that we chose. Let p be a rational prime, $\mathfrak{q}|p$ a prime of K, and $\mathfrak{P}|\mathfrak{p}$ a prime of M. Then there is a bijection

$$D(\mathfrak{P}|p)\backslash\mathrm{Gal}(M/\mathbf{Q})/H \cong \{\text{primes of } K \text{ lying over } p\}$$

where $H = \mathrm{Gal}(M/K)$, given by

$$\sigma \mapsto \sigma\mathfrak{p}.$$

Indeed, this map is well-defined because H fixes K and by definition of decomposition groups, and proving it is a bijection amounts to the fact that $D(\mathfrak{P}|p)\backslash\mathrm{Gal}(M/\mathbf{Q})$ is in bijection with the set of primes of M lying over p (because of transitivity of the Galois action on those primes), and that the $\mathrm{Gal}(M/K)$-orbits of primes of M lying over p are in bijection with the primes of K lying over p. Moreover,

$$e(\sigma\mathfrak{p}|p)[\kappa(\sigma\mathfrak{p}) : \kappa(p)] = \frac{e(\sigma\mathfrak{P}|p)[\kappa(\sigma\mathfrak{P}) : \kappa(p)]}{e(\sigma\mathfrak{P}|\sigma\mathfrak{p})[\kappa(\sigma\mathfrak{P}) : \kappa(\sigma\mathfrak{p})]} = \frac{|D(\mathfrak{P}|\mathfrak{p})|}{|D(\sigma\mathfrak{P}|\sigma\mathfrak{p})|} = \frac{|D(\mathfrak{P}|\mathfrak{p})|}{|D(\sigma\mathfrak{P}|p) \cap H|} = \frac{|D(\mathfrak{P}|\mathfrak{p})|}{|\sigma D(\mathfrak{P}|\mathfrak{p})\sigma^{-1} \cap H|},$$

which is just the size of σ considered as an orbit of $D(\mathfrak{P}|\mathfrak{p})$ acting on $\mathrm{Gal}(M/\mathbf{Q})/H$. Similarly,

$$e(\sigma\mathfrak{p}|p) = \frac{e(\sigma\mathfrak{P}|p)}{e(\sigma\mathfrak{P}|\sigma\mathfrak{p})} = \frac{|I(\mathfrak{P}|\mathfrak{p})|}{|I(\sigma\mathfrak{P}|\sigma\mathfrak{p})|} = \frac{|I(\mathfrak{P}|\mathfrak{p})|}{|I(\sigma\mathfrak{P}|p) \cap H|} = \frac{|I(\mathfrak{P}|\mathfrak{p})|}{|\sigma I(\mathfrak{P}|\mathfrak{p})\sigma^{-1} \cap H|},$$

which is the size of the $I(\mathfrak{P}|\mathfrak{p})$-orbit of σ considered as an element of $\mathrm{Gal}(M/\mathbf{Q})/H$.

Having assembled the double coset machinery of splitting of primes in non-Galois extension contained in Galois extension, we can now analyze further the relationship between splitting of rational primes in K and the Galois group of $M/\mathbf{Q}$.

Since $\mathrm{Gal}(M/\mathbf{Q})$ is a transitive subgroup of S_5, and H is the subgroup of $\mathrm{Gal}(M/\mathbf{Q})$ that fixes $\alpha = \alpha_5$, the set $\mathrm{Gal}(M/\mathbf{Q})/H$ is identified with the set $\{\alpha_1, \ldots, \alpha_5\}$ via the map taking σ to $\sigma\alpha$. Therefore, if p is unramified in M, the set

$$D(\mathfrak{P}|\mathfrak{p})\backslash\mathrm{Gal}(M/\mathbf{Q})/H = \langle \mathrm{Frob}_{\mathfrak{P}} \rangle \backslash \{\alpha_1, \ldots, \alpha_5\}$$

has size equal to the number of cycles in the cycle decomposition of $\mathrm{Frob}_{\mathfrak{P}}$. The size of the orbit of a particular element of $\{\alpha_1, \ldots, \alpha_5\}$ is just the size of the corresponding cycle. Put again: for a prime p that is unramified in M, if p splits into r primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in K with inertia degrees $f_1, \ldots, f_r$, then r is the number of cycles in $\mathrm{Frob}_{\mathfrak{P}}$, and the f_i's are the lengths of those cycles. For example, let $p = 2$. In $\mathbf{F}_2[X]$, we have a factorization into irreducibles

$$f(X) = X^5 - X - 1 = (X^2 + X + 1)(X^3 + X^2 + 1)$$

which implies that 2 is unramified in K (and therefore in M for example by the identity involving inertia above) and that $\mathrm{Frob}_{\mathfrak{P}} \in \mathrm{Gal}(M/\mathbf{Q})$ has cycle decomposition $(\cdot\cdot)(\cdot\cdot\cdot)$ where $\mathfrak{P}$ is any prime of M lying over 2. In particular, $\mathrm{Gal}(M/\mathbf{Q})$ contains a 2-cycle (a third power of any such Frobenius). It also contains a 5-cycle, since it is a transitive subgroup of S_5 (this implies $\mathrm{Gal}(M/\mathbf{Q})$ contains a 5-cycle by Cauchy's theorem), so we conclude that $\mathrm{Gal}(M/\mathbf{Q}) = S_5$ via the permutation action on $\{\alpha_1, \ldots, \alpha_n\}$. This is convenient because it means conjugacy classes in $\mathrm{Gal}(M/\mathbf{Q})$ are quite big, so for the purposes of writing down Artin L-functions we only care about the cycle type decompositions of the Frobenii rather than anything more specific.

By factoring the polynomial $X^5 - X - 1$ modulo p for unramified primes p (that is, $p \neq 19, 151$), we can compute these cycle types (as proved above). For example:

$$\begin{aligned}
p = 2 &\implies \mathrm{Frob}_p = (\cdot\cdot)(\cdot\cdot\cdot) \\
p = 3 &\implies \mathrm{Frob}_p = (\cdot\cdot\cdot\cdot\cdot) \\
p = 5 &\implies \mathrm{Frob}_p = (\cdot\cdot\cdot\cdot\cdot) \\
p = 7 &\implies \mathrm{Frob}_p = (\cdot\cdot)(\cdot\cdot\cdot) \\
p = 11 &\implies \mathrm{Frob}_p = (\cdot\cdot\cdot\cdot\cdot) \\
p = 13 &\implies \mathrm{Frob}_p = (\cdot\cdot\cdot\cdot\cdot) \\
p = 17 &\implies \mathrm{Frob}_p = (\cdot\cdot\cdot)(\cdot)(\cdot).
\end{aligned}$$

(I did this by factoring $X^5 - X - 1$ modulo p for each of these p using PARI/GP). It isn't surprising that we didn't find some cycle types here: for example the conjugacy class $\{1\}$ will only occur for $1/|S_5| < 1\%$ of primes and the conjugacy class $(\cdot\cdot)$ occurs for only $10/|S_5| < 10\%$ of primes. Let us also consider the ramified prime 19, just for fun. The prime factorization of $f(X)$ in $\mathbf{F}_{19}[X]$ is

$$(X+6)^2(X^3 + 14X^2 + 12X + 6),$$

so 19 decomposes in K as $\mathfrak{p}_1^2\mathfrak{p}_2$ where $f(\mathfrak{p}_1|19) = 1$ and $f(\mathfrak{p}_2|19) = 3$. Let $\mathfrak{P}_i$ be a prime of M lying over $\mathfrak{p}_i$. In particular, there are two orbits of $D(\mathfrak{P}_1|19) \subset \mathrm{Gal}(M/\mathbf{Q}) \cong S_5$ acting on $1 \in \mathrm{Gal}(M/\mathbf{Q})/H \cong \{\alpha_1, \ldots, \alpha_5\}$, one of size 2 and one of size 3. By enumerating all the subgroups of S_5, we can check that there are only three conjugacy classes of subgroups with this property:

$$\langle (123), (12), (45)\rangle, \langle (123), (12)(45)\rangle, \langle (123), (45)\rangle.$$

Since p is tamely ramified in K, it is tamely ramified in the Galois closure M (see for example [Con2004]), which implies $I(\mathfrak{P}_1|19)$ is cyclic. The orbits of $I(\mathfrak{P}_1|19)$ inside the orbit $\{1,2,3\}$ of $D(\mathfrak{P}_1|19)$ are all of size 1, and there is just one orbit of $I(\mathfrak{P}_1|19)$ inside the orbit $\{4,5\}$ of $D(\mathfrak{P}_1|19)$. Therefore, $I(\mathfrak{P}_1|19) = \langle (45)\rangle$, which rules out $D(\mathfrak{P}_1|19) = \langle (123), (12)(45)\rangle$. The quotient of $D(\mathfrak{P}_1|19)$ by $I(\mathfrak{P}_1|19)$ must be cyclic, which rules out $D(\mathfrak{P}_1|19) = \langle (123), (12), (45)\rangle$ and leaves only the possibility (up to conjugacy)

$$D(\mathfrak{P}_1|19) = \langle (123), (45)\rangle \supset \langle (45)\rangle = I(\mathfrak{P}_1|19),$$

so we can choose for instance $\mathrm{Frob}_{\mathfrak{P}_1} = (123)$ for the purposes of computation of Euler factors.

Let $\rho : \mathrm{Gal}(M/\mathbf{Q}) \to \mathrm{GL}_4(\mathbf{C}) = \mathrm{GL}(V)$ be the stadard representation of S_5. Either by the explicit definition of the standard representation ρ or by ad hoc determination of eigenvalues, we have the following chart of values relevant to writing down the Euler factors:

σ	$\psi_{\mathrm{std}}(\sigma)$	$\mathrm{charpoly}(\rho(\sigma))$	Euler factor of $L(s,\rho)$ at unramified p when $\mathrm{Frob}_p = \sigma$
1	4	$(X-1)^4$	$(1-p^{-s})^4$
$(\cdot\cdot)$	2	$(X-1)^3(X+1)$	$(1-p^{-s})^3(1+p^{-s})$
$(\cdot\cdot)(\cdot\cdot)$	0	$(X-1)^2(X+1)^2$	$(1-p^{-s})^2(1+p^{-s})^2$
$(\cdot\cdot\cdot)$	1	$(X-1)^2(X^2+X+1)$	$(1-p^{-s})^2(1+p^{-s}+p^{-2s})$
$(\cdot\cdot)(\cdot\cdot\cdot)$	-1	$(X-1)(X+1)(X^2+X+1)$	$(1-p^{-s})(1+p^{-s})(1+p^{-s}+p^{-2s})$
$(\cdot\cdot\cdot\cdot)$	0	$(X-1)(X+1)(X^2+1)$	$(1-p^{-s})(1+p^{-s})(1+p^{-2s})$
$(\cdot\cdot\cdot\cdot\cdot)$	-1	$X^4+X^3+X^2+X+1$	$1+p^{-s}+p^{-2s}+p^{-3s}+p^{-4s}$

(here we have used that the denominator of the Euler factor at p is $p^{-4s}\text{charpoly}(\rho(\text{Frob}_p))(p^s))$. Combining with our previous calculation of Frobenii for $p < 19$, we see that the first few unramified Euler factors are

$$\frac{1}{(1-2^{-s})(1+2^{-s})(1+2^{-s}+2^{-2s})} \cdot \frac{1}{1+3^{-s}+3^{-2s}+3^{-3s}+3^{-4s}} \cdot \frac{1}{1+5^{-s}+5^{-2s}+5^{-3s}+5^{-4s}}$$
$$\cdot \frac{1}{(1-7^{-s})(1+7^{-s})(1+7^{-s}+7^{-2s})} \cdot \frac{11}{1+11^{-s}+11^{-2s}+11^{-3s}+11^{-4s}} \cdot \frac{1}{1+13^{-s}+13^{-2s}+13^{-3s}+13^{-4s}}$$
$$\cdot \frac{1}{(1-17^{-s})^2(1+17^{-s}+17^{-2s})}.$$

To compute the ramified Euler factor at $p = 19$, we need to write down the action of $\text{Frob}_{19} = (123)$ on the (45)-fixed points of ρ_{st}. The definition of ρ as being inside the permutation representation is

$$V = \left\{(x_1, \ldots, x_5) \in \mathbf{C}^5 : \sum_{i=1}^{5} x_i = 0\right\}$$

and thus

$$V^{I(\mathfrak{P}_1|19)} = V^{(45)} = \left\{(x_1, \ldots, x_5) : x_4 = x_5, \sum_{i=1}^{5} x_i = 0\right\}.$$

is 3-dimensional. This has a nice basis $(1, 0, 0, -1/2, -1/2)$, $(0, 1, 0, -1/2, -1/2)$, $(0, 0, 1, -1/2, -1/2)$

The Frobenius we chose was (123), which has matrix

$$M := \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

in this basis. There are 3 eigenvalues which must be 3rd roots of unity and add up to zero. The only choice is for them to be $1, \zeta_3, \zeta_3^2$, and hence the Euler factor at 19 is

$$\frac{1}{(1-19^{-s})(1+19^{-s}+19^{-2s})}.$$

Note that the fact that the ramification gets rid of one dimension corresponds to the fact that the total degree in p^{-s} downstairs is 3 instead of 4 like all the other Euler factors we did so far.

As in Lemma 2.2 in the abelian case, convergence to a holomorphic function on the region $\text{Re}(s) > 1$, and not much else, comes for free from looking at the Euler product:

Lemma 3.2. Let M/K be a finite Galois extension, and let ρ be a finite-dimensional complex representation of $\text{Gal}(M/K)$. The Artin L-series $L^{\text{Artin}}(s, \rho)$ converges to a holomorphic function on the region $\text{Re}(s) > 1$.

Proof. Let $\chi = \text{Tr}\rho$ be the character of ρ. We can safely ignore the finitely many Euler factors for the $\mathfrak{p}$ with $I(\mathfrak{P}|\mathfrak{p}) \neq 1$. As in the proof of Lemma 2.2 (where the taking of logarithms is hidden by citing [SS2003]), we take the logarithm, finding that the log of the Euler factor at $\mathfrak{p}$ is

$$\begin{aligned} -\log\det(1 - \text{N}\mathfrak{p}^{-s}\rho(\text{Frob}_{\mathfrak{P}})) &= \sum_{i=1}^{\dim\rho} -\log\left(1 - \text{N}\mathfrak{p}^{-s}a_{\mathfrak{p},\rho}^{(i)}\right) \\ &= \sum_{i=1}^{\dim\rho}\sum_{m=1}^{\infty} \frac{1}{m}\text{N}\mathfrak{p}^{-ms}(a_{\mathfrak{p},\rho}^{(i)})^m \\ &= \sum_{m=1}^{\infty} \frac{1}{m}\text{N}\mathfrak{p}^{-ms}\sum_{i=1}^{\dim\rho}(a_{\mathfrak{p},\rho}^{(i)})^m \end{aligned}$$

$$= \sum_{m\geq 1}^{\infty} \frac{\chi(\mathrm{Frob}_{\mathfrak{P}}^m)}{m\mathrm{N}\mathfrak{p}^{ms}},$$

where the $a_{\mathfrak{p},\rho}^{(i)}$, $1 \leq i \leq \dim\rho$, denote the eigenvalues[3] of $\rho(\mathrm{Frob}_{\mathfrak{P}})$ (the choice of $\mathfrak{P}$ doesn't matter since conjugating a matrix doesn't affect its eigenvalues).

All that remains to do is to prove the absolute convergence of the sum

$$\sum_{\mathfrak{p}\nmid\Delta_{M/K}} \sum_{m=1}^{\infty} \frac{\chi(\mathrm{Frob}_{\mathfrak{P}}^m)}{m\mathrm{N}\mathfrak{p}^{ms}}.$$

The quantities $\chi(\mathrm{Frob}_{\mathfrak{P}}^m)$, being the sum of $\dim\rho$ roots of unity (the eigenvalues $a_{\mathfrak{p},\rho}^{(i)}$ are roots of unity because $\mathrm{Gal}(M/K)$ is finite), are of absolute value at most $\dim\rho$, so the result follows from the convergence of

$$\sum_{m=1}^{\infty}\sum_{\mathfrak{p}} \frac{1}{\mathrm{N}\mathfrak{p}^{ms}}.$$

when $\mathrm{Re}(s) > 1$. This convergence is for example a consequence of the fact (from the proof of Theorem 2.9) that the $m \geq 2$ terms converge and that the $m = 1$ term is clearly bounded in absolute value by $\zeta_K(\mathrm{Re}(s))$. □

This still leaves open the question of whether $L^{\mathrm{Artin}}(s,\rho)$ can be extended meromorphically to the left of $\Re(s) = 1$ with functional equation and expected poles (this is called Artin's conjecture). In the previous section we used the fact that ρ was 1-dimensional in order to apply class field theory and connect the Artin L-function to a Hecke L-function, which itself has these desirable properties.

The strategy for the rest of this section will be to try and attack Artin's conjecture by thinking about all the different ways that we can build higher-dimensional representations from lower-dimensional representations, showing that this process is compatible in some way with taking Artin L-functions, and then deducing some information about meromorphic continuation from the 1-dimensional case.

For example, for the process of taking direct sums of representations, we have

Proposition 3.3. Given two finite-dimensional complex representations $\rho_i : \mathrm{Gal}(M/K) \to \mathrm{GL}(V_i)$, $i = 1, 2$, we have

$$L^{\mathrm{Artin}}(s, \rho_1 \oplus \rho_2) = L^{\mathrm{Artin}}(s,\rho_1)L^{\mathrm{Artin}}(s,\rho_2).$$

Proof. This can be read off of the Euler products for both sides, since (by definition of the direct sum)

$$\begin{aligned}\big(1-\mathrm{N}\mathfrak{p}^{-s}(\rho_1\oplus\rho_2)(\mathrm{Frob}_{\mathfrak{P}})\big)\,|_{(V_1\oplus V_2)^{I(\mathfrak{P}|\mathfrak{p})}} &= \big((1-\mathrm{N}\mathfrak{p}^{-s}\rho_1(\mathrm{Frob}_{\mathfrak{P}}))\oplus(1-\mathrm{N}\mathfrak{p}^{-s}\rho_1(\mathrm{Frob}_{\mathfrak{P}}))\big)\,|_{V_1^{I(\mathfrak{P}|\mathfrak{p})}\oplus V_2^{I(\mathfrak{P}|\mathfrak{p})}}\\ &= (1-\mathrm{N}\mathfrak{p}^{-s}\rho_1(\mathrm{Frob}_{\mathfrak{P}}))|_{V_1^{I(\mathfrak{P}|\mathfrak{p})}}\oplus(1-\mathrm{N}\mathfrak{p}^{-s}\rho_1(\mathrm{Frob}_{\mathfrak{P}}))|_{V_2^{I(\mathfrak{P}|\mathfrak{p})}}\end{aligned}$$

and determinants turn direct sums of operators into products. □

We also remark that technically we have not given the full detail of why we have taken care of the 1-dimensional case. *A priori*, there might be a difference between the Artin L-function of a character of an abelian group (which is what we used class field theory to prove Artin's conjecture for in the previous section) and the Artin L-function of a (1-dimensional) character of a non-abelian group (which only *factors through* an abelian quotient). In fact, there is no difference between these two L-functions, as shown by the following lemma in the special case where ρ is 1-dimensional and $\mathrm{Gal}(M'/K)$ is abelian.

Lemma 3.4. Let $K \subset M' \subset M$ be finite extensions with the property that M'/K and M/K are Galois. Let $\pi : \mathrm{Gal}(M/K) \to \mathrm{Gal}(M'/K)$ be the natural projection, and let $\rho : \mathrm{Gal}(M'/K) \to \mathrm{GL}_d(\mathbf{C})$ be a complex representation of $\mathrm{Gal}(M'/K)$. Then we have an equality of L-functions for $\mathrm{Gal}(M'/K)$ and $\mathrm{Gal}(M/K)$

$$L^{\mathrm{Artin}}(s,\rho) = L^{\mathrm{Artin}}(s,\rho\circ\pi)$$

[3] Take Jordan normal form.

Proof. It suffices to prove equality of Euler factors at each $\mathfrak{p}$. For each $\mathfrak{p}$, choose a prime $\mathfrak{P}'$ of M' lying over $\mathfrak{p}$, and a prime $\mathfrak{P}$ of M lying over $\mathfrak{P}'$. The Euler factor at $\mathfrak{p}$ of the right hand side is

$$\frac{1}{\det((1-\mathrm{N}\mathfrak{p}^{-s}\rho(\mathrm{Frob}_{\mathfrak{P}}|_{\mathrm{Gal}(M'/K)}))|_{V^{I(\mathfrak{P}|\mathfrak{p})}}}=\frac{1}{\det((1-\mathrm{N}\mathfrak{p}^{-s}\rho(\mathrm{Frob}_{\mathfrak{P}'}))|_{V^{I(\mathfrak{P}'|\mathfrak{p})}}},$$

since $\mathrm{Frob}_{\mathfrak{P}}|_{\mathrm{Gal}(M'/K)}=\mathrm{Frob}_{\mathfrak{P}'}$ (general fact about Frobenii) and $I(\mathfrak{P}|\mathfrak{p})$ surjects onto $I(\mathfrak{P}'|\mathfrak{p})$ via π (general fact from algebraic number theory, see e.g. [Gun2019]). This is exactly the Euler factor at $\mathfrak{p}$ of the left hand side. □

Finally, we prove the same property is true for induction of characters. This proof is taken straight from [Lan1994, XII, §3].

Proposition 3.5. Let $K\subset M'\subset M$ be finite extensions with the property that M/K is Galois. Let $\rho:\mathrm{Gal}(M/M')\to\mathrm{GL}(V)$ be a finite-dimensional complex representation of $\mathrm{Gal}(M/M')$. Then there is an equality of L-functions for $\mathrm{Gal}(M/M')$ and $\mathrm{Gal}(M/K)$

$$L^{\mathrm{Artin}}(s,\rho)=L^{\mathrm{Artin}}(s,\mathrm{Ind}_{\mathrm{Gal}(M/M')}^{\mathrm{Gal}(M/K)}\rho).$$

Proof. Let χ be the character of ρ, and $\mathrm{Ind}\chi:=\mathrm{Tr}\mathrm{Ind}_{\mathrm{Gal}(M/M')}^{\mathrm{Gal}(M/K)}\rho$ the character of the induction of ρ. Recall that (e.g. as a consequence of Frobenius reciprocity), if $\{c\}$ is any system of representatives for $\mathrm{Gal}(M/M')\backslash\mathrm{Gal}(M/K)$, then for all $g\in\mathrm{Gal}(M/K)$,

$$(\mathrm{Ind}\chi)(g)=\sum_c\chi(cgc^{-1}),$$

where $\chi(cgc^{-1})$ is understood to be zero when cgc^{-1} is not in $\mathrm{Gal}(M/M')$.

We first construct a convenient choice of $\{c\}$. Fix a prime $\mathfrak{p}$ of K, let $\mathfrak{P}'_1,\dots,\mathfrak{P}'_{r_\mathfrak{p}}$ be the primes of M' lying over $\mathfrak{p}$, and for each $i=1,\dots,r_\mathfrak{p}$, let $\mathfrak{P}_i$ be an arbitrary choice of prime of M lying over $\mathfrak{P}'_i$. Choose elements $\sigma_1,\dots,\sigma_{r_\mathfrak{p}}\in\mathrm{Gal}(M/K)$ such that $\sigma_i\mathfrak{P}_1=\mathfrak{P}_i$, and hence $D(\mathfrak{P}_i|\mathfrak{p})=\sigma_iD(\mathfrak{P}_i|\mathfrak{p})\sigma_i^{-1}$, $I(\mathfrak{P}_i|\mathfrak{p})=\sigma_iI(\mathfrak{P}_i|\mathfrak{p})\sigma_i^{-1}$, and $\sigma_i\mathrm{Frob}_{\mathfrak{P}_1}\sigma_i^{-1}=\mathrm{Frob}_{\mathfrak{P}_i}$. For each $i=1,\dots,r_\mathfrak{p}$, we have

$$[D(\mathfrak{P}_i|\mathfrak{p}):D(\mathfrak{P}_i|\mathfrak{P}'_i)]=e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p}),$$

so we can choose representatives $\{\gamma_{i,j}\}_{1\le j\le e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p})}$ for $D(\mathfrak{P}_i|\mathfrak{P}'_i)\backslash D(\mathfrak{P}_i|\mathfrak{p})$. We claim that the elements

$$\{\gamma_{i,j}\sigma_i\}_{1\le i\le r_\mathfrak{p},1\le j\le e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p})}$$

are a system of representatives for $\mathrm{Gal}(M/M')\backslash\mathrm{Gal}(M/K)$. Indeed, there are $r_\mathfrak{p}e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p})=[M':K]$ of them, so it suffices to show that the cosets $\mathrm{Gal}(M/M')\gamma_{i,j}\sigma_i$ are different for different pairs (i,j). If there were pairs (i_1,j_1) and (i_2,j_2) such that

$$\mathrm{Gal}(M/M')\gamma_{i_1,j_1}\sigma_{i_1}=\mathrm{Gal}(M/M')\gamma_{i_2,j_2}\sigma_{i_2},$$

then we would have

$$\gamma_{i_1,j_1}\sigma_{i_1}\sigma_{i_1}^{-1}\gamma_{i_2,j_2}^{-1}\in\mathrm{Gal}(M/M').$$

Since $\gamma_{i,j}$ is always in $D(\mathfrak{P}_i|\mathfrak{p})$, we have $\gamma_{i,j}\mathfrak{P}_i=\mathfrak{P}_i$, and we have $\sigma_i^{-1}\mathfrak{P}_i=\mathfrak{P}_1$, so

$$\gamma_{i_1,j_1}\sigma_{i_1}\sigma_{i_1}^{-1}\gamma_{i_2,j_2}^{-1}\mathfrak{P}_{i_2}=\mathfrak{P}_{i_1}.$$

But $\mathfrak{P}_{i_1}$ and $\mathfrak{P}_{i_2}$ live over different primes of M', which $\mathrm{Gal}(M/M')$ acts trivially on, so we already have $i_1=i_2=:i$. Thus $\sigma_{i_1}=\sigma_{i_2}$, and we are left with $\gamma_{i,j_1}\gamma_{i,j_2}^{-1}\in\mathrm{Gal}(M/M')$. But the $\{\gamma_{i,j}\}_{1\le j\le e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p})}$ were chosen to be a system of representatives for $D(\mathfrak{P}_i|\mathfrak{P}'_i)\backslash D(\mathfrak{P}_i|\mathfrak{p})$, and $D(\mathfrak{P}_i|\mathfrak{P}'_i)=D(\mathfrak{P}_i|\mathfrak{p})\cap\mathrm{Gal}(M/M')$, so we conclude that $j_1=j_2$, finally proving that the $\gamma_{i,j}\sigma_i$ are a full system of representatives for $\mathrm{Gal}(M/M')\backslash\mathrm{Gal}(M/K)$.

We remark, too, that for a prime $\mathfrak{P}_i|\mathfrak{p}$, determinant of the operator

$$\left(1 - \mathrm{N}\mathfrak{p}^{-s}(\mathrm{Ind}\rho)(\mathrm{Frob}_{\mathfrak{P}_i})\right)|_{(\mathrm{Ind}V)^{I(\mathfrak{P}_i|\mathfrak{p})}}$$

is the same as the determinant of the operator on $\mathrm{Ind}V$ given by

$$\left(1 - \mathrm{N}\mathfrak{p}^{-s}(\mathrm{Ind}\rho)(\mathrm{Frob}_{\mathfrak{P}_i})\right) \circ \pi_{I(\mathfrak{P}_i|\mathfrak{p})},$$

where $\pi_{I(\mathfrak{P}_i|\mathfrak{p})}$ is the natural projection $V \to V^{I(\mathfrak{P}_i|\mathfrak{p})}$ given by

$$v \mapsto \frac{1}{e(\mathfrak{P}_i|\mathfrak{p})} \sum_{g \in I(\mathfrak{P}_i|\mathfrak{p})} \mathrm{Ind}(\rho)(g)v.$$

Therefore, the log of the Euler factor at $\mathfrak{p}$ of Artin L-function for $\mathrm{Ind}\rho$ is

$$\begin{aligned}
\log \frac{1}{\det\left(1 - \mathrm{N}\mathfrak{p}^{-s}(\mathrm{Ind}\rho)(\mathrm{Frob}_{\mathfrak{P}_1})\right) \circ \pi_{I(\mathfrak{P}_1|\mathfrak{p})}} &= \sum_{m=1}^{\infty} \frac{\mathrm{Tr}((\mathrm{Ind}\rho)(\mathrm{Frob}_{\mathfrak{P}_1}^m) \circ \pi_{I(\mathfrak{P}_1|\mathfrak{p})})}{m\mathrm{N}\mathfrak{p}^{ms}} \\
&= \sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{p})} \sum_{g \in I(\mathfrak{P}_1|\mathfrak{p})} \frac{(\mathrm{Ind}\chi)(\mathrm{Frob}_{\mathfrak{P}_1}^m g)}{m\mathrm{N}\mathfrak{p}^{ms}} \\
&= \sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{p})} \sum_{g \in I(\mathfrak{P}_1|\mathfrak{p})} \sum_{i=1}^{r_{\mathfrak{p}}} \sum_{j=1}^{e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p})} \frac{\chi(\gamma_{i,j}\sigma_i \mathrm{Frob}_{\mathfrak{P}_1}^m g \sigma_i^{-1}\gamma_{i,j}^{-1})}{m\mathrm{N}\mathfrak{p}^{ms}},
\end{aligned}$$

where we recall that the convention is that $\chi(\sigma) = 0$ when $\sigma \notin \mathrm{Gal}(M/M')$. Since $\sigma_i I(\mathfrak{P}_1|\mathfrak{p})\sigma_i^{-1} = I(\mathfrak{P}_i|\mathfrak{p})$ and $\sigma_i \mathrm{Frob}_{\mathfrak{P}_1}\sigma_i^{-1} = \mathrm{Frob}_{\mathfrak{P}_i}$, this is the same as

$$\sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{p})} \sum_{i=1}^{r_{\mathfrak{p}}} \sum_{j=1}^{e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p})} \sum_{g \in I(\mathfrak{P}_i|\mathfrak{p})} \frac{\chi(\gamma_{i,j}\mathrm{Frob}_{\mathfrak{P}_i}^m g \gamma_{i,j}^{-1})}{m\mathrm{N}\mathfrak{p}^{ms}}.$$

Since $\gamma_{i,j} \in D(\mathfrak{P}_i|\mathfrak{p})$, where $I(\mathfrak{P}_i|\mathfrak{p})$ is a normal subgroup, we can further simplify to

$$\sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{p})} \sum_{i=1}^{r_{\mathfrak{p}}} \sum_{j=1}^{e(\mathfrak{P}'_i|\mathfrak{p})f(\mathfrak{P}'_i|\mathfrak{p})} \sum_{g \in I(\mathfrak{P}_i|\mathfrak{p})} \frac{\chi(\mathrm{Frob}_{\mathfrak{P}_i}^m g)}{m\mathrm{N}\mathfrak{p}^{ms}}$$

which is just

$$\sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{P}'_1)} \sum_{i=1}^{r_{\mathfrak{p}}} f(\mathfrak{P}'_i|\mathfrak{p}) \sum_{g \in I(\mathfrak{P}_i|\mathfrak{p})} \frac{\chi(\mathrm{Frob}_{\mathfrak{P}_i}^m g)}{m\mathrm{N}\mathfrak{p}^{ms}}.$$

The problem now is to understand when the input to χ is in $\mathrm{Gal}(M/M')$. Let $1 \leq i \leq r_{\mathfrak{p}}$. If there exists $g \in I(\mathfrak{P}_i|\mathfrak{p})$ such that $\mathrm{Frob}_{\mathfrak{P}_i}^m g \in \mathrm{Gal}(M/M')$, then we would have $\mathrm{Frob}_{\mathfrak{P}_i}^m g \in D(\mathfrak{P}_i|\mathfrak{p}) \cap \mathrm{Gal}(M/M') = D(\mathfrak{P}_i|\mathfrak{P}'_i)$ and therefore the reduction modulo $\mathfrak{P}_i$ of $\mathrm{Frob}_{\mathfrak{P}_i}^m g$, which *a priori* lived in $\mathrm{Gal}(\kappa(\mathfrak{P}_i)|\kappa(\mathfrak{p}))$, actually fixes $\kappa(\mathfrak{P}'_i)$. Since $g \in I(\mathfrak{P}_i|\mathfrak{p})$, its mod-$\mathfrak{P}_i$ reduction is trivial, so we have really concluded that $\mathrm{Frob}_{\mathfrak{P}_i}^m$ mod $\mathfrak{P}_i$ fixes $\kappa(\mathfrak{P}'_i)$. Since

$$\mathrm{Gal}(\kappa(\mathfrak{P}_i)/\kappa(\mathfrak{P}'_i)) = \left\langle \mathrm{Frob}_{\mathfrak{P}_i}^{f(\mathfrak{P}'_i|\mathfrak{p})} \right\rangle \subset \mathrm{Gal}(\kappa(\mathfrak{P}_i)/\kappa(\mathfrak{p})) = \left\langle \mathrm{Frob}_{\mathfrak{P}_i} \right\rangle,$$

it follows that $f(\mathfrak{P}'_i|\mathfrak{p})$ divides m. In other words, the terms where $f(\mathfrak{P}'_i|\mathfrak{p}) \nmid m$ all vanish.

When $f(\mathfrak{P}'_i|\mathfrak{p})|m$, we have

$$\mathrm{Frob}_{\mathfrak{P}_i}^m I(\mathfrak{P}_i|\mathfrak{p}) \cap \mathrm{Gal}(M/M') = \mathrm{Frob}_{\mathfrak{P}_i}^m I(\mathfrak{P}_i|\mathfrak{p}) \cap D(\mathfrak{P}_i|\mathfrak{P}'_i)$$

$$= (\mathrm{Frob}_{\mathfrak{P}_i}(M/M'))^{\frac{m}{f(\mathfrak{P}_i'|\mathfrak{p})}} I(\mathfrak{P}_i|\mathfrak{P}_i'),$$

where the first equality is because $D(\mathfrak{P}_i|\mathfrak{P}_i') = D(\mathfrak{P}_i|\mathfrak{p}) \cap \mathrm{Gal}(M/M')$ and the second one is because a subset of $D(\mathfrak{P}_i|\mathfrak{P}_i')$ that is invariant under right-multiplication by $I(\mathfrak{P}_i|\mathfrak{P}_i')$ is determined by what subset of $\mathrm{Gal}(\kappa(\mathfrak{P}_i)/\kappa(\mathfrak{P}_i'))$ it projects down to modulo $\mathfrak{P}_i$. Our Euler factor at $\mathfrak{p}$ is therefore

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{P}_1')} \sum_{i=1}^{r_\mathfrak{p}} f(\mathfrak{P}_i'|\mathfrak{p}) \sum_{g \in I(\mathfrak{P}_i|\mathfrak{p})} \frac{\chi(\mathrm{Frob}_{\mathfrak{P}_i}^m g)}{m \mathrm{N}\mathfrak{p}^{ms}}. &= \sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{P}_1')} \sum_{i=1}^{r_\mathfrak{p}} f(\mathfrak{P}_i'|\mathfrak{p}) \sum_{g \in I(\mathfrak{P}_i|\mathfrak{p})} \frac{\chi(\mathrm{Frob}_{\mathfrak{P}_i}^{mf(\mathfrak{P}_i'|\mathfrak{p})} g)}{mf(\mathfrak{P}_i'|\mathfrak{p}) \mathrm{N}\mathfrak{p}^{mf(\mathfrak{P}_i'|\mathfrak{p})s}}. \\
&= \sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{P}_1')} \sum_{i=1}^{r_\mathfrak{p}} \sum_{g \in I(\mathfrak{P}_i|\mathfrak{p})} \frac{\chi(\mathrm{Frob}_{\mathfrak{P}_i}^{mf(\mathfrak{P}_i'|\mathfrak{p})} g)}{m(\mathrm{N}\mathfrak{P}_i')^{ms}}. \\
&= \sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{P}_1')} \sum_{i=1}^{r_\mathfrak{p}} \sum_{g \in I(\mathfrak{P}_i|\mathfrak{P}_i')} \frac{\chi((\mathrm{Frob}_{\mathfrak{P}_i}(M/M'))^m g)}{m(\mathrm{N}\mathfrak{P}_i')^{ms}}. \\
&= \sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{P}_1')} \sum_{i=1}^{r_\mathfrak{p}} \sum_{g \in I(\mathfrak{P}_i|\mathfrak{P}_i')} \frac{\chi((\mathrm{Frob}_{\mathfrak{P}_i}(M/M'))^m g)}{m(\mathrm{N}\mathfrak{P}_i')^{ms}}. \\
&= \sum_{i=1}^{r_\mathfrak{p}} \sum_{m=1}^{\infty} \frac{1}{e(\mathfrak{P}_1|\mathfrak{P}_1')} \sum_{g \in I(\mathfrak{P}_i|\mathfrak{P}_i')} \frac{\chi((\mathrm{Frob}_{\mathfrak{P}_i}(M/M'))^m g)}{m(\mathrm{N}\mathfrak{P}_i')^{ms}}. \\
&= \sum_{i=1}^{r_\mathfrak{p}} \log \frac{1}{\det((1 - \mathrm{N}\mathfrak{P}_i' \rho(\mathrm{Frob}_{\mathfrak{P}_i}(M/M')))|_{V^{I(\mathfrak{P}_i|\mathfrak{P}_i')}})} \\
&= \log \prod_{i=1}^{r_\mathfrak{p} f} \frac{1}{\det((1 - \mathrm{N}\mathfrak{P}_i' \rho(\mathrm{Frob}_{\mathfrak{P}_i}(M/M')))|_{V^{I(\mathfrak{P}_i|\mathfrak{P}_i')}})}.
\end{aligned}$$

This is exactly the product of all the Euler factors of $L^{\mathrm{Artin}}(s, \rho)$ at the primes of M' over $\mathfrak{p}$, so we have shown that

$$L^{\mathrm{Artin}}(s, \rho) = L^{\mathrm{Artin}}(s, \mathrm{Ind}_{\mathrm{Gal}(M/M')}^{\mathrm{Gal}(M/K)} \rho)$$

□

It turns out that all representations of possibly non-abelian groups can be reached essentially by combining Proposition 3.3, Lemma 3.4, and Proposition 3.5, with the disclaimer that sometimes we may need to divide instead of multiplying when we apply Proposition 3.3. The reason for this is the following celebrated result of Brauer.

Theorem 3.6 (Brauer's theorem)**.** Let G be a finite group, and suppose that χ is the character of a representation ρ of G. Then

$$\chi \in \mathbf{Z}\left[\left\{\mathrm{Ind}_H^G \psi\right\}_{H,\psi}\right],$$

where H runs over all subgroups of G and ψ runs over all 1-dimensional characters of H.

Proof. See [Ser1978]. □

Example 5. Let $G = S_3$. The character table of G is

Conjugacy class	Size of conjugacy class	ψ_{triv}	ψ_{alt}	ψ_{std}
$(\cdot)(\cdot)(\cdot)$	1	1	1	2
$(\cdot\cdot)(\cdot)$	3	1	-1	0
$(\cdot\cdot\cdot)$	2	1	1	-1

The characters ψ_{triv} and ψ_{alt} are already 1-dimensional, so the only thing to do is to write ψ_{std} as a $\mathbf{Z}$-linear combination of inductions. In fact, it turns out that it is already induced, as we could have known already using the fact that S_3 is nilpotent. Since ψ_{std} is of dimension 2, the place to go looking for characters it might be induced from is the subgroup $A_3 \subset S_3$, which is cyclic of order 3. Let $\sigma = (123)$ be a generator of A_3, and for $i = 0, 1, 2$ let χ_i be the character of A_3 taking $\sigma \mapsto \zeta_3^i$. Then by Frobenius reciprocity we have

$$\begin{aligned}
\dim_{\mathbf{C}} \operatorname{Hom}_{S_3}(\psi_{\text{std}}, \operatorname{Ind}_{A_3}^{S_3} \chi_i) &= \dim_{\mathbf{C}} \operatorname{Hom}_{A_3}(\psi_{\text{std}}|_{A_3}, \chi_i) \\
&= \langle \psi_{\text{std}}|_{A_3}, \chi_i \rangle_{A_3} \\
&= \frac{1}{3} \sum_{j=0}^{2} \psi_{\text{std}}(\sigma^j) \chi_i(\sigma^j) \\
&= \frac{1}{3} \sum_{j=0}^{2} \psi_{\text{std}}(\sigma^j) \zeta_3^{ij} \\
&= \frac{1}{3} \left(2 - \zeta_3^i - \zeta_3^{2i}\right) \\
&= 1
\end{aligned}$$

when $i = 1, 2$. By Maschke's theorem, Schur's lemma, and the fact that $\dim \psi_{\text{std}} = \dim \operatorname{Ind}_{A_3}^{S_3}(\text{anything})$, we conclude that ψ_{std} is precisely the induction from A_3 to S_3 of either one of the nontrivial characters of A_3.

Example 6. Let $G = S_4$. In this case G is no longer nilpotent, but we might as well hope that every character will be induced. In fact, we know this *a priori*, because S_4 is what Serre calls "hypersolvable" (at least I think this is a reasonable translation): $A_4 \subset S_4$ is normal, and it contains inside it a copy of the Klein 4 group which is also normal in S_4. The character table of G is

Conjugacy class	Size of conjugacy class	ψ_{triv}	ψ_{alt}	ψ_{std}	$\psi_{\text{std}\otimes\text{alt}}$	ψ
$(\cdot)(\cdot)(\cdot)(\cdot)$	1	1	1	3	3	2
$(\cdot\cdot)(\cdot)(\cdot)$	6	1	-1	1	-1	0
$(\cdot\cdot)(\cdot\cdot)$	3	1	1	-1	-1	2
$(\cdot\cdot\cdot)(\cdot)$	8	1	1	0	0	-1
$(\cdot\cdot\cdot\cdot)$	6	1	-1	-1	1	0

where the last column is deduced for instance from the decomposition of the regular representation. If a character of dimension 3 is induced from a 1-dimensional subgroup, that subgroup has index 3. For example by Sylow theorem, the only index-3 subgroup of S_4 up to conjugacy is D_8 (choose the ordering $1, 2, 3, 4$ of the set S_4 acts on in order to fix one particular copy of D_8). The nontrivial dimension-1 characters of D_8 (the only relevant ones for us) are the following:

Conjugacy class	Size of conjugacy class	χ_1	χ_2	χ_3
$\{1\}$	1	1	1	1
$\{r, r^3\} = \{(1234), (2143)\}$	2	1	-1	-1
$\{rs, r^3s\} = \{(12)(34), (23)(41)\}$	2	-1	1	-1
$\{r^2\} = \{(13)(24)\}$	1	1	1	1
$\{s, r^2s\} = \{(13)(2)(4), (24)(1)(3)\}$	2	-1	-1	1

(it just depends on the choice of order-4 subgroup of D_8 to choose as the kernel). By Frobenius reciprocity, we compute (leaving out the ones that are zero to save space)

$$\begin{aligned}
\left\langle \psi_{\text{std}}, \operatorname{Ind}_{D_8}^{S_4} \chi_3 \right\rangle &= \langle \psi_{\text{std}}|_{D_8}, \chi_3 \rangle_{D_8} = 1 \\
\left\langle \psi_{\text{std}\otimes\text{alt}}, \operatorname{Ind}_{D_8}^{S_4} \chi_1 \right\rangle &= \langle \psi_{\text{std}}|_{D_8}, \chi_1 \rangle_{D_8} = 1
\end{aligned}$$

and so $\psi_{\mathrm{std}} = \mathrm{Ind}_{D_8}^{S_4}\chi_3$ and $\psi_{\mathrm{std}\otimes\mathrm{alt}} = \mathrm{Ind}_{D_8}^{S_4}\chi_1$. For the 2-dimensional character ψ, we know it will be induced from the index-2 subgroup $A_4 \subset S_4$. The dimension-1 characters of A_4 must factor through $A_4 \to A_4^{\mathrm{ab}} = A_4/[A_4, A_4]$. Since A_4 is not abelian, $[A_4, A_4]$ is nontrivial. Also, A_4 contains a normal subgroup H of order 4, namely the copy of Klein 4 generated by (12)(34) and (13)(24). Since the quotient by this subgroup has order 3 and is therefore abelian, H contains $[A_4, A_4]$. But all the nontrivial subgroups of the Klein 4 group are either of order 2 or the whole thing, and A_4 has no normal subgroups of order 2, so we deduce that $[A_4, A_4] = H$, and therefore that the nontrivial 1-dimensional characters of A_4 come directly from those of the quotient $A_4/H \cong \mathbf{Z}/3\mathbf{Z}$, i.e.

Conjugacy class	Size of conjugacy class	φ_1	φ_2
$\{1\}$	1	1	1
$\{(12)(34), (13)(24), (14)(23)\}$	3	1	1
$\{(123)(4), (243)(1), (134)(2), (142)(3)\}$	4	ζ_3	ζ_3^2
$\{(132)(4), (234)(1), (143)(2), (124)(3)\}$	4	ζ_3^2	ζ_3

and we compute again by Frobenius reciprocity that $\langle \psi, \mathrm{Ind}_{A_4}^{S_4}\varphi_1\rangle = \langle \psi, \mathrm{Ind}_{A_4}^{S_4}\varphi_2\rangle = 1$ and therefore that the 2-dimensional character ψ of S_4 can be induced from either one of the 1-dimensional characters of any $D_8 \subset S_4$. This proves Artin's conjecture for S_4 extensions.

Example 7. Let $G = S_5$, and let ψ_{std} be the character of the standard representation. It is 4-dimensional and irreducible by the general theory. If ψ_{std} was a $\mathbf{Z}_{\geq 0}$-linear combination of induced 1-dimensional characters, then it would decompose as a direct sum of inductions of 1-dimensional representations of subgroups of G. Since ψ_{std} is irreducible of dimension 4, that would imply it is induced from a 1-dimensional representation of an index-4 subgroup of S_5. But S_5 has no index-4 subgroups: indeed, S_5 has no subgroup of order 15 (such a group would have to be cyclic by Sylow theorems, and S_5 contains no element of order 15), and any group of order 30 contains a subgroup of order 15 (such a group H has an element of order 2 by Cauchy's theorem, which becomes an odd permutation when viewed as an element of $S_{|H|}$, so $H \cap A_{|H|} = \ker \mathrm{sgn}|_H$ would do the trick). Therefore, we have found our first example of a situation where negative coefficients are required in Brauer's theorem, and therefore where Artin's conjecture does not follow simply from a certain instance of Brauer's theorem.

Example 8. This is a continuation of the previous example, where we now try to explictly write ψ_{std} as a $\mathbf{Z}$-linear combination of inductions of 1-dimensional characters of subgroups of $G = S_5$. I don't know how to do this by hand without enumerating all the subgroups of S_5 and then using Frobenius reciprocity to compute all the inductions to S_5 of all the 1-dimensional characters of those subgroups. Since I am not very enthusiastic about doing this by hand, I wrote a SAGE script to do it for me. This involved enumerating all conjugacy classes of subgroups of S_5, inducing all the 1-dimensional characters of those subgroups to S_5, and then computing the Smith normal form of a 7×57 integer matrix. Ultimately, the answer is actually simple:

$$\psi_{\mathrm{std}} = -\psi_{\mathrm{triv}} + \mathrm{Ind}_{S_4}^{S_5}(1_{S_4}),$$

where $S_4 \subset S_5$ can be chosen to be any of the 5 obvious choices. Fan Zhou has explained to me that this is a special case of the Pieri rule. The other case of the Pieri rule is

$$\psi_{\mathrm{alt}\otimes\mathrm{std}} = -\psi_{\mathrm{alt}} + \mathrm{Ind}_{S_4}^{S_5}(\mathrm{alt}_{S_4}).$$

In fact, now that my SAGE script is up and running, we might as well finish off the remaining three characters of S_5. The full character table is

Conjugacy class	Size of conjugacy class	ψ_{triv}	ψ_{alt}	ψ_{std}	$\psi_{\mathrm{std}\otimes\mathrm{alt}}$	$\psi_{5,1}$	$\psi_{5,2}$	ψ_6
$\{1\}$	1	1	1	4	4	5	5	6
$(\cdot\cdot)$	10	1	-1	2	-2	1	-1	0
$(\cdot\cdot)(\cdot\cdot)$	15	1	1	0	0	1	1	-2
$(\cdot\cdot\cdot)$	20	1	1	1	1	-1	-1	0
$(\cdot\cdot)(\cdot\cdot\cdot)$	20	1	-1	-1	1	1	-1	0
$(\cdot\cdot\cdot\cdot)$	30	1	-1	0	0	-1	1	0
$(\cdot\cdot\cdot\cdot\cdot)$	24	1	1	-1	-1	0	0	1

where I got the last three columns off of SAGE. Doing the Smith normal form computation in SAGE as before, we get

$$\begin{aligned}\psi_{5,1} &= \mathrm{Ind}_{AGL_1(\mathbf{F}_5)}^{S_5}\chi_1 - \psi_{\mathrm{alt}}\\ \psi_{5,2} &= \mathrm{Ind}_{AGL_1(\mathbf{F}_5)}^{S_5}1 - \psi_{\mathrm{triv}}\\ \psi_6 &= \mathrm{Ind}_{AGL_1(\mathbf{F}_5)}^{S_5}\chi_2\end{aligned}$$

where the character $\chi_1 : AGL_1(\mathbf{F}_5) \to \mathbf{C}^\times$ is given by composing the natural projection $AGL_1(\mathbf{F}_5) \to \mathbf{F}_5^\times$ with the unique nontrivial real-valued 1-dimensional character of $\mathbf{F}_5^\times$ (the one taking a primitive root to -1), and the character $\chi_2 : AGL_1(\mathbf{F}_5) \to \mathbf{C}^\times$ is given by composing that same projection with either one of the complex-values characters of $\mathbf{F}_5^\times$ (the ones taking a primitive root to $\pm i$).

To be precise, the fact that the coefficients could be negative means that we do not quite prove Artin's conjecture in this way, instead proving the statement with "holomorphic" replaced with "meromorphic":

Corollary 3.7. Let M/K be a finite Galois extension, and $\rho : \mathrm{Gal}(M/K) \to \mathrm{GL}(V)$ a finite-dimensional complex representation. Then $L^{\mathrm{Artin}}(s,\rho)$ has meromorphic continuation to the entire complex plane.

Proof. Let χ be the character of ρ. By Theorem 3.6, there are subgroups $\mathrm{Gal}(M/M_i) = H_i \subset \mathrm{Gal}(M/K)$ equipped with 1-dimensional characters $\psi_i : H_i \to \mathbf{C}^\times$, and integers $a_i \in \mathbf{Z}$, $i = 1,\dots,r$, such that

$$\chi = \sum_{i=1}^r a_i \mathrm{Ind}_{\mathrm{Gal}(M/M_i)}^{\mathrm{Gal}(M/K)}\psi_i.$$

Noticing that the definition of the Artin L-series only depends on the character of the representation ρ and extending the definition of $L^{\mathrm{Artin}}(s,\psi)$ to arbitrary class functions $f : \mathrm{Gal}(M/K) \to \mathbf{C}$ as

$$L^{\mathrm{Artin}}(s,f) := \prod_{\mathfrak{p}} \exp\left(\sum_{m=1}^\infty \frac{1}{e(\mathfrak{P}|\mathfrak{p})}\sum_{g\in I(\mathfrak{P}|\mathfrak{p})}\frac{f(\mathrm{Frob}_{\mathfrak{p}}^m g)}{m\mathrm{N}\mathfrak{p}^{ms}}\right),$$

we have convergence of these on $\Re(s) > 1$ as well as the generalization of Proposition 3.3 given by

$$L^{\mathrm{Artin}}\left(s, \sum_i a_i f_i\right) = \prod_i L^{\mathrm{Artin}}(s,f_i)^{a_i}.$$

Applying this to $f_i = \mathrm{Ind}_{\mathrm{Gal}(M/M_i)}^{\mathrm{Gal}(M/K)}\psi_i$, and applying Proposition 3.5, we get

$$\begin{aligned}L^{\mathrm{Artin}}(s,\rho) &= \prod_{i=1}^r L^{\mathrm{Artin}}(s, \mathrm{Ind}_{\mathrm{Gal}(M/M_i)}^{\mathrm{Gal}(M/K)}\psi_i)^{a_i}\\ &= \prod_{i=1}^r L^{\mathrm{Artin}}(s,\psi_i)^{a_i},\end{aligned}$$

and the meromorphic continuation of each of these r factors follows from Lemma 3.4 and Corollary 2.6. ☐

Example 9. Using the fact that the regular representation decomposes as $\bigoplus_\rho (\dim \rho)\rho$ where ρ runs over the irreducible representations of $\mathrm{Gal}(M/K)$, we have

$$\zeta_M(s) = L^{\mathrm{Artin}}(s, \rho_{\mathrm{regular}}) = \prod_\rho L^{\mathrm{Artin}}(s, \rho)$$

up to an entire nonvanishing function that doesn't vanish at 1 (coming from Euler factors at non-split primes, which have multiplicatively bounded contribution). This is the non-abelian analog of Proposition 2.7.

The functional equation satisfied by the (non-abelian) Artin L-function $L^{\mathrm{Artin}}(s, \rho)$ extended meromorphically to the whole complex plane in Corollary 3.7 can also be written down explicitly using the decomposition

$$L^{\mathrm{Artin}}(s, \rho) = \prod_{i=1}^{r} L^{\mathrm{Artin}}(s, \psi_i)^{a_i}$$

from Corollary 3.7 and the functional equations satisfied by the abelian L-functions $L^{\mathrm{Artin}}(s, \psi_i)$ (which are just the functional equations satisfied by Hecke L-functions by class field theory).

The following will definitely NOT be in the talk. Artin was able to analyze this more closely and write down, somewhat more explicitly in terms of ρ as opposed to the data that comes from Brauer's theorem, the functional equation satisfied by the meromorphic continuation of $L(s, \rho)$ to $\mathbf{C}$, namely

Theorem 3.8. Let M/K be a finite Galois extension of number fields and $\rho : \mathrm{Gal}(M/K) \to \mathrm{GL}(V)$ a finite-dimensional irreducible complex representation with character χ. Let

$$\Lambda(s, \rho) := (|\Delta_K|^{\dim \rho} \mathrm{N}(\mathfrak{f}(\chi)))^{s/2} L(s, \rho) \prod_{v|\infty} L_v(s, \rho),$$

where for all complex places v of K,

$$L_v(s, \rho) := 2(2\pi)^{-s}\Gamma(s)^{\dim \rho},$$

and for all real places v of K,

$$L_v(s, \rho) := \left(\pi^{-s/2}\Gamma(s/2)\right)^{\frac{\chi(1)+\chi(c_v)}{2}} \left(\pi^{-(s+1)/2}\Gamma((s+1)/2)\right)^{\frac{\chi(1)-\chi(c_v)}{2}}$$

and c_v is the generator of $\mathrm{Gal}(L_w/K_v)$, which is always trivial or $\mathbf{Z}/2\mathbf{Z}$. Then there is a constant $W(\chi)$ of absolute value 1 such that Λ satisfies the functional equation

$$\Lambda(s, \chi) = W(\chi)\Lambda(1 - s, \overline{\chi}).$$

Proof. See for example [Neu1999, §VII.12], and [Ser1968, Ch. VI] for the necessary theory of the Artin conductor $\mathfrak{f}(\chi)$. □

Langlands [Lan] and Deligne [Del1973] did some work on the Artin root number $W(\chi)$, which I know nothing about.

References

[Art1924] Emil Artin, *Über eine neue art von L-Reihen*, Abh. Math. Sem. Univ. Hamburg **3** (1924), no. 1, 89–108. MR3069421

[Art1931] ______, *Zur theorie der L-reihen mit allgemeinen gruppencharakteren*, Abh. Math. Sem. Univ. Hamburg **8** (1931), no. 1, 292–306. MR3069563

[Con2004] Brian Conrad, *Tame ramification and composite fields*, 2004. Handout from University of Michigan course Math 676. http://math.stanford.edu/~conrad/676Page/handouts/tamecomp.pdf.

[Del1973] Pierre Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 501–597. MR0349635

[Gun2019] Fabian Gundlach, *Math 223a: Algebraic number theory*, 2019. Notes from Harvard course taken by Kenz Kallal. https://scholar.harvard.edu/files/kallal/files/kallal_223a_notes.pdf.

[Kis2019] Mark Kisin, *Math 129: Algebraic number theory*, 2019. Notes from Harvard course taken by Kenz Kallal. https://scholar.harvard.edu/files/kallal/files/kallal_129_notes.pdf.

[Lan1994] Serge Lang, *Algebraic number theory*, Second, Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994. MR1282723

[Lan] Robert Langlands, *On the functional equation of the artin L-functions*. Unpublished(?) manuscript. https://publications.ias.edu/sites/default/files/a-ps.pdf.

[Neu1999] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. MR1697859

[Ser1968] Jean-Pierre Serre, *Corps locaux*, Publications de l'Université de Nancago, No. VIII, Hermann, Paris, 1968. Deuxième édition. MR0354618

[Ser1978] ______, *Représentations linéaires des groupes finis*, revised, Hermann, Paris, 1978. MR543841

[Ser1981] ______, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. **54** (1981), 323–401. MR644559

[SS2003] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, Princeton, NJ, 2003. MR1976398

[Tat1967] J. T. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 1967, pp. 305–347. MR0217026

[Woo2011] Melanie Wood, *How to determine the splitting type of a prime*, 2011. Online notes. https://people.math.harvard.edu/~mmwood/Splitting.pdf.