ARITHMETIC CYCLES, MODULAR FORMS, AND L-FUNCTIONS (DARMONFEST)

1. Lilienfeldt (notes by: Kenz Kallal)

I have made every attempt to provide enough technical detail to make exactly correct statements, but do not guarantee that every statement in these notes is correct on the nose. For those purposes I defer to the actual literature on the topic. Lilienfeldt's talk was about his recent preprint with Shnidman [LS24] which generalizes Gross–Zagier and Zhang's higher weight Gross–Zagier formula. It was situated in the broader context of a series of works by Bertolini, Darmon, Lilienfeldt, and Prasanna involving the generalized Heegner cycles, and came along with an explanation of what is in those articles and how this fits in.

1.1. **Recall of Gross–Zagier and Zhang's higher weight height formula.** Let f be a normalized newform of weight $2k \geq 2$ for $\Gamma_0(N)$, where $N \geq 3$. Let K be an imaginary quadratic field, and let χ be a character $\mathrm{Cl}(K) \to \mathbf{C}^\times$. In other words, χ is the same as an unramified Hecke character for K of infinity-type $(0,0)$. Assume that K has odd discriminant, and that all rational primes p dividing N also split in K (this is the *Heegner hypothesis*). In this context, the Heegner hypothesis is important for at least two reasons:

(1) It ensures the existence of Heegner points of discriminant Δ_K in $X_0(N)$, that is, cyclic degree-N isogenies of elliptic curves $E \to E'$ such that both E and E' have CM by $\mathcal{O}_K$. In particular, there exists an integral ideal $\mathfrak{n} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{n} \cong \mathbf{Z}/N$; so we can always form the CM elliptic curve $A := \mathbf{C}/\mathcal{O}_K$ and take the isogeny with kernel $A[\mathfrak{n}]$.
(2) It ensures (via the functional equation, which can be proven by the Rankin–Selberg method, as in [GZ86, IV]) that the Rankin–Selberg L-function $L(f \times \chi, s)$ has a zero at the central point $s = k$. It therefore makes sense to study the central derivative $L'(f \times \chi, k)$.

The whole point of this subject is that the two types of objects (1) and (2) whose existence is ensured by the Heegner hypothesis are supposed to be related to each other via some height pairing. As the context becomes more and more general, more and more technically sophisticated arithmetic geometry is required to define and study that height. The most nontrivial part of that analysis occurs at the archimedean places and typically involves explicit computation with Green's functions.

1.1.1. *Gross–Zagier: the case $k = 1$, and Heegner points.* Fix a Heegner point $x \in X_0(N)(\mathbf{C})$, for example the point $(A := \mathbf{C}/\mathcal{O}_K, A[\mathfrak{n}])$. By basic CM theory (e.g. [Sil94]) we really have $x \in X_0(N)(H)$, where H is the Hilbert class field of K. The curve $X_0(N)$ admits an embedding into its Jacobian $J_0(N)$ given on closed points by $p \mapsto (p) - (\infty)$ (the image of any p is a divisor on $X_0(N)$ of degree 0). We abusively denote by x the point in $J_0(N)(H)$ corresponding to $x \in X_0(N)(H)$. By class field theory the character χ can also be interpreted as a character of $\mathrm{Gal}(H/K)$, so (using the group law on $J_0(N)$) we can form the average against χ^{-1}

$$C_\chi := \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi(\sigma)^{-1} \sigma(x) \in J_0(N)(H) \otimes \mathbf{C}.$$

Up to a factor of the class number of K, the object C_χ is just the projection of x to the χ-isotypic component of $J_0(N)(H) \otimes \mathbf{C}$ under the action of $\mathrm{Gal}(H/K)$. There is a further action of the relevant Hecke algebra $\mathbf{T}$, and one also needs to project C_χ to the f-isotypic component of $\mathbf{T}$ acting on $J_0(N)$ to have any hope of obtaining a Heegner object that is related to the L-function of $f \times \chi$. We call

Date: September 13, 2026.

this projection $C_{\chi,f}$, and note that its definition involves projecting our fixed Heegner point x to the $(f \times \chi)$-isotypic component of $J_0(N)(H) \otimes \mathbf{C}$ under $\mathbf{T} \times \mathrm{Gal}(H/K)$.

The Jacobian is an abelian variety over $\mathbf{Q}$, so the H-points $J_0(N)(H)$ admit a Néron–Tate height pairing [Ser97], a quadratic form which is denoted by $\widehat{h}$. The celebrated theorem of Gross–Zagier [GZ86] is the precise relationship between $C_{\chi,f}$ and $L'(f \times \chi, k)$ that was promised, restricted to the case $k = 1$.

Theorem 1.1 (Gross–Zagier, 1986). *Under the hypotheses of this section and the added restriction $k = 1$ (so that f is a modular form of weight 2), we have*

$$L'(f \times \chi, 1) = \frac{8\pi^2 (f,f)_{\mathrm{Pet}}}{|\mathrm{Cl}(K)| u_K^2} \cdot \widehat{h}(C_{\chi,f}),$$

where u is half the number of roots of unity in K (so that $u = 1$ for all but finitely many possible K).

Remark 1.2. Since it will set the stage for what follows, I choose to emphasize the following point right now. The relevant Heegner object in Gross–Zagier is essentially the image under a certain projection operator of an element of $J_0(N)(H)$. The significance of $J_0(N)$ here is twofold, with an emphasis on the second point:

(1) It allowed us to access the Néron–Tate height, which is at the very least one of the simplest possible heights to define, so that Theorem 17.1 could be stated.

(2) Since f is of weight 2, Shimura (though I will just cite Deligne [Del71]) associated to f an ℓ-adic Galois representation which is the f-isotypic component under $\mathbf{T}$ of the cuspidal part of $H^1_{\text{ét}}(X_0(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$. By "Kummer theory" in étale cohomology, this is dual to the rational ℓ-adic Tate module of $J_0(N)$, where again the ℓ-adic Galois representations associated to f can be found by applying the projection to the f-isotypic component under $\mathbf{T}$.

These two points suggest a structure for the general theory (e.g. as we progressively loosen the hypotheses on f and χ): The relevant "Heegner objects" will be special cycles (after projection to the $(f \times \chi)$-isotypic part) on some variety whose ℓ-adic cohomology has $(f \times \chi)$-isotypic part equal to the tensor product of the ℓ-adic Galois representations associated to f and χ. A more refined thing to say is that there is a motive $h(f \times \chi) = h(f) \otimes h(\chi)$ attached to the data of $f \times \chi$; the Heegner objects will be cycles on the variety that defines that motive, after projection under the correspondence that is also part of the definition of that motive (see [BDP14, §2.1] for definitions regarding motives). Finally, one will have to define and study a height pairing that applies to those cycles, spread out to an integral model of the relevant variety (i.e. the height pairing will come from some form of arithmetic intersection theory). This is the program that is carried out in [Zha97] and [LS24].

1.1.2. *Zhang's higher weight formula: the case $k > 1$, and Heegner cycles on Kuga–Sato varieties.* We now consider the case where $k > 1$, with the exact same setup, including the Heegner hypothesis. The relevant generalization of Theorem 17.1 is due to Zhang [Zha97]. Following the program set up in Remark 17.2, we first state what the right cycles are and where they live. These now seem to be called *Heegner cycles*, though Zhang sometimes called them CM-cycles in his paper.

Deligne [Del71] associated to f a system of ℓ-adic Galois representations for all ℓ not dividing the level of f. The point of that work is that these Galois representations are found in the étale cohomology of certain ℓ-adic local systems on the modular curve $X_0(N)$. From now on we augment our notation a bit by letting M be a positive integer divisible by N and construct the right variety over $X(M)$, which in turn has a natural projection map $\pi_{M,N} : X(M) \to X_0(N)$.

Scholl [Sch90] refined [Del71] by associating to f a Grothendieck motive[1] $h(f)$ over $\mathbf{Q}$ with coefficients in the field of definition L_f of f. The data defining $h(f)$ are:

[1]See [BDP14, §2.1] for the exact definition of Grothendieck motive. The main point is that the ℓ-adic realization of this motive, which is just the projection under $\varepsilon_f$of the ℓ-adic cohomology of W_{2k-2}, is precisely the ℓ-adic Galois

- The Kuga–Sato variety W_{2k-2}, which is a desingularization of the $(2k-2)$-th power of the universal generalized elliptic curve $\mathcal{E}/X(M)$ fibered over $X(M)$. Here the notion of generalized elliptic curve is that of [DR73], and this desingularization is the one defined in [Del71, Lemma 5.4, 5.5]. Since $\mathcal{E}$ is smooth over $Y(M)$, for all points $p \in Y(M)$, the fiber $W_{2k-2}|_p$ is just $\mathcal{E}|_p^{2k-2}$.
- A certain idempotent[2] $\varepsilon_f \in \mathrm{Corr}^0(W_{2k-2}, W_{2k-2})_{L_f}$ which can actually be defined as an explicit element of the group ring $\mathbf{Q}[\mathrm{Aut}\, W_{2k-2}]$ (see [Sch90, §1.1.2]) composed with the extra projection to the f-isotypic subspace under $\mathbf{T}$.

Therefore, following Remark 17.2, the Heegner cycles we are looking for should live on the Kuga–Sato variety W_{2k-2}. By adding up all the relevant ε_f (or just directly using the idempotent of [Sch90, §1.1.2] instead of also taking the f-isotypic component), one obtains an idempotent ε_{2k} defining the motive of weight-$2k$ cusp forms.

Definition 1.3. Fix as before a Heegner point $x \in X(M)$ of discriminant $|\Delta_K|$. This time, we choose x to be such that $\pi_{M,N}(x)$ is given by some integral ideals $\mathfrak{a}, \mathfrak{n}$ with $\mathcal{O}_K/\mathfrak{n} \cong \mathbf{Z}/N$, as before: its underlying curve is $A^{\mathfrak{a}} := \mathbf{C}/\mathfrak{a}^{-1}$, together with the isogeny with kernel $A^{\mathfrak{a}}[\mathfrak{n}]$. Also one has to make sure that $\mathfrak{a}$ is coprime to M. Since W_{2k-2} is fibered over $X(M)$ via $\pi : W_{2k-2} \to X(M)$, we will define our starting cycle to be in $\pi^{-1}(x)$, which is $\mathcal{E}|_x^{2k-2} = (A^{\mathfrak{a}})^{2k-2} = (A^{\mathfrak{a}})^{k-1} \times (A^{\mathfrak{a}})^{k-1}$. There is a reasonably canonical choice of cycle in here that has to do with the CM structure of $A^{\mathfrak{a}}$, namely the graph of the endomorphism $[\sqrt{\Delta_K}] : A^{\mathfrak{a}} \to A^{\mathfrak{a}}$ given by the CM-by-$\mathcal{O}_K$ that $A^{\mathfrak{a}}$ admits, and the choice of $\sqrt{\Delta_K} \in \mathcal{O}_K$. The *Heegner cycle* $S_{k,\chi}(x)$ corresponding to our original choice of x is precisely the projectors ε_{2k} and $\sum_{\sigma \in \mathrm{Gal}(H/K)} \chi(\sigma)^{-1}\sigma$ applied to this graph in the fiber over x.

To be precise, the cycle $S_{k,\chi}$ has codimension k in W_{2k-2} (you pick up one complementary dimension from $X(M)$ and one more from each copy of $A^{\mathfrak{a}}$ involved in the codomain of $[\sqrt{\Delta_K}]$), i.e.

$$S_{k,\chi}(x) \in \mathrm{CH}^k(W_{2k-2}/H)_{\mathbf{C}}.$$

(the complex coefficients are required here in order to accommodate the values of χ).

If x denotes a CM point in $X_0(N)$ instead of $X(M)$, then by $S_{k,\chi}(x) \in \mathrm{CH}^k(W_{2k-2}/H)_{\mathbf{C}}$ we mean

$$S_{k,\chi}(x) := \frac{1}{\sqrt{\deg \pi_{N,M}}} \sum_i S_{k,\chi}(x_i),$$

where the x_i's are the preimages of x under $\pi_{M,N}$.

These are the appropriate generalizations to the case $k > 1$ of the Heegner divisors C_χ used in Section 17.1.1. The reason for the extra parameter M is that having M be such that $M = N_1N_2$ with $(N_1, N_2) = 1$ and $N_1, N_2 \geq 3$ guarantees the existence of a sufficiently nice $\mathbf{Z}[\zeta_M]$-integral model of $\mathcal{E} \to X(M)$ — see [Zha97, Theorem 2.1.1], which is necessary for the arithmetic intersection theory.

Now it is time to discuss the appropriate intersection theory. Zhang proved that the image of $S_{k,\chi}(x)$ in $H^{2k}(W_{2k-2}(\mathbf{C}), \mathbf{C})$ vanishes, so that there is a Green's current associated to it. Note here that W_{2k-2} has arithmetic dimension $2k$, so k is the middle dimension: after producing integral models by taking Zariski closures from the generic fiber, we should be able to pair two Heegner cycles together via an arithmetic intersection theory to get a number. Indeed, the existence of Green currents for the Heegner cycles allow Zhang (after also choosing $\mathcal{O}_H$-integral models of the cycles) to upgrade $S_{k,\chi}(x)$ to an element $\widehat{S_{k,\chi}}(x)$ of the arithmetic Chow group of $W_{2k-2}/\mathcal{O}_H$, which admits an arithmetic intersection theory due to Gillet–Soulé [GS90]. For this to be well-behaved, Zhang verified that the cycles $S_{k,\chi}$ have trivial intersection number with all other cycles of the

representation associated to f by Deligne — this is what [Sch90, Theorem 1.2.1], which provides the exact link between $(W_{2k-2}, \varepsilon_f)$ and the f-isotypic component under $\mathbf{T}$ of the cuspidal part of $H^1_{\text{ét}}(X_0(N)_{\overline{\mathbf{Q}}}, \mathrm{Sym}^{k-2} R^1\pi_*\mathbf{Q}_\ell)$ (Deligne's thing), says.

[2]See [BDP14, §2.1] for the exact definition of this ring of correspondences.

appropriate codimension supported in the special fibers of $W_{2k-2}/\mathcal{O}_H$: as a result, all the relevant Gillet–Soulé intersection pairings do not depend on the choice of integral model of the cycle (which is ambiguous up to adding some fibral divisors). The same observation is also necessary in [GZ86] (see [Sri, Proposition 5.2]).

I abusively denote the Gillet–Soulé intersection pairing by $\langle -, - \rangle^{\mathrm{BB}}$ when applied to these cycles, because (thanks in part to the well-definedness mentioned above) it is known to coincide with the conjectural Beilinson–Bloch height pairing [Bei87, Blo84]. For example, the notation $\langle S_{k,\chi}(x), S_{k,\chi}(x)\rangle^{\mathrm{BB}}$ is really the Gillet–Soulé arithmetic intersection pairing of the arithmetic Chow group element $\widehat{S}_{k,\chi}(x)$ with itself, and this doesn't depend on the choice of integral model implicit in the upgrade $\widehat{S}_{k,\chi}(x)$.

Zhang proved the following generalization of Theorem 17.1:

Theorem 1.4 (Zhang 1997)**.** *In the general situation where $k \geq 1$, the heights of Heegner cycles on W_{2k-2} satisfy (up to scalars having to do with the right normalizations of the Heegner cycles)*

$$L'(f \times \chi, k) = \frac{2^{4k-1}\pi^{2k}(f,f)_{\mathrm{Pet}}}{(2k-2)!u_K^2|\mathrm{Cl}(K)|\sqrt{|\Delta_K|}}\langle S'_{f,\chi}(x), S'_{f,\chi}(x)\rangle^{\mathrm{BB}}$$

for all CM points x in $X_0(N)$ of the form used in Definition 17.3. Here $S'_{f,\chi}$ denotes the f-isotypic component under the Hecke algebra of the image of $S_{k,\chi}$ in the quotient of $\mathrm{Span}_{(m,N)=1}(T_m S_{k,\chi})$ *by the kernel of the height pairing.*

Zhang's proof of Theorem 17.4 is a calculation after splitting up the Gillet–Soulé pairing into local contributions. The decomposition into local height pairings is only possible when the cycles involved (which, in the details of the proof, are $\widehat{S}_{\chi,f}(x)$ and its conjugate under the Hecke action T_m for $m \geq 1$) do not intersect in the generic fiber (else the local intersection pairings will not all be defined). When they do intersect in the generic fiber, one has to be more careful when defining the local heights by considering limits along archimedean and nonarchimedean deformations along the base, as in [Zha97, Theorem 1.2.2]. This is exactly the same as the extra technicalities in deformation theory that must be dealt with in Gross–Zagier when "$r_{\mathcal{A}}(m) \neq 0$."

1.2. **The generalization of Lilienfeldt–Shnidman: more general χ, and generalized Heegner cycles.** David Lilienfeldt's lecture at Darmonfest was about his recent preprint [LS24], which is joint with Ari Shnidman. The main theorem of the lecture was the even further generalization of the program described in Remark 17.2 to the case where the algebraic Hecke character χ is not necessarily of finite order. In the previous section, χ was taken to be an algebraic unramified Hecke character of $\mathbf{A}_K^\times$ with infinity type $(0,0)$. Now, we let it be an algebraic unramified Hecke character of $\mathbf{A}_K^\times$ with infinity type $(2t, 0)$, where $0 < 2t \leq 2k-2$. We retain the Heegner hypothesis and the setup in which f is a newform for $\Gamma_0(N)$ of weight $2k$.

To access the L-function for $f \times \chi$, the relevant motivic information is not just $h(f)$ but also the motive $h(\chi)$ that is associated to χ. For the exact meaning of this, see [Del79, Conjecture 8.1] and [Sch88, Ch. 1]. In [BDP12, §2] and [BDP14, §2.2] exactly the necessary material about how to think about this motive as having defining datum involving the $2t$-th power of a CM elliptic curve is explained, which I follow below anyway. The basic points here are as follows:

(1) Weil [Wei56] gave a recipe to associate a system of ℓ-adic Galois representations to algebraic Hecke characters. That is, if $\chi|_{\mathbf{A}_K^{\times,\infty}}$ has values in a number field $E_\chi \supset K$, then for all ℓ there is an associated ℓ-adic Galois representation $\mathrm{Gal}(\overline{K}/K) \to (E_\chi \otimes \mathbf{Q}_\ell)^\times$. These should be related to the ℓ-adic realizations of the motive $h(\chi)$.
(2) Let A/F be an elliptic curve with CM by $\mathcal{O}_K$, where F is a number field containing the Hilbert class field of K. By CM theory (see for instance [Sil94, §II.9]) there is then an assocated algebraic Hecke character $\psi_A : \mathbf{A}_F \to \mathbf{C}^\times$ such that $L(s, A) = L(s, \psi_A)L(s, \overline{\psi}_A)$

(the L-function on the left is the one that only has to do with H^1). The infinity type of ψ_A is the pullback to F of some CM type on K (see [KS25, §1.1]). In other words, it is just the sum of the complex embeddings of F that lie over a choice (which we make once and for all depending on that of χ to make the next point true) of complex embedding of K.

(3) We can also form the Hecke character $\chi \circ N_K^F$ from χ, whose infinity type is exactly the same as that of ψ_A^{2t}, namely the pullback to F of $(2t, 0)$. In particular, $(\chi \circ N_K^F) \cdot \psi_A^{-2t}$ is a finite-order Hecke character for F, which can be made trivial if F is taken to be large enough (as its associated Galois character has finite image). In other words, F can be taken such that
$$\chi \circ N_K^F = \psi_A^{2t}.$$

(4) It follows that we can define a motive associated to χ using the variety A^{2t}, namely (taking $K = \mathbf{Q}(\sqrt{-D})$)
$$(A^{2t}, e_{2t}, 0)$$
where
$$e_{2t} := \left(\left(\frac{\sqrt{-D} + [\sqrt{-D}]}{2\sqrt{-D}} \right)^{\boxtimes 2t} + \left(\frac{\sqrt{-D} - [\sqrt{-D}]}{2\sqrt{-D}} \right)^{\boxtimes 2t} \right) \circ \left(\frac{1 + [-1]}{2} \right)^{\boxtimes 2r}.$$
This motive is defined over F and has coefficients in $\mathbf{Q}$, and it can be extended to a motive $h(\chi)$ defined over F with coefficients in K by having K act via the diagonal CM action (using Language A of [Del79, p. 9] instead of Language B). The point of the idempotent that projects to the ($[-1] = -1$)-eigenspace is that it isolates $H^1_{dR}(A/F)$ from $H^0_{dR}(A/F)$ and $H^2_{dR}(A/F)$ (it kills H^0 and H^2 since $[-1]$ acts by 1 on those). So in the Künneth decomposition of $H^\bullet_{dR}(A^{2t}/F)$, this idempotent projects onto $H^1_{dR}(A/F)^{\otimes 2t}$. The point of the idempotents projecting to the $[\sqrt{-D}] = \pm\sqrt{-D}$ eigenspaces is that they isolate the $(*, 0)$ and $(0, *)$ parts in the Hodge decomposition after $\otimes\mathbf{C}$. So the base change to $\mathbf{C}$ of the de Rham realization of the motive $h(\chi)$ (recall that we are using Language A) is
$$H^{2t,0}_{dR}(A^{2t}/\mathbf{C}) \oplus H^{0,2t}_{dR}(A/\mathbf{C}).$$
All of this comes from $H^{1,0}(A/\mathbf{C})^{\otimes 2t}$ or $H^{0,1}(A/\mathbf{C})^{\otimes 2t}$, since A is a curve. In particular, the de Rham realization of $h(\chi)$ is of rank 1 over $F \otimes K$ and everything is concentrated in degree $2t$. The ℓ-adic étale realization of $h(\chi)$ is then of rank 1 over $K \otimes \mathbf{Q}_\ell$. More precisely, Künneth still applies to ℓ-adic cohomology, and the idempotent e_{2t} singles out (for the same reasons outlined above except applied to étale cohomology) the part of $H^{2t}(A^{2t}_{/\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$ given (over a suitable extension of $\mathbf{Q}_\ell$) by the subset $(L_+)^{\otimes 2t} \oplus (L_-)^{\otimes 2t} \subset H^1(A_{/\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)^{\otimes 2t}$, where $L_\pm$ is the line in a suitable base change of $H^1(A_{/\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$ given by the $\pm\sqrt{-D}$-eigenspace of the action of $[\sqrt{-D}]$. Note that even though $L_+, L_- \subset H^1(A_{/\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \otimes \overline{\mathbf{Q}}_\ell$ are not necessarily individually defined[3] over $\mathbf{Q}_\ell$, their sum is, because the idempotent defining it is fixed by $\mathrm{Gal}(K/\mathbf{Q})$. This is the same reason why that idempotent actually does *a priori* define a motive with coefficients in $\mathbf{Q}$ instead of K, even though its two summands individually do not.

(5) Since the motive $h(\chi)$ is defined over F, its ℓ-adic étale realization $(L_+)^{\otimes 2t} \oplus (L_-)^{\otimes 2t}$, which is a rank-1 free module over $K \otimes \mathbf{Q}_\ell$, admits an action of $\mathrm{Gal}(\overline{F}/F)$, in other words it is a character
$$\mathrm{Gal}(\overline{F}/F) \to (K \otimes \mathbf{Q}_\ell)^\times.$$

[3]This depends on the splitting behaviour of ℓ in K. The group $H^1(A_{/\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$ is a rank-1 free $K \otimes \mathbf{Q}_\ell$-module. So if ℓ is split in K, then that cohomology group is just $\mathbf{Q}_\ell \oplus \mathbf{Q}_\ell$ with $[\sqrt{-D}]$

The main theorem of complex multiplication says that this character is exactly the same as the one coming from ψ_A^{2t} via (1), so (up to definitional issues regarding what it means to descend from F to K) $h(\chi)$ is the correct motive to associate to χ.

(6) One also has a theta series θ_χ associated to χ, which is of weight $2t+1$. The motive $h(\theta_\chi)$ associated to it by Scholl turns out to have the same ℓ-adic realizations as $h(\chi)$, considered as 2-dimensional $\mathbf{Q}_\ell$-representations of $\mathrm{Gal}(\overline{F}/F)$: see [BDP14, Proposition 2.7].

Having put our hands on idempotents defining a suitable motive for χ, we are ready to define the *generalized Heegner cycles* that will generalize the role of the Heegner cycles in Theorem 17.4. They were originally defined by Bertolini–Darmon–Prasanna [BDP13].

Definition 1.5. The *generalized Kuga–Sato variety* that is relevant to our starting datum is

$$X_{t,k} = W_{2k-2} \times_H A^{2t}.$$

The *generalized Heegner cycles* are the following generalization of Definition 17.3. Take the Heegner point $Q \in (X(M))(H)$ and its fiber

$$\pi^{-1}(Q) = \mathcal{E}|_Q^{2k-2} \times_H A^{2t} = (A^{\mathfrak{a}})^{2k-2} \times A^{2t} = (A^{\mathfrak{a}} \times A^{\mathfrak{a}})^{k-t-1} \times (A \times A^{\mathfrak{a}})^{2t}.$$

Here we have retained the notation from Definition 17.3, $A^{\mathfrak{a}} = \mathbf{C}/\mathfrak{a}^{-1}$ and $A = \mathbf{C}/\mathcal{O}_K$, so that there is a standard isogeny $\varphi^{\mathfrak{a}} : A \to A^{\mathfrak{a}}$. In particular, we can define the dimension-$(k+t-1)$ cycle

$$\gamma_Q^{\mathfrak{a}} := (\text{graph of } [\sqrt{-D}])^{k-t-1} \times (\text{graph of } \varphi^{\mathfrak{a}})^{2t} \subset (A^{\mathfrak{a}} \times A^{\mathfrak{a}})^{k-t-1} \times (A \times A^{\mathfrak{a}})^{2t} = \pi^{-1}(Q),$$

and the *generalized Heegner cycle* coming from this choice of data is

$$\Delta^{\mathfrak{a}} := (\varepsilon_{2k} \boxtimes e_{2t}) \cdot \gamma^{\mathfrak{a}} \in \mathrm{CH}^{k+t}(X_{k,t}/H)_{\mathbf{C}}.$$

To define the cycles that actually intervene for $x \in X_0(N)$ defined using $\mathfrak{n}$ as in Definition 17.3, we have to take a χ-weighted average. To be more precise, let $\epsilon, \overline{\epsilon}$ be the two summands defining $\varepsilon_{2k} \boxtimes e_{2t}$ coming from the two summands of e_{2t}. Then one defines

$$Z^{\mathfrak{a}} := \sum_{Q \in \pi_{M,N}^{-1}(x)} \epsilon \gamma_Q^{\mathfrak{a}}.$$

and

$$\overline{Z}^{\mathfrak{a}} := \sum_{Q \in \pi_{M,N}^{-1}(x)} \overline{\epsilon} \gamma_Q^{\mathfrak{a}},$$

and finally the generalized Heegner cycles that are actually used are

$$\Delta_{k,\chi} = \frac{1}{\sqrt{\deg \pi_{M,N}}} \sum_{\mathfrak{a}} \left(\chi(\mathfrak{a})^{-1} Z^{\mathfrak{a}} + \chi(\overline{\mathfrak{a}})^{-1} \overline{Z}^{\mathfrak{a}} \right),$$

where $\mathfrak{a}$ runs over representatives of the elements of the class group of K with $\mathfrak{a}$ coprime to M. Generalizing the analogous fact for the Heegner cycles, this is a null-homologous element of $\mathrm{CH}^{k+t}(X)_{\mathbf{C}}$ [LS24, Lemma 5.1].

The theorem presented in Lilienfeldt's talk was:

Theorem 1.6 (Lilienfeldt–Shnidman, 2024)**.** *With starting data χ, f where f is a newform of weight $2k$ and χ an unramified algebraic Hecke character for the imaginary quadratic field K of infinity type $(2t, 0)$ where $0 < t \le k-1$, and where the Heegner hypothesis is satisfied, the L-function $L(f \times \chi, s)$ vanishes to odd order at its center $s = k+t$, and we have*

$$L'(f \times \chi, k+t) = \frac{4^{k+t}\pi^{2k}(f,f)_{\mathrm{Pet}}}{(k-t-1)!(k+t-1)!|\mathrm{Cl}(K)|u_K^2|\Delta_K|^{k-t-1/2}} \langle \Delta'_{f,\chi}, \Delta'_{f,\chi} \rangle^{\mathrm{BB}}$$

where $\Delta'_{f,\chi}$ is the f-isotypic component for the Hecke algebra of the image of $\Delta_{k,\chi}$ in the quotient of $\mathrm{Span}_{(m,N)=1}(T_m \Delta_{k,\chi})$ *by the kernel of the height pairing.*

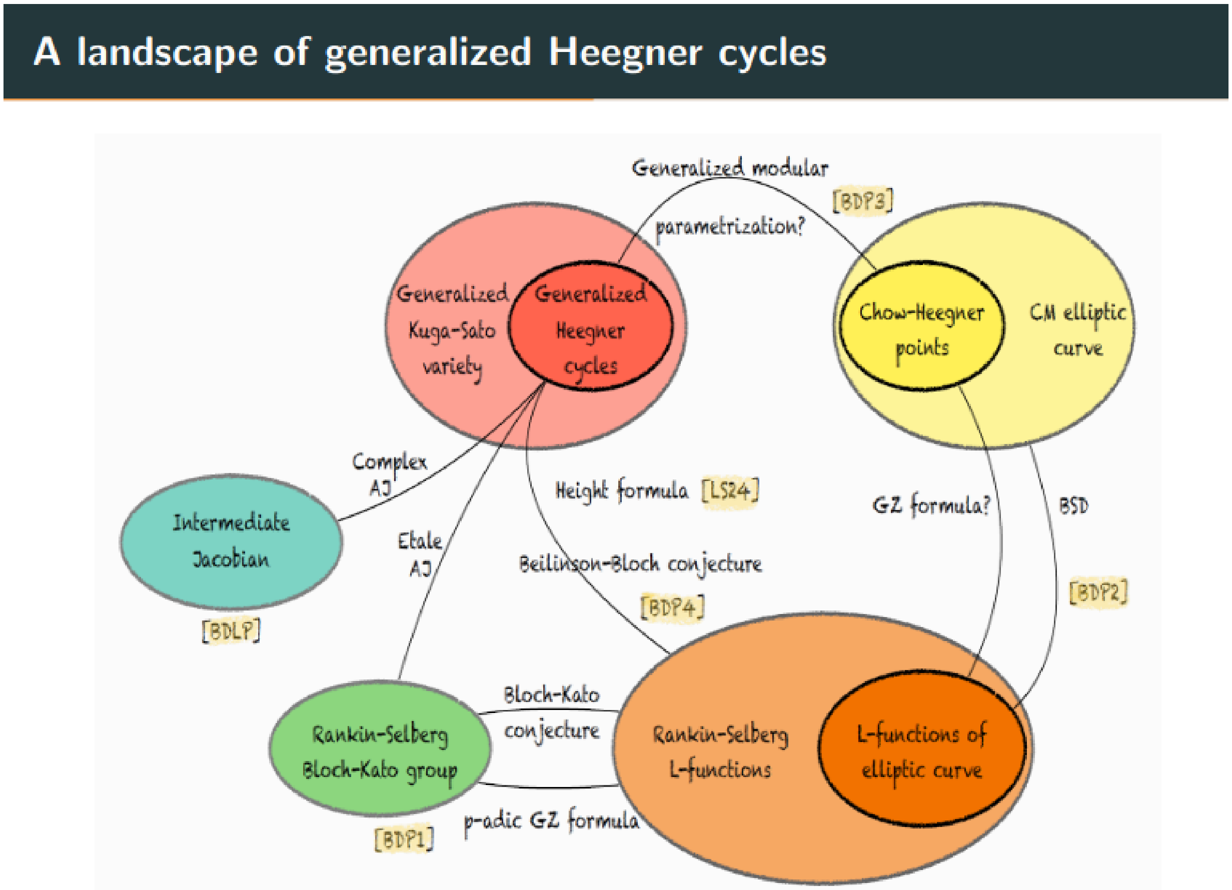

FIGURE 1. The diagram from Lilienfeldt's slides

So the Gross–Zagier formula has been generalized to the case where f has arbitrary even weight (as opposed to weight 2) and χ is an arbitrary unramified algebraic Hecke character for K of infinity type $(2t, 0)$ (as opposed to $(0, 0)$).

1.3. **Broader context of BD(L)P works.** The paper [LS24] that Lilienfeldt spoke about fits into the context of a series of five other papers by Bertolini–Darmon–Prasanna and Bertolini–Darmon–Lilienfeldt–Prasanna [BDP13, BDP12, BDP14, BDP17, BDLP21] that all involve the generalized Heegner cycles. Since the conference was in honor of Darmon's 60th birthday and these papers constitute a very influential part of his research program (as evidenced by 258 citations on Google Scholar for [BDP13] and 20 papers containing the words "generalized Heegner cycles" in their title at the time of Darmonfest), Lilienfeldt gave a general overview of the contributions made by the BD(L)P papers to Beilinson–Bloch–Kato for generalized Kuga–Sato varieties using the generalized Heegner cycles, using the diagram in Figure 1 (copied here directly from Lilienfeldt's slides).

Here is a slightly more detailed explanation of each link in Figure 1.

(1) [BDP13]: defined for the first time the generalized Heegner cycles of Definition 17.5, and established a p-adic analog of Theorem 17.6 where χ is algebraic and has infinity type of a certain form (not necessarily finite-order but also not the same ones as in Theorem 17.6). For cusp eigenforms f of weight $k \geq 2$, they use the étale Abel–Jacobi map for rational primes p not dividing N and split in K to map the generalized Heegner cycles in the generalized Kuga–Sato variety $W_{k-2} \times A^{k-2}$ to the Bloch–Kato Selmer group $H^1_f(F, H^{2k-3}((W_{k-2} \times$

$A^{k-2})_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p)(k-1))$. Using the standard comparison theorems, one ends up with the images of the generalized Heegner cycles in $(\mathrm{Fil}^{k-1} H^{2k-3}_{dR}(W_{k-2} \times A^{k-2})/F)^\vee$. The main theorem of [BDP13] is that those images, when evaluated on specific differential forms, produce (up to normalization by the appropriate period) special values outside the range of classical interpolation of p-adic L-functions interpolating the central values $L(f \times \chi^{-1}, 0)$ for p-adically varying χ. The proof uses integration à la Coleman of p-adic modular forms.

(2) [BDP12]: generalizes a theorem of Rubin by starting with an arbitrary algebraic Hecke character χ for K of infinity type $(1, 0)$ satisfying certain technical conditions, associating to it a CM abelian variety B_χ/K by a recipe due to Casselman [Shi71, Theorem 6], and then proving a relationship between a p-adic L-value at χ^* (the Hecke character $\chi \circ c$, where c denotes complex conjugation of ideals) and the formal group logarithm of a certain $P_\chi \in B_\chi(K) \otimes \mathbf{Q}$. Thus a non-torsion point for B_χ is produced, when the order of vanishing of a certain L-function is equal to exactly 1 — hence the "BSD" label in Lilienfeldt's diagram.

(3) [BDP14]: introduced the idea that there should be a "generalized modular parametrization" $\mathrm{CH}^{k-1}(W_{k-2} \times A^{k-2})_{0,\mathbf{Q}} \to A \otimes \mathbf{Q}$, generalizing the modular parametrization $\mathrm{CH}^1(X_0(N))_{0,\mathbf{Q}} = J_0(N) \to A$; and that this generalized modular parametrization should come from a special algebraic cycle $\Pi^?$ in $\mathrm{CH}^{k-1}(W_{k-2} \times A^{k-2} \times A)_{\mathbf{Q}}$. Assuming such a cycle with the right properties exists, one obtains the Chow–Heegner points in A by taking the image of the generalized Heegner cycles. The Chow–Heegner points generalize the usual Heegner points on elliptic curves that occur when $k = 2$ as images of Heegner points on a modular curve under the modular parametrization coming from the modularity theorem for that elliptic curve [BCDT01]. Even though the existence of the $\Pi^?$ they need is conditional on the Tate conjecture, by passing through the p-adic Abel–Jacobi map of [BDP13] and the Kummer map in Galois cohomology, they non-conditionally produce a local map

$$\mathrm{CH}^{k-1}(W_{k-2})_{0,F_v} \to A(F_v) \otimes L_{\theta_\chi}$$

for places v of F over p. Their main theorem is that the images of the generalized Heegner cycles can be lifted to global rational points of A. There is a criterion for that global point to be of infinite order in terms of the non-vanishing of certain L-values. This uses both [BDP13, BDP12]. They also provide a complex-analytic analog via the complex Abel–Jacobi map to the intermediate Jacobian,

$$\mathrm{CH}^{k-1}(W_{k-2} \times A^{k-2})_{0,\mathbf{C}} \to \frac{\mathrm{Fil}^{k-1} H^{2k-3}_{dR}(W_{k-2} \times A^{k-2}/\mathbf{C})^\vee}{\mathrm{Im} H_{2k-3}((W_{k-2} \times A^{k-2})(\mathbf{C}), \mathbf{Z})},$$

which ends up providing Chow–Heegner points in $A(\mathbf{C})$ corresponding to the generalized Heegner cycles, by explicit integration of differential forms. The Chow–Heegner points are conjectured to actually be in $A(H) \otimes \mathbf{Q}$, and numerical evidence for this is gathered.

(4) [BDLP21]: contains the bulk of the proof of the explicit formula for the complex Abel–Jacobi map in terms of integration of differential forms which is used in [BDP14] to carry out the explicit verification in special cases of the conjecture that the values of the complex Abel–Jacobi map on generalized Heegner cycles are in $A(H) \otimes \mathbf{Q}$. Note that the reference to "[BDP14, §3]" in the introduction to [BDLP21] should be "[BDP14, §4]." The results of [BDLP21] have been further generalized by Lilienfeldt in a more recent work [Lil24].

(5) [BDP17]: The Beilinson–Bloch and Bloch–Kato conjectures together predict that the Abel–Jacobi map

$$\mathrm{CH}^d(X)_{0,\mathbf{Q}_\ell} \to H^1_f(F, H^{2d-1}(X_{/\overline{\mathbf{Q}}}, \mathbf{Q}_\ell(d)))$$

is an isomorphism of $\mathbf{Q}_\ell$ vector spaces of dimension equal to the order of vanishing of $L(h^{2d-1}(X), s)$ at the central point $s = d$. The main result of Lilienfeldt's talk Theorem 17.6 constitutes partial progress towards this goal when $d = k + t$, because in the case where the central order of vanishing of $L(h(f \times \chi), s)$ is exactly one, it says that the generalized

Heegner cycles have self-height-pairing that is nonzero, and hence produce nonzero elements of $\mathrm{CH}^{k+t}(X)_{0,K}$. The fourth BDP paper [BDP17] also uses the generalized Heegner cycles in $X = W_{r+i} \times A^{r-i}$ ($i \equiv r - 1 \bmod 2$) to produce evidence of a further refined version of Beilinson–Bloch–Kato conjecture involving the *coniveau filtrations* on the Chow group and on the ℓ-adic cohomology groups $H^{2j-1}(X_{/\overline{\mathbf{Q}}}, \mathbf{Q}_\ell(j))$ compatible under the ℓ-adic Abel–Jacobi map (see [Blo85, Jan00]). The refined Beilinson–Bloch–Kato conjecture says that the rank of each graded piece of the Chow group is predicted by the order of vanishing of the L-function for the graded pieces of the relevant motive. And [BDP17] provides evidence for this by using the generalized Heegner cycles to produce, under certain L-function nonvanishing conditions, two nontrivial elements of the first graded piece of a Chow group of a certain generalized Kuga–Sato variety.

We take the opportunity here to remark that there is obvious flexibility available in the definition of generalized Heegner cycles. We refer to the various papers above for the exact definition that is convenient in each application.

1.4. New inputs from arithmetic intersection theory in the Lilienfeldt–Shnidman proof.

Finally, I provide some information about the proof of Theorem 17.6. Just as in [GZ86] and [Zha97], by using the Rankin–Selberg method, the statement is reduced to calculating

$$\langle \Delta_{k,\chi}, T_m \Delta_{k,\chi} \rangle^{\mathrm{BB}},$$

and (after upgrading to integral models of all the cycles) splitting it into local components which are computed individually. Again as in [GZ86] and [Zha97], after upgrading to integral models $\Delta_{k,\chi,\mathcal{O}_F}$, one wants to decompose into local heights. When the cycles involved intersect in the generic fiber, this becomes more involved and one needs to consider certain limits along deformations along the base. Lilienfeldt–Shnidman carry this out using the archimedean local heights for local systems due to Brylinksi [Bry89], applied to the PVHS on $X_0(N)_{\mathbf{C}}$ given by

$$(\mathrm{Sym}^{2k-2} R^1 \pi_{\mathcal{E},*} \mathbf{Z})(k-1) \otimes (\varepsilon_{2k} \otimes e_{2t}) H^{2t}(A^{2t}(\mathbf{C}), \mathbf{Z})(t).$$

Explicit calculations using Brylinski's formalism, including explicit determination of the required Green's kernels in terms of Jacobi functions of the second kind constitute the bulk of [LS24]. The nonarchimedean computations are done with the same definitions and tools as [Zha97].

References

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939. MR 1839918

[BDLP21] Massimo Bertolini, Henri Darmon, David Lilienfeldt, and Kartik Prasanna, *Generalised Heegner cycles and the complex Abel-Jacobi map*, Math. Z. **298** (2021), no. 1-2, 385–418. MR 4257092

[BDP12] Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *p-adic Rankin L-series and rational points on CM elliptic curves*, Pacific J. Math. **260** (2012), no. 2, 261–303. MR 3001796

[BDP13] ———, *Generalized Heegner cycles and p-adic Rankin L-series*, Duke Math. J. **162** (2013), no. 6, 1033–1148, With an appendix by Brian Conrad. MR 3053566

[BDP14] ———, *Chow-Heegner points on CM elliptic curves and values of p-adic L-functions*, Int. Math. Res. Not. IMRN (2014), no. 3, 745–793. MR 3163566

[BDP17] ———, *p-adic L-functions and the coniveau filtration on Chow groups*, J. Reine Angew. Math. **731** (2017), 21–86, With an appendix by Brian Conrad. MR 3709060

[Bei87] Alexander Beilinson, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–25. MR 923131

[Blo84] Spencer Bloch, *Height pairings for algebraic cycles*, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), vol. 34, 1984, pp. 119–145. MR 772054

[Blo85] ———, *Algebraic cycles and values of L-functions. II*, Duke Math. J. **52** (1985), no. 2, 379–397. MR 792179

[Bry89] Jean-Luc Brylinski, *Heights for local systems on curves*, Duke Math. J. **59** (1989), no. 1, 1–26. MR 1016879

[Del71] Pierre Deligne, *Formes modulaires et représentations l-adiques*, Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363, Lecture Notes in Math., vol. 175, Springer, Berlin, 1971, pp. Exp. No. 355, 139–172. MR 3077124

[Del79] ———, *Valeurs de fonctions L et périodes d'intégrales*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., vol. XXXIII, Amer. Math. Soc., Providence, RI, 1979, With an appendix by N. Koblitz and A. Ogus, pp. 313–346. MR 546622

[DR73] Pierre Deligne and Michael Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., vol. Vol. 349, Springer, Berlin-New York, 1973, pp. 143–316. MR 337993

[GS90] Henri Gillet and Christophe Soulé, *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math. (1990), no. 72, 93–174. MR 1087394

[GZ86] Benedict Gross and Don Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320. MR 833192

[Jan00] Uwe Jannsen, *Equivalence relations on algebraic cycles*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 225–260. MR 1744947

[KS25] Guido Kings and Johannes Sprang, *Eisenstein-Kronecker classes, integrality of critical values of Hecke L-functions and p-adic interpolation*, Ann. of Math. (2) **202** (2025), no. 1, 1–109. MR 4927734

[Lil24] David T.-B. G. Lilienfeldt, *Heegner cycles in Griffiths groups of Kuga-Sato varieties*, Math. Res. Lett. **31** (2024), no. 2, 459–509. MR 4819388

[LS24] David T. B. G. Lilienfeldt and Ari Shnidman, *Derivatives of Rankin–Selberg L-functions and heights of generalized Heegner cycles*, 2024, `https://arxiv.org/abs/2408.04375`.

[Sch88] Norbert Schappacher, *Periods of Hecke characters*, Lecture Notes in Mathematics, vol. 1301, Springer-Verlag, Berlin, 1988. MR 935127

[Sch90] Anthony. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), no. 2, 419–430. MR 1047142

[Ser97] Jean-Pierre Serre, *Lectures on the Mordell-Weil theorem*, third ed., Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997, With a foreword by Brown and Serre. MR 1757192

[Shi71] Goro Shimura, *On the zeta-function of an abelian variety with complex multiplication*, Ann. of Math. (2) **94** (1971), 504–533. MR 288089

[Sil94] Joseph Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994. MR 1312368

[Sri] Vijay Srinivasan, *Modular curves and local heights*, MIT seminar notes, `https://math.mit.edu/~vijayrs/files/modcurves.pdf`.

[Wei56] André Weil, *On a certain type of characters of the idèle-class group of an algebraic number-field*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, Science Council of Japan, Tokyo, 1956, pp. 1–7. MR 83523

[Zha97] Shouwu Zhang, *Heights of Heegner cycles and derivatives of L-series*, Invent. Math. **130** (1997), no. 1, 99–152. MR 1471887