

# ALGEBRAIC THEORY OF INDEFINITE THETA FUNCTIONS

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ABSTRACT. We describe an algebraic theory of theta functions attached to a nondegenerate *but not necessarily definite* quadratic form. Following Kudla and Millson, to handle the general case, we index such functions by homology classes in the automorphism group of this form. Our approach permits these classes to have integral or torsion coefficients; it also gives a new proof of the result of Kudla and Millson and sheds some new light on the classical algebraic theory of  $\theta$ -functions.

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## 1. INTRODUCTION

Let  $B$  be a symmetric  $m \times m$  matrix of integers, and consider the associated symmetric bilinear pairing  $\langle x, y \rangle = x^t B y$  on  $\mathbf{Z}^m$ , with associated quadratic form  $Q(x) = \langle x, x \rangle / 2$ . We suppose that  $Q$  has signature  $(r, s)$  where  $r + s = m$ , and will write  $\mathbf{d}$  for the absolute value of the determinant of  $B$ , always assumed nonzero.

In the positive-definite case  $s = 0$ , the theory of theta series assures that the series

$$(1.1) \quad \theta(q) = \sum_{n \in \frac{1}{2}\mathbf{Z}} Z_n q^n, \quad q := \exp(2\pi i \tau)$$

is a modular form, where  $Z_n$  is the number of solutions to  $Q(v) = n$  with  $v$  an integer vector. That is to say: taking  $\tau$  in the upper half-plane, the function  $\theta(e^{2\pi i \tau})$  satisfies, in addition to its obvious periodicity  $\tau \mapsto \tau + 2$ , a non-obvious symmetry under other fractional linear transformations, including  $\tau \mapsto \frac{\tau}{4\mathbf{d}\tau + 1}$ .

In the case of  $Q$  indefinite, that is, when  $s > 0$ , this number  $Z_n$  is typically infinite. Nonetheless, an inspection of classical formulae<sup>1</sup> suggests the desirability of carrying over some part of the classical theory associated to (1.1). One way to do so, due to Siegel [53], replaces  $\sum Z_n q^n$  by a *non-holomorphic* function of  $q$  (and we briefly discuss the rich history of other approaches to this question in §1.3). But, following Kudla–Millson [38], it is also possible to preserve the holomorphic character of (1.1), at the cost of an increased algebraic complexity — by accounting for the increased *symmetry* of the situation. If we write  $\Gamma$  for the group of unimodular integral transformations preserving  $Q$ , that is

$$\Gamma = \{m \times m \text{ unimodular integral matrices } \gamma \text{ such that } \gamma^t B \gamma = B\}$$

then the number of *orbits* of  $\Gamma$  on solutions  $v$  to  $Q(v) = n$  is finite whenever  $n \neq 0$ . One wants  $Z_n$  to measure the size of the quotient  $\frac{Q^{-1}(n)}{\Gamma}$ ; if the stabilizers are finite, a naïve count will do, but in general one needs to “count” by taking a fundamental class in the sense of topology. The resulting analogue of  $Z_n$  lives inside a cohomology group<sup>2</sup>:

$$Z_n \in H^s(\Gamma, \mathbf{Z})$$

and, as we explain in §1.1, counts the number of solutions to  $Q = n$  inside certain conical regions (the full detail of this is resumed in Section V). The main result of this paper is:

$$(1.2) \quad \begin{aligned} & \text{(See Theorem V.1 for details; see Section 2 and Section 4 for proof sketches.)} \\ & \text{The series } \sum Z_n q^n \text{ of (1.1) remains a modular form, with the same weight and} \\ & \text{level as in the positive definite case – at least after enlarging the coefficients of} \\ & \text{the cohomology from } \mathbf{Z} \text{ to } \mathbf{Z} \left[ \frac{1}{2\mathbf{d}} \right]. \end{aligned}$$

<sup>1</sup>For example: the Fourier coefficients of the weight 2 Eisenstein series are related to the divisor function, which we can think of as the number of representations of an integer  $n$  by the two-variable indefinite form  $x_1, x_2 \mapsto x_1 x_2$ .

<sup>2</sup>at the cost of possibly replacing  $\Gamma$  with a slightly smaller finite-index subgroup

The classes  $Z_n$  were defined by Kudla and Millson, and in fact they proved the above statement *with  $\mathbf{C}$  instead of  $\mathbf{Z}[\frac{1}{2d}]$  as coefficient ring for cohomology*. Our main goal was to obtain a result that equally detects torsion phenomena: that is to say, if  $H^s(\Gamma, \mathbf{Z}[\frac{1}{2d}])$  contains a factor of  $\mathbf{Z}/p$  for some prime  $p$ , then (1.2) is asserting the existence of a *mod  $p$*  modular form associated to  $Q$ .<sup>3</sup> Such a conclusion does not seem to be available using the transcendental methods of Kudla–Millson; and, indeed, our paper also gives a new algebraic proof of their result.<sup>4</sup>

The key novel points of our argument involve the construction and exploitation of a flat connection on the coherent cohomology of a  $\Gamma$ -equivariant line bundle on a certain family of abelian varieties. It is well known in closely related contexts (and in particular in conformal field theory) that a *projectively* flat *projective* connection exists [2, 32, 50, 55, 60]; but it is crucial for us to remove the word “projective,” and in a completely explicit fashion.

We remark that for all  $g \geq 1$  Kudla–Millson produce a degree- $g$  Siegel modular form with coefficients in  $H^{gs}(\Gamma, \mathbf{C})$  whose Fourier coefficient at a symmetric matrix  $T$  is similarly a generalized count of ways to represent  $T$  as  $T = \underline{v}^t B \underline{v}$  where  $\underline{v}$  is a  $m \times g$  integral matrix. The theorem actually proved in this paper only applies to the  $g = 1$  case; we expect that generalizing further to all  $g$  is accessible for our methods, at the cost of increasing the amount of notation and generalizing the results of [34].

**1.1. What does  $Z_n$  actually count?** Rather than describing  $Z_n$  directly, we will describe their pairings with a homology class. Thus, let us fix a coefficient ring  $\mathbf{k}$  in which  $2d$  is invertible, and a homology class  $h \in H_s(\Gamma, \mathbf{k})$ ; in place of interpreting  $Z_n$  directly, we will discuss how to interpret  $\langle Z_n, h \rangle$ .

Let us suppose (as we will in fact impose in the paper, and as Kudla–Millson also do) that  $\Gamma$  is torsion-free, which can always be arranged by passing to a finite index subgroup. The homology class  $h$  can be interpreted as a linear combination of  $q$ -cycles  $\sum a_i \Delta_i$  inside the locally symmetric space  $\Gamma \backslash D$ , where  $a_i \in \mathbf{k}$  and where  $D$  is the space of maximal positive definite subspaces of  $\mathbf{R}^m$  (with respect to  $Q$ ). Lift each  $q$ -cycle  $\Delta_i$  to a cycle in  $D$ ; as we vary over this cycle, the associated positive definite space “sweeps out” a conical region  $\mathcal{R}_i$  inside  $\mathbf{R}^m$ . Then – modulo certain transversality issues that will be discussed in detail in the main text –

$$\langle Z_n, h \rangle = \sum a_i \cdot (\text{number of integral } v \in \mathcal{R}_i \text{ with } Q(v) = n).$$

Note that, for each  $i$ , the count of  $v$  is finite.

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<sup>3</sup>A power series  $\sum a_m q^m$  with coefficients modulo a prime  $\geq 5$  is a modular form if there exist integers  $A_m$ , with  $A_m$  equal to  $a_m$  modulo  $p$ , and  $\sum A_m q^m$  a modular form in the usual sense. A more intrinsic definition is given by means of algebraic geometry.

<sup>4</sup>By “algebraic proof” we mean that it falls within the broad framework of the the algebraic theory of  $\theta$ -functions, developed in [11, 15, 43, 44, 47], and resting on the algebraic geometry of the moduli space of elliptic curves.

In words, then,  $\langle Z_n, h \rangle$  remains a count of solutions to  $Q = n$ , but inside a region that depends on  $h$ . It is true, but not obvious, that this count does not depend on the choice of cycle representing  $h$ . We might say that homology classes for  $\Gamma$  index reasonable ways of “regularizing” the count of solutions to  $Q = n$ .

Now, our main theorem guarantees that  $\sum Z_n q^n$  is modular. The modularity of such a series, whose coefficients lie in a cohomology group, is perhaps at first an unfamiliar notion (for definition, see [Section V](#)); but it implies, in particular, that the series after pairing with  $h$

$$(1.3) \quad \langle \theta, h \rangle := \sum_n \langle Z_n, h \rangle q^n$$

defines a modular form over  $\mathbf{k}$  (cf. [\[38, Theorem 2\]](#)). In fact, each  $Z_n$  is determined by its pairing with all classes  $h$  (for varying coefficient rings  $\mathbf{k}$ ), and similarly  $\theta$  is determined by the collection of all the scalar-valued forms  $\langle \theta, h \rangle$ . For the majority of the paper, we will prefer to work with these scalar-valued forms, and will switch to the  $Z_n$  formalism only at the very end.

### 1.2. Is there really an algebraic theory of $\theta$ -functions, even in the definite case?

In the course of our proof of [\(1.2\)](#), we reinterpret the Kudla–Millson theta series as an algebro-geometric object over the arithmetic moduli of elliptic curves. Perhaps the main difficulty we had in trying to understand this object already exists in the positive definite case – an unsatisfactory point concerning the foundation of the algebraic theory of  $\theta$ -functions (cf. [\[11, 15, 43, 44, 47\]](#)).

As we see it, a satisfactory algebraic theory of positive definite theta functions should interpret

*There is a modular form  $\theta$  of weight  $m/2$  with  $q$ -expansion  $\sum q^{Q(n)}$*

in terms of the algebraic geometry of the moduli of elliptic curves, along the lines of the following dictionary, at least when  $\mathbf{d} = 1$ :

- (I) modular form: – a section  $\theta$  of a line bundle on a moduli space  $Y$  of elliptic curves (equipped with a suitable auxiliary level structure).
- (II) weight  $m/2$ : – an explicit isomorphism of the *square* of that line bundle with the  $m$ th power  $\omega^m$  of the Hodge bundle  $\omega$ .<sup>5</sup>
- (III) with  $q$ -expansion  $\sum q^{Q(n)}$ : the isomorphism of (II) sends  $\theta^2$  to a global section of  $\omega^m$  with  $q$ -expansion  $(\sum q^{Q(n)})^2$ .

One wants an analogue of (I), (II), (III) to hold, too, in higher dimensions, replacing families of elliptic curves by abelian schemes; the story is entirely parallel but to maintain

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<sup>5</sup>We ignore here certain torsion effects that are related to the roots of unity in the transformation law of the  $\theta$ -function

simplicity and continuity of notation, we stick to the elliptic case, which is also the case relevant to our proof of (1.2).

There is a clear and agreed-upon definition for the line bundle of (I) which we will describe in detail in Section 2.2 and will sketch here; to simplify the discussion recall that we suppose that  $\mathbf{d} = 1$ . Form the  $m$ -fold self-product of the universal elliptic curve; it is an abelian scheme over  $Y$ . The choice of  $B$  determines a line bundle on this abelian scheme; pushing it down to  $Y$  we obtain a line bundle  $\Theta$ ; the *dual* of  $\Theta$  is the desired bundle of (I). The notion of “evaluation at the identity section” gives rise to the desired section  $\theta$  of this dual bundle.

(When  $\mathbf{d} > 1$ , the bundle of (I) becomes a vector bundle, and the isomorphism of (II) becomes a bundle map – no longer an isomorphism — depending on the choice of theta-characteristic.)

However, we have not been able to find any treatment in the literature that simultaneously provides (II) and (III), if the word “explicit” in point (II) is understood in the moduli-theoretic sense, i.e.

for each base scheme  $S$  equipped with the data that defines a map to the moduli space  $Y$  — i.e., we are given an elliptic curve  $\mathcal{E} \rightarrow S$  endowed with appropriate level structures — write down explicit formulas that convert the section  $s^2$  of the squared line bundle  $\Theta^{-2}$  on  $S$  to a section of  $\omega^m$  on  $S$ .

The closest work seems to be Moret-Bailly [43], which assures us that isomorphisms as in (II) and (III) exist over  $\mathbf{Z}$ . However, they are not explicitly given in the sense just described.

Indeed, to explicitly construct the isomorphism of (II) seems to be a subtle problem (it is related to the various roots of unity in the transformation law of classical  $\theta$  functions) and we will not solve it as stated. But we shall offer an ersatz solution that can be used for many of the same purposes as a genuine one. In place of constructing an isomorphism of the squared line bundle  $\Theta^{-2}$  with  $\omega^m$ , we will provide

$$(1.4) \quad \text{an explicit flat connection on their difference } \Theta^2 \otimes \omega^m.$$

The desired isomorphism is then a flat section for this flat connection, which we exhibit explicitly near the cusp.

In this proof strategy, the weight of the  $\theta$ -function is encoded by the power of  $\omega$  required to construct our flat connection, and the level of the  $\theta$ -function is determined by the level structure required to trivialize the monodromy of that flat connection.

**Warning:** In later text, we will often work with the twist  $\Theta_{\text{norm}} = \Theta \otimes \omega^{1/2}$ . Therefore, we will work interchangeably with the following data and their  $\Gamma$ -equivariant versions:

- a flat connection on  $\Theta \otimes \omega^{1/2}$ , and a canonical homomorphism  $\Theta \rightarrow \mathcal{O}$ .

- a flat connection on  $\Theta_{\text{norm}}$ , and a canonical homomorphism  $\Theta_{\text{norm}} \rightarrow \omega^{1/2}$ .

**1.3. A brief history of indefinite  $\theta$  functions.** There is a rather rich and long history of generalizations of the theory of  $\theta$  functions to indefinite quadratic forms. The most relevant to us is the work of Kudla–Millson, which we have already mentioned. For completeness, we will briefly survey a few other approaches to this question.

Hecke [30, 31] produced an analytic theory of theta series for indefinite rank-2 quadratic lattices. His construction applied to an ideal  $\Lambda$  in a real quadratic field  $K$ , and he constructed a weight-1 modular theta series for the corresponding signature- $(1, 1)$  quadratic form  $Q = N_{\mathbf{Q}}^K$ . The idea of Hecke’s construction is to sum  $q^{Q(\lambda)/2}$  for  $\lambda$  in a shift of  $\Lambda$ , but with three modifications that ensure well-definedness and modularity of the series:

- (1) Sum the quantity  $q^{Q(\lambda)/2}$  only over  $\lambda$  restricted to the positive-definite cone for  $Q$  in  $K$ .
- (2) Further restrict  $\lambda$  to a fundamental domain for the action of the subgroup of  $\mathcal{O}_K^\times$  preserving the relevant shifted ideal.
- (3) Sum the  $q^{Q(\lambda)/2}$  weighted by a sign depending on which connected component of the positive-definite cone  $\lambda$  lies in.

The three basic points of its construction are directly generalized by the construction of the Kudla–Millson classes  $Z_m$  whose definition was sketched in Section 1.1, and which will be defined in detail in Section V.

Siegel [53, 54] and Maass [41] later continued the program by constructing (in general, non-holomorphic) degree- $g$  Siegel modular forms whose coefficients, like the higher-degree version of Kudla–Millson mentioned previously, were intended to encode how many ways a given rank- $g$  quadratic form can be represented by a signature- $(r, s)$  quadratic form  $Q$ . Those constructions work by averaging a certain *nonholomorphic* theta kernel that depends on  $Q$ , and were later reinterpreted by Weil [57, 58] in representation-theoretic terms. The approach of Kudla and Millson [38] is structurally similar, but makes use of a theta kernel valued in differential forms, which is integrated over cycles in the locally symmetric space associated to  $\Gamma$ .

It is also worth mentioning the work of Zagier [62], who investigated the signature- $(n, 1)$  case in concrete terms, finding that the concrete interpretation of  $\langle Z_m, h \rangle$  in terms of summing over positive-definite cones could also apply in the case where  $h$  is not a singular chain on the locally symmetric space for  $\Gamma$ . In that case, Zagier found that one naturally obtains a nonholomorphic real-analytic modular form, and used analytic techniques of Vignéras [51] to prove the modularity property. That line of investigation was continued in a series of works by Funke [19], Kudla–Funke [20, 21], Funke–Millson [22], and many others.

In all of the work mentioned above, the tool being used to establish modularity is the Poisson summation formula, in one guise or another. It would be difficult to imagine how to use such techniques in the proof of our main result, because the  $q$ -expansions whose

modularity must be established are not necessarily valued in characteristic zero. That is why the method of this paper instead uses an algebro-geometric formulation of the heat equation.

Indeed, the two-variable theta functions were first written down by Fourier [16] for the purpose of solving the heat equation on a circle; in particular, they are uniquely characterized by the fact that they solve the heat equation, together with initial conditions. There are at least two proofs of modularity of theta functions in the literature that start from here. One is due to Couwenberg [13] and is closely related to the argument of this paper. The other appears in Weil's book [59]. This leads to a natural question: since both the heat equation and the oscillator representation can be used to prove that theta functions are modular, what is the relationship between the two concepts? The answer is that *the monodromy of the heat equation is the finite oscillator representation* (cf. [Theorem III<sub>b</sub>.6](#))

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## 2. A LOW-TECH INTRODUCTION TO THE IDEAS OF THE PROOF

In this section we will explain some of the ideas of the proof in a “low-tech” way, emphasizing differential equations and complex variables rather than flat connections and algebraic geometry. We shall start with the one-variable case (§2.1), and then proceed to the many-variable case but keeping  $B$  positive definite (§2.2) and then finally outline what is to be done when  $B$  is indefinite (§2.3).

A more detailed account of the entire argument, using the algebro-geometric language, will be given in §4.

**2.1. The one-variable case.** Let us restrict in this section to the simplest and most classical case, which is  $m = 1$ ,  $B = 1$ . The relevant theta function is then the classical series

$$\theta_3(q) := \sum_{n \in \mathbb{Z}} q^{n^2/2}.$$

We will prove its modularity, following Couwenberg [13]; the nontrivial statement to be established is

$$(2.1) \quad \theta_3(\tau') = (-i\tau)^{1/2} \theta_3(\tau).$$

where  $\tau' := \frac{-1}{\tau}$ . To ease our life, we will use both  $q$  and  $\tau$  as notation, and similarly  $q'$  and  $\tau'$ :

$$q = e^{2\pi i \tau}, \quad \tau' = -\frac{1}{\tau}, \quad q' = e^{2\pi i \tau'},$$

A key idea from the classical theory of theta-functions is that it is useful to study Jacobi's two-variable function  $\vartheta_3(z, q) = \sum q^{n^2/2} z^n$  even if one is primarily interested only in  $\theta_3(q) = \vartheta_3(1, q) = \sum q^{n^2/2}$ . The advantage of doing so is that  $\vartheta_3(z, q)$  can be characterized more easily than  $\theta_3(q)$ ; namely, up to an absolute constant, it is characterized by the following two properties:

(a)  $\vartheta_3(z, q)$  is quasiperiodic under  $z \leftarrow zq$ :

$$\vartheta_3(q^n z, q) = q^{-n^2/2} z^{-n} \vartheta_3(z, q).$$

(b)  $\vartheta_3(z, q)$  is annihilated by the differential “heat” operator

$$\heartsuit = D_q - \frac{1}{2} D_z^2$$

where  $D_q = q \frac{d}{dq}$  and similarly for  $D_z$ .

Note that (a) is *not enough*. It characterizes  $z \mapsto \vartheta_3(z, q)$  up to a constant that could depend on  $q$ . It is, rather, the heat equation (b) that forces independence of  $q$ , thus characterizing  $\vartheta_3(z, q)$  up to an overall constant. The idea of the proof of (2.1) is to verify that both sides satisfy these characterizing properties.

Just as for  $q, \tau$  it is convenient to work simultaneously with  $z$  and the corresponding additive coordinate  $w$ , as well as  $z', w'$ . Let us write

$$z = e^{2\pi i w}, \quad w' = -\frac{w}{\tau}, \quad z' = e^{2\pi i w'}.$$

We will make use of the fundamental solution

$$g_t(w) = t^{-1/2} \cdot e^{-\pi \frac{w^2}{t}}$$

of the heat equation. That is to say: regarding  $t$  as time and  $w$  as position, it describes the evolution of temperature if initialized at time 0 as a  $\delta$ -function.

We claim that, for any fixed  $q$ ,

$$\vartheta_3(q, z) \text{ and } g_{-i\tau}(w) \vartheta_3(q', z')$$

are proportional, considered as functions of  $z$  (or  $w$ ) alone. For one readily verifies that the latter function has exactly the same periodicity in  $w$  as the former. However, an examination of Fourier expansions shows that this periodicity condition characterizes  $\vartheta_3(q, z)$ , as a function of  $z = e^{2\pi i w}$ , up to a constant.

Therefore,  $\vartheta_3(z, q)$  and  $g_{-i\tau}(w) \cdot \vartheta_3(z', q')$  differ by a function  $F(\tau)$  that depends only on  $\tau$ . But the heat operator  $\heartsuit$  annihilates both  $\vartheta_3$  and the Gaussian  $g_{-i\tau}(w)$ . The first fact is easy to check: it kills the series  $\vartheta_3$  term-by-term; the second is the mathematical manifestation of the physical property of  $g$  noted above). From this we deduce that  $F(\tau)$  is independent of  $\tau$ ; that  $F \equiv 1$  follows by taking  $w = 0, \tau = i$  and computing. Finally, specializing to  $z = 1$ , we recover (2.1).  $\square$

Thus, the factor  $\sqrt{-\tau}$  in (2.1) has a “physical” interpretation for  $\tau$  purely imaginary: it is the inverse density of the heat kernel at the origin – a rough measure of the length scale over which diffusion has occurred – after a time  $-i\tau$  has elapsed.

We emphasize that the reader should consider the two properties that  $\heartsuit$  annihilates  $\vartheta_3$  and the Gaussian  $g_{-i\tau}(w)$  to be special cases of the essential properties of the flat connection that we will construct, and they will be revisited in more generality and heavier algebro-geometric language later for the purposes of the proof of our main theorem. In particular, interpreting (ii) algebraically in terms of the Gauss–Manin connection and the universal vector extension will be crucial to pinning down the lowest order term of the flat connection. That is carried out in [Section II<sub>b</sub>](#).

**2.2. Generalizing our prior discussion to arbitrary  $B$ .** Having already explained an analytic version of our argument in the one-variable case, we now turn to the more variables, but considering only  $B$  positive definite. We now seek to prove

$$\theta_B(q) = \sum_{\underline{n} \in \mathbf{Z}^m} q^{Q(\underline{n})}.$$

modular of weight  $m/2$ .

Again, this is a specialization of a two-variable function. The two-variable theta function  $\vartheta_3(z, q)$  considered in [Section 2](#) is classically generalized [\[46\]](#) by a function of  $z_1, \dots, z_m, q$ , given by

$$(2.2) \quad \vartheta_B(z, q) := \sum_{\underline{n} \in \mathbf{Z}^m} q^{Q(\underline{n})} \underline{z}^{B\underline{n}}.$$

Here we write  $\underline{z}^{\underline{n}} = z_1^{n_1} \cdots z_m^{n_m}$ . We recover  $\theta_B(q)$  by specializing at  $\underline{1} = (1, \dots, 1) \in (\mathbf{C}^\times)^m$ .

The properties (a) and (b) of Jacobi’s  $\vartheta_3$  stated in [Section 2](#) generalize in a straightforward manner to:

$$(a_B) \quad \vartheta_B(q^{n_1} z_1, \dots, q^{n_m} z_m) = q^{-Q(\underline{n})} \underline{z}^{-B\underline{n}} \vartheta_B(\underline{z}) \text{ for all } \underline{n} \in \mathbf{Z}^m.$$

$$(b_B) \quad \vartheta_B(\underline{z}, q) \text{ is annihilated by the explicit differential operator,}$$

$$\heartsuit^B = D_q - \frac{1}{2} \sum_{i,j} (B^{-1})_{ij} D_{z_i} D_{z_j}.$$

If  $\mathbf{d} = 1$ , then locally in  $q$ ,  $\vartheta_B$  is uniquely determined up to a scalar by  $(a_B)$  and  $(b_B)$ , and the proof in [Section 2](#) applies verbatim to show that  $\theta_B$  is modular of weight  $m/2$ ; the heat kernel on the real line is replaced by the heat kernel on  $\mathbf{R}^m$  which is considered to have the metric defined by  $B$ , and so the argument  $\frac{w^2}{\tau}$  of the exponential is now replaced by  $\frac{\sum_{i,j} B_{ij} w_i w_j}{\tau}$ ; the prefactor  $\tau^{-1/2}$  becomes  $\tau^{-m/2}$ .

In general, when  $\mathbf{d} > 1$ , the quasiperiodicity condition  $(a_B)$  does not uniquely determine  $\vartheta_B(-, q)$  up to scalar: there is a  $\mathbf{d}$ -dimensional vector space of functions of  $z_1, \dots, z_m$  satisfying  $(a_B)$ . This vector space has, for basis, the various functions analogous to (2.2), but allowing

to replace  $\mathbf{Z}^m$  by any coset of  $\mathbf{Z}^m$  within  $B^{-1}\mathbf{Z}^m$ ; we will later call the group of such cosets the *discriminant group*, noting that  $B^{-1}\mathbf{Z}^m$  is exactly the dual of  $\mathbf{Z}^m$  with respect to the bilinear form  $B$ . Such functions are called *theta functions with characteristic* — see for instance [46]. Locally in  $q$  they are a basis for the complex  $\mathbf{d}$ -dimensional vector space of functions satisfying  $(a_B)$  and  $(b_B)$ . This is the generalization of the fact that  $(b)$  uniquely determines  $\vartheta_3$  up to a scalar that is constant in  $q$ .

To execute the strategy of Section 2 now becomes more technically burdensome, as one must keep track of what pulling back under  $\tau \mapsto -1/\tau$  does to the local basis of solutions  $\vartheta_x$  to  $(a_B)$  and  $(b_B)$ . This will be encoded, in our geometric argument, by means of a phenomenon of *monodromy*.

**2.3. The case of  $B$  indefinite.** In order to handle the case of  $B$  indefinite, it will be necessary to reformulate our discussion geometrically.

First let us consider the case  $m = 1, B = 1$ . Then, for each  $\tau$ , the function  $z \mapsto \vartheta_3(z, q)$  is a global section of a line bundle  $\mathcal{L}_\tau = \mathcal{L}_q$  on the elliptic curve  $E_q := \mathbf{C}^\times / q^{\mathbf{Z}} = \mathbf{C} / (\mathbf{Z} + \tau\mathbf{Z})$  engineered so that its global sections are exactly the holomorphic functions of  $z \in \mathbf{C}^\times$  satisfying (a).<sup>6</sup> This line bundle can be verified to have degree one, and, therefore, has a one-dimensional space of sections, so that two nonvanishing sections are proportional.

For general  $m$ , and  $B$  positive definite, there is a similar picture, where we now replace the role of  $E_q$  by  $E_q^m$ ; and  $\vartheta_B$  now defines a section of a certain line bundle  $\mathcal{L}_q^B$  on  $E_q^m$  that is determined by  $B$ .

Now, this geometric interpretation makes perfect sense whether or not  $B$  is indefinite: the family  $(E_q^m, \mathcal{L}_q)$  does not depend, for its definition, on  $B$  being positive definite. However, the series (2.2) is divergent, and indeed  $\mathcal{L}_q^B$  no longer has global sections — but it *does* have higher cohomology, which from many points of view behaves very similarly to sections. Moreover, the group  $\Gamma$  of automorphisms of  $B$  acts on all  $(E_q^m, \mathcal{L}_q^B)$ , and all the cohomology of  $\mathcal{L}_q^B$  promotes uniquely to  $\Gamma$ -equivariant cohomology. What we do, then, is simply to redo the proof above, recast inside algebraic geometry, and using  $\Gamma$ -equivariant cohomology in place of sections.

In place of (2.2) we write down explicit formulas for higher cocycles; these formulas actually look very much like (2.2), with the one change that one sums not over all  $n$ , but only over a subset which renders the sum absolutely convergent. The recipe is similar to the one described briefly in Section 1.1 except one also has to add additional flexibility due to the  $z$ -variables. This is what we do in Section I and Section IV<sub>0</sub>.

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<sup>6</sup>More formally, then, the total space of  $\mathcal{L}_q$  is given by taking the trivial bundle over  $\mathbf{C}^\times$ , and quotienting by the action of  $\mathbf{Z}$ , where a generator sends  $(z, t)$  to  $(qz, q^{-1/2}z^{-1}t)$ .

What is less familiar is what to do about (b) or (b)<sub>B</sub>, that is, the use of the differential operator  $\mathfrak{D}$ . We will re-interpret this operator in terms of a certain flat connection (Section II<sub>b</sub>). This connection originates in work of Welters [60] and was further developed by others in various contexts; it is well-understood *except for its lowest order term*. Our contribution here will be to fix the lowest-order term algebraically and explicitly; the resulting connection will be flat and defined without any scalar ambiguity. This yields an algebraic reinterpretation of  $\mathfrak{D}$ , and it enables us to run the same proofs as in §2.1 or §2.2, just replacing “killed by  $\mathfrak{D}$ ” by “flat for the connection.”

### 3. NOTATIONS

We will set up notation used throughout the rest of this paper. Our main goal is to construct an abelian scheme  $\mathcal{A}$  equipped with line bundle  $\mathcal{L}^B$  over a certain modular curve  $Y^*$ .

**3.1. Quadratic form and rings.** As in the introduction,  $B$  is a nonsingular symmetric integer-entried  $m \times m$  matrix with absolute value of determinant  $\mathfrak{d} \neq 0$ . We can identify it with a bilinear form on  $\mathbf{Z}^m$ ,  $\langle \underline{x}, \underline{y} \rangle := \underline{x}^t B \underline{y}$ . We denote by  $Q(x) = \langle x, x \rangle / 2$  the associated quadratic form, and denote its signature by  $(r, s)$ , so that  $r + s$  equals  $m$ . Let  $\Gamma$  be a torsion-free subgroup<sup>7</sup> of integral unimodular automorphisms of  $B$  and we put

$$\tilde{\Gamma} = \Gamma \rtimes \mathbf{Z}^m,$$

the group of “rigid motions of the lattice  $\mathbf{Z}^m$  for the form  $B$ .”

We put  $\Delta = \mathfrak{d}$  or  $2\mathfrak{d}$  according to whether  $\mathfrak{d}$  is odd or even. (This is to get slightly sharper results about level structure in the odd case, but the reader will lose very little by just assuming that  $\Delta = 2\mathfrak{d}$  throughout, or by assuming that  $\mathfrak{d}$  is odd which we do in the main body of this paper until Section B). Let  $\zeta$  be a  $\Delta$ -th root of unity.

Put  $\mathbf{Z}' = \mathbf{Z}[\frac{1}{2\mathfrak{d}}, \zeta]$ . By the letter  $\mathbf{k}$  we will denote a  $\mathbf{Z}'$ -algebra, i.e. a ring on which  $2\mathfrak{d}$  is invertible and equipped with a fixed  $\Delta$ th root of unity. It will be used as a coefficient ring for homology.

**3.2. Moduli of elliptic curves with a rigidified symmetric line bundle.** A *rigidified* line bundle  $\mathcal{L}$  on a family of elliptic curves  $E \rightarrow S$  (defined following [36]) is one that comes with a trivialization at the identity, that is a choice of isomorphism  $r : e^* \mathcal{L} \cong \mathcal{O}_S$ , where  $e : S \rightarrow E$  is the identity section.

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<sup>7</sup>There should be no problem using the full automorphism group, at the cost of inverting the order of any torsion elements in the coefficient ring  $\mathbf{k}$ .

A *symmetric rigidified* line bundle is one equipped with a negation isomorphism  $\varphi : \mathcal{L} \cong [-1]^*\mathcal{L}$  which is “compatible” with the rigidification (that is:  $e^*\varphi$  is the trivial automorphism of  $\mathcal{O}_S$  after identifying  $e^*\mathcal{L}$  with  $\mathcal{O}_S$  using  $r$ ).

A symmetric rigidified line bundle gives rise to a quadratic form  $E_s[2] \rightarrow \pm 1$  on each geometric fiber  $E_s$  by looking at the fibers of  $\mathcal{L}$  there (see [44, §2] for the full detail). There are precisely two isomorphism classes of quadratic forms on a vector space of fixed dimension over  $\mathbf{F}_2$  (distinguished by the Arf invariant), and the isomorphism class of the form is locally constant on  $S$ .

We can speak of “the moduli stack of elliptic curves equipped with a rigidified symmetric line bundle of degree one” (cf. [11, 45, 49]). *Here, and elsewhere when we work with related moduli of elliptic curves, we will work over the base ring  $\mathbf{Z}'$ , unless noted otherwise.* It is a four-fold cover of the usual moduli-stack and splits into two components in the way just described. Concretely, this division corresponds to whether a nonzero section of the line bundle vanishes at the origin of the elliptic curve, or at a nontrivial 2-torsion point.

The latter component is identified in an evident way with  $Y_0(2)$ , the moduli space of elliptic curves with a nontrivial 2-torsion point  $Q$ . Namely, given any elliptic curve  $E$  with two-torsion point  $Q$ , the line bundle  $\mathcal{O}_E(Q)$  can be given a rigidification and symmetric structure in an obvious way, and these fit together to give a rigidified symmetric line bundle on the universal elliptic curve over  $Y_0(2)$ . For later use, let us describe the geometry near the cusps, of which there are two:

- the “cusp at infinity”, which in a local coordinate  $q$  is described by the family of elliptic curves  $E_q = \mathbf{C}^\times/q^{\mathbf{Z}}$  with two-torsion point  $Q = -1$  degenerating as  $q \rightarrow 0$ ; the associated line bundle  $\mathcal{L}_q$  is described by means of the transformation law of

$$\vartheta_2 = \sum_{n \in \frac{1}{2} + \mathbf{Z}} q^{n^2/2} z^n$$

under  $z \mapsto qz$  with rigidification arising from evaluating at  $z = 1$ . This really does describe the fiber at  $q$  of  $\mathcal{L}$ , namely  $\mathcal{L}_q = \mathcal{O}_{E_q}(Q)$ , since  $\vartheta_2(-, q)$  has a zero at  $z = Q = -1$  and is regular everywhere else on  $E_q$ .

- the “cusp at zero”, which in a local coordinate  $\sqrt{q}$ <sup>8</sup> is described by the family of elliptic curves  $\mathbf{C}^\times/q^{\mathbf{Z}}$  with two-torsion point  $Q = -q^{1/2}$ ; the associated line bundle is described by the transformation law of

$$\vartheta_3 = \sum_{n \in \mathbf{Z}} q^{n^2/2} z^n$$

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<sup>8</sup>Algebro-geometrically, it is preferable to write the local coordinate as  $u = q^{1/2}$ , and describe the family of curves as  $\mathbf{C}^\times/u^2\mathbf{Z}$  with torsion point  $u$ . However, this makes the comparison with the entire literature wearying, so, on the grounds that what is good enough for Jacobi is also good enough for us, we will allow ourselves to use  $\sqrt{q}$  as the coordinate.

under  $z \mapsto qz$  with rigidification arising from evaluating at  $z = 1$ . This really does describe the fiber at  $q$  of  $\mathcal{L}$ , namely  $\mathcal{L}_q = \mathcal{O}_{E_q}(Q)$ , since  $\vartheta_3(-, q)$  has a zero at  $z = Q = -q^{1/2}$  and is regular everywhere else on  $E_q$ .

**3.3. The modular curve  $Y^*$ .** We shall now introduce a degree 8 cover  $Y^*$  of  $Y_0(2)$ , again understood to be a  $\mathbf{Z}'$ -scheme. It is closely related to Legendre's normal form for elliptic integrals. The level structure corresponding to  $Y^*$  is the index 8 subgroup of matrices  $\Gamma^*$  whose reduction modulo 4 is either trivial or equal to the reduction of  $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ :

$$(3.1) \quad \Gamma^* = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : a \equiv d \equiv 1, b \equiv 2c \equiv 0 \pmod{4} \right\}$$

$Y^*$  has two key (if technical) advantages over  $Y_0(2)$  for us:

- $Y^*$  is a scheme and not a stack and
- pulling back to  $Y^*$  will annihilate the monodromy homomorphism associated to our flat connection.

In principle, we could carefully use  $Y_0(2)$  in many places, and only switch to  $Y^*$  when we need to – but why bother?

The corresponding modular curve  $Y^*$  parameterizes

(3.2) elliptic curves with points  $\tilde{P}, Q$  such that  $2\tilde{P}, Q$  form a basis for 2-torsion,

or “a 2-torsion basis  $P, Q$  and a choice of  $\frac{P}{2}$ .” To identify its complex points with the quotient of the upper-half plane by  $\Gamma^*$ , we associate to a point  $\tau$  in the upper-half plane the elliptic curve  $\mathbb{C}/\langle 1, \tau \rangle$ , equipped with  $\tilde{P} = \frac{\tau}{4}, Q = \frac{1}{2}$ ; then  $w \mapsto w(c\tau + d)$  identifies the corresponding datum for  $\frac{a\tau+b}{c\tau+d}$  with this one.

**3.4. The cusps of  $Y^*$ .** We will summarize, for later use, facts about the cusps of  $Y^*$ .

The covering map  $Y^* \rightarrow Y_0(2)$  is the one given by forgetting  $\tilde{P}$ . Correspondingly,  $Y^*$  has four cusps; one of them lies above the infinite cusp of  $Y_0(2)$ , corresponding in the coordinates of §3.2 to taking  $\tilde{P} = q^{1/4}$ , with  $q^{1/4}$  as local coordinate. This one has width 4 and ramification index 4 above the cusp  $\infty$  of  $Y_0(2)$ . There aren't any others above  $\infty$ , because the loop  $\tau \mapsto \tau + 1$  takes  $q^{1/4}$  to  $iq^{1/4}$ , then  $-q^{1/4}$ , then  $-iq^{1/4}$  — so all four possibilities for a square-root of  $q^{1/2}$  or  $-q^{1/2}$  are accounted for by this family. The other possibilities  $\pm q^{3/4}$  are also accounted for by this, under the automorphism  $z \mapsto z^{-1}$  (which of course preserves the 2-torsion point  $Q$ ).

The remaining three cusps of  $Y^*$  lie above the zero cusp of  $Y_0(2)$ , corresponding to the choices  $\tilde{P} = i, iq^{1/2}, iq^{1/4}$ . The obvious choice  $\tilde{P} = -i$  doesn't give a new isomorphism class of marked elliptic curves since the isomorphism  $z \mapsto z^{-1}$  takes  $i$  to  $-i$ . The two choices  $\tilde{P} = \pm i$  correspond to the choice of  $P = -1$ , and both have no ramification over the cusp 0 — they

are of width 2. The choice  $\tilde{P} = iq^{1/4}$  accounts for both possible square roots of the other choice  $P = -q^{1/2}$ , since the loop  $\tau \mapsto \tau + 2$  near the cusp 0 of  $Y_0(2)$  takes  $iq^{1/4}$  to  $-iq^{1/4}$ .

We summarize the cusp situation in this table, which we will use in the proofs later on:

TABLE 1. Cusps of  $Y^*$ 

| Name             | $\mathbf{P}_Q^1$ representative | width | $Q$       | $2\tilde{P}$ | $\tilde{P}$ | Image in $X_0(2)$ |
|------------------|---------------------------------|-------|-----------|--------------|-------------|-------------------|
| $\infty^*$       | $\infty$                        | 4     | $-1$      | $q^{1/2}$    | $q^{1/4}$   | $\infty$          |
| $0_i^*$          | 0                               | 2     | $q^{1/2}$ | $-1$         | $i$         | 0                 |
| $0_{iq^{1/2}}^*$ | 2                               | 2     | $q^{1/2}$ | $-1$         | $iq^{1/2}$  | 0                 |
| $0_{iq^{1/4}}^*$ | 1                               | 4     | $q^{1/2}$ | $-q^{1/2}$   | $iq^{1/4}$  | 0                 |

**3.5. Curves associated to  $Y^*$ .** We will later denote by  $Y^*(\Delta)$  the modular curve over  $\mathbf{Z}'$  which parameterizes an elliptic curve with  $Y^*$ -level structure, plus choices of  $\Delta$ -torsion points  $P, Q$  with  $\langle P, Q \rangle = \zeta$ .

**3.6. Tensoring a line bundle with a bilinear form.** An abelian variety with symmetric rigidified line bundle  $(A, \mathcal{L}_A)$  can be “tensoried with” the symmetric bilinear form  $B$  on  $\mathbf{Z}^n$  to give a line bundle  $\mathcal{L} \otimes B$  on  $A \otimes \mathbf{Z}^n = A^n$ , well defined up to isomorphism. The association  $B \mapsto \mathcal{L} \otimes B$  (or  $\mathcal{L}^B$  for short) is characterized up to isomorphism by the following properties

- (I) it is additive, i.e.  $\mathcal{L}^{B+B'}$  is the sum of  $\mathcal{L}^B$  and  $\mathcal{L}^{B'}$ ;
- (II) when  $B$  is a rank one matrix, so that  $B_{ij} = \ell_i \ell_j$  for some  $\ell_1, \dots, \ell_n$ , then  $\mathcal{L}^B$  is obtained by pulling back  $\mathcal{L}$  along

$$A^n \longrightarrow A, (a_1, \dots, a_n) \mapsto \sum \ell_i a_i.$$

For example, if the matrix of  $B$  is diagonal with  $r$  entries  $+1$  and  $s$  entries equal to  $-1$ , then the line bundle  $(A, \mathcal{L}) \otimes B$  is given by

$$(3.3) \quad (A^m, \lambda_1 \dots \lambda_r \lambda_{r+1}^{-1} \dots \lambda_s^{-1}),$$

where  $\lambda_i = \pi_i^* \mathcal{L}$  is the pull-back of  $\mathcal{L}$  via the  $i$ th projection map  $\pi_i : A^m \rightarrow A$ .

The resulting construction is equivariant under the automorphisms  $\Gamma$  of the bilinear form  $B$ ; that is for  $\gamma \in \Gamma$ , we have a unique rigidified isomorphism  $[\gamma] : \gamma^* \mathcal{L}_B \simeq \mathcal{L}_{\gamma B}$ , coherently in  $\gamma$ . Indeed, the two line bundles here are abstractly isomorphic by the (evident) functoriality of the construction above, and then this abstract isomorphism is realized by a unique rigidified isomorphism. The rigidification guarantees the coherence  $[\gamma_1][\gamma_2] = [\gamma_1 \gamma_2]$ .

**Remark.** A different description of  $\mathcal{L}_B$  comes from the theorem of the cube (cf. *biextensions*, [10]). Put  $\langle x, y \rangle = \mathcal{L}_{x+y} \mathcal{L}_x^{-1} \mathcal{L}_y^{-1}$ , which we think of as a function from  $x, y$  to rank one modules over the relevant base ring. The theorem of the cube asserts that it is “bilinear” in  $x, y$  – where one replaces the equalities that usually define bilinearity by preferred isomorphisms. Moreover,

$\langle x, x \rangle$  admits a preferred square root, namely  $\mathcal{L}_x$  itself. The fiber of  $\mathcal{L}_B$  at  $(x_1, \dots, x_n)$  is then given by tensoring together  $\langle x_i, x_j \rangle^{b_{ij}/2}$  (using the preferred square root to handle  $i = j$ ).

Unwinding the paragraph above gives an explicit formula for  $(A, \mathcal{L}) \otimes B$ , namely,

$$\left( \bigotimes_{i=1}^m \pi_i^* \mathcal{L}^{b_{ii}} \right) \otimes \left( \bigotimes_{1 \leq i < j \leq m} (m_{ij}^* \mathcal{L} \otimes p_i^* \mathcal{L}^{-1} \otimes p_j^* \mathcal{L}^{-1})^{b_{ij}} \right),$$

and the reader who prefers this viewpoint is welcome to verify any of our assertions using this instead.

**3.7. The abelian scheme  $\mathcal{A}$  and line bundle  $\mathcal{L}^B$  associated to the bilinear form  $B$ .** The construction of [Section 3.6](#) applies in families. Therefore, beginning with the line bundle  $\mathcal{L}$  from [Section 3.2](#), which lives on the universal elliptic curve  $\mathcal{E}$  over  $Y^*$ , we obtain by tensoring with the bilinear form  $B$

- an abelian scheme  $\pi : \mathcal{A} \longrightarrow Y^*$ , and
- a line bundle  $\mathcal{L}^B$  on  $\mathcal{A}$

by means of the rule

$$(\mathcal{A}, \mathcal{L}^B) = (\mathcal{E} \otimes \mathbf{Z}^m, \mathcal{L} \otimes B).$$

Much of this section works after base change from  $Y^*$ . So let  $Y$  be either  $Y^*$  or  $Y^*(\Delta)$ , and overload the meaning of  $\pi$ ,  $\mathcal{L}^B$ ,  $\mathcal{E}$  to mean the same thing constructed over  $Y$  (there is no difference between constructing it over  $Y$  and pulling it back from  $Y^*$ ). Later we will change the base to  $\mathbf{Z}'$  and replace  $Y$  with  $Y_{\mathbf{Z}'}$ .

We now follow [\[52\]](#) in setting up some basic properties of  $\mathcal{L}^B$ .

Recall that a line bundle  $L$  on an abelian variety induces a map  $\phi_L$  from that abelian variety to its dual, whereby a point  $x$  is sent to “ $x$ -differenced” line bundle, cf. [\[?Mumford\]](#), and the line bundle is called *nondegenerate* if this is an isogeny (equivalently, has a finite kernel). Then:

**Lemma 3.1.** *The following hold on  $Y$ .*

- (i) *The homomorphism  $\phi_{\mathcal{L}} : \mathcal{E} \rightarrow \mathcal{E}^\vee$  is an isomorphism; in particular,  $\mathcal{L}$  is nondegenerate.*
- (ii) *The post-composition of the map  $\phi_{\mathcal{L}^B} : \mathcal{A} \rightarrow \mathcal{A}^\vee$  with the inverse of the similar map  $\phi_{\mathcal{L}}^{\times m} : \mathcal{A} \rightarrow \mathcal{A}^\vee$  (that is, obtained by taking the product of  $m$  copies of the map from (i)) is exactly the same as the  $Y$ -scheme map*

$$B : \mathcal{E}^m \rightarrow \mathcal{E}^m$$

*given by  $(z_1, \dots, z_m) \mapsto (\prod z_j^{b_{ij}})_{i=1}^m$ . In particular,  $\mathcal{L}^B$  is also nondegenerate.*

Note that  $\Gamma$  acts on  $(\mathcal{A}, \mathcal{L}^B)$ , covering the trivial action on  $Y$ .<sup>9</sup> We introduce the *theta bundle* as the relative  $s$ -th cohomology of this line bundle:

$$\text{bundle } \Theta \text{ of } \vartheta\text{-functions} = R^s \pi_* \mathcal{L}^B.$$

**Lemma 3.2.**  $\Theta$  is a vector bundle of rank  $d$ .

**3.8. Square roots of  $\omega$ .** The bundle  $\omega := e^* \Omega_{\mathcal{E}/Y}^1$  denotes the usual Hodge bundle on a modular curve, so that sections of  $\omega^k$  correspond to weight  $k$  modular forms.

In general, to speak of  $\omega^k$  for  $k$  half-integral, one would like to have a square root of  $\omega$ . Nonetheless, we will frequently use the notation  $\omega^k$  for  $k$  half-integral even without having fixed such a choice of square root, and we want to discuss our conventions around this.

In [Section II<sub>a</sub>](#), we will define a notion of “connection on  $\Theta \otimes \omega^{m/2}$ ” without defining  $\omega^{1/2}$  itself even if  $m$  is odd. Most of the time, when we speak of half-integral powers of  $\omega$ , it will only be used through this notion. Thus, for example, in the next paragraph we shall define a “vector bundle” by twisting  $\Theta$  by  $\omega^{m/2}$ ; but all we will really be using is the concept of what it means to have a connection on that twist.

However, our *final* statements should be taken to mean, when  $m$  is odd, that we have genuinely constructed a square root of  $\omega^m$  — this will arise in the course of the proofs; see [Section III<sub>b</sub>.4](#).

**3.9. Other notations associated to  $\Theta$ -bundles.** These will be defined at the relevant points in the text, but we briefly discuss them here as a convenient reference.

Firstly, using the convention of [§3.8](#), we define

$$\text{normalized bundle } \Theta_{\text{norm}} \text{ of } \theta\text{-functions} = R^s \pi_* \mathcal{L}^B \otimes \omega^{m/2} = \Theta \otimes \omega^{m/2}$$

another vector bundle of rank  $d$  over  $Y$ .

Next, we already noted that  $\Gamma$  acts on the line bundle  $\mathcal{L}^B$ , covering the trivial action on the base. This allows one to define a version  $\Theta^\Gamma$  where we do the pushforward  $\Gamma$ -equivariantly; see [Section IV](#) for details.

All these bundles come with certain canonical sections of importance; in particular the notation  $\underline{\vartheta}$  will denote a certain basis of sections of  $\Theta$  which exists after pullback to a cusp, to be defined in [Section I](#).

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<sup>9</sup>It also follows from the Remark of the previous section that, if  $B$  is even, this bundle  $\Theta$  descends to the level 1 modular curve. From the current point of view, this accounts in part for the distinguished role of even lattices in the theory of  $\theta$ -functions. However, we will not make any explicit use of this.

## 4. AN OVERVIEW OF OUR ARGUMENT

The precise statement, slightly more refined than (1.2), to be proved is this:

- (4.1) there is a modular form of level  $\Gamma^*(\Delta)$  with coefficients in  $H^s(\Gamma, \mathbf{Z}[\frac{1}{2d}])$  whose  $q$ -expansion at zero is  $\sum Z_m q^m$ .

We have given a classical account of the argument in §2. We now give a more detailed account, adopting from the start algebro-geometric language, which is intended also as an overview of the rest of the paper. We will first give a short version of the argument (§4.1), then a longer version (§4.2), divided into five steps (I)–(V) which parallel the division of the rest of the paper.

Note that our argument will give a good deal of detail about the level structure implicit in the above statement – the form above arises from a vector-valued form at full level  $\Gamma^*(\Delta)$  that transforms in the finite oscillator representation of §III<sub>0</sub>. However, in the present discussion, we will give only a concise statement for simplicity.

Similarly, throughout the paper, we do not aim for the sharpest statements about level structures at 2. It will be clear from the presentation and the methods that the level structure can frequently be reduced at the prime 2. But to maintain simplicity of statement, we will not do so, and rather present results that work in all cases.

**4.1. Short version of the argument.** Recall from Section 3.6 the rank  $d$  vector bundle  $\Theta$  on the modular curve  $Y^*(\Delta)$ . We will

- (I) Construct sections  $\underline{\vartheta} = (\vartheta_x)_{x \in B^{-1}\mathbf{Z}^m/\mathbf{Z}^m}$  for  $\Theta$  over a cusp neighbourhood.
- (II) Construct a flat connection on a twist  $\Theta_{\text{norm}} := \Theta \otimes \omega^{m/2}$  of  $\Theta$ .
- (III) Show that the sections  $\underline{\vartheta} \cdot (\frac{dq}{q})^{m/2}$  are flat and so extend over the whole modular curve  $Y^*(\Delta)$ .

In the definite case  $s = 0$  we could now proceed as follows;

- Apply an evaluation map  $\Theta \rightarrow \mathcal{O}_{Y^*(\Delta)}$  to  $\underline{\vartheta} \cdot (\frac{dq}{q})^{m/2}$  to obtain a collection of modular forms of weight  $m/2$ , which are  $\theta$ -functions with characteristic attached to  $Q$ .

This doesn't work in the indefinite case. The problem is that the evaluation map from  $\Theta$  arises eventually from using the rigidification of  $\mathcal{L}$  to evaluate sections. But when we try to evaluate higher cohomology classes at a point, the result is zero: a point has no higher cohomology. *However*, we can introduce a variant  $\Theta^\Gamma$  of  $\Theta$  – still a vector bundle on the modular curve – by introducing  $\Gamma$ -equivariance; and this variant comes with

$$(4.2) \quad \Theta^\Gamma \rightarrow H^s(\Gamma, \mathbf{Z}') \otimes \mathcal{O}_{Y^*(\Delta)}$$

because a point *does* have nonzero  $\Gamma$ -equivariant higher cohomology. Accordingly, we proceed in the general case as follows:

- (IV) We promote  $\vartheta_0$  to a section  $\tilde{\vartheta}_0$  of the equivariant refinement  $\Theta^\Gamma$ .<sup>10</sup>
- (V) We prove that the evaluation map (4.2) carries  $\tilde{\theta}_0 \cdot (\frac{dq}{q})^{m/2}$  to  $\sum Z_m q^m$ .

This concludes the proof: we have shown that the  $q$ -series  $\sum Z_n q^n$  is a modular form of weight  $n/2$  valued in the cohomology group  $H^s(\Gamma, \mathbf{Z}')$ .

Since the inclusion of  $\mathbf{Z}[\frac{1}{2d}]$  into  $\mathbf{Z}'$  induces an injection on cohomology, and the  $Z_n$  lie in the image of this inclusion, this means that  $\sum Z_n q^n$  is already modular when considered to have values in  $H^s(\Gamma, \mathbf{Z}[\frac{1}{2d}])$ , without needing the extra roots of unity in the statement.

**4.2. Longer version of the argument.** We now spell out steps (I)–(V) in more detail.

- (I) Constructions of sections, in Section I. We construct an explicit basis of sections  $\underline{\vartheta} = (\vartheta_x)_{x \in \kappa_B}$  of  $\Theta$  in a formal neighbourhood of certain cusps of  $Y^*(\Delta)$  of width  $w$ , where  $x$  is indexed by the discriminant group  $\kappa_B$ , namely the order- $d$  kernel of  $B$  acting on the  $m$ -torus  $(\mathbf{R}/\mathbf{Z})^m$ .

For this we need to use toric uniformization. Near the cusp, the universal abelian variety can be exhibited as a quotient of an  $m$ -dimensional torus by  $\mathbf{Z}^m$ , and correspondingly we can realize the sections of  $\Theta$  in terms of the invariants (or cohomology) of  $\mathbf{Z}^m$  acting on sections of a suitable line bundle on the  $m$ -dimensional torus:

$$(4.3) \quad \text{sections of } \Theta \text{ over cusp} = \text{sth cohomology of } \mathbf{Z}^m \text{ on } \mathcal{O}^{\text{an}}$$

where  $\mathcal{O}^{\text{an}}$  is a completed version of the ring of functions on  $\mathbb{G}_m^m$  over  $\mathbf{Z}'((q^{1/w}))$ . This result is due to Faltings–Chai [15] for  $s = 0$  and to the first-named author of this paper for general  $s$  [34]; see Section IV.1 for further discussion.

The desired sections  $\vartheta_x$  are produced by writing down *explicit*  $q$ -cocycles that look very much like the usual formulas in the classical theory of  $\theta$ -functions. The only difference, however, is that when one usually sums over  $\mathbf{Z}^m$ , here we sum over *regions* in  $\mathbf{Z}^m$  cut out by linear inequalities determined by the input to the cocycle.

- (II) Flat connection, carried out in Section II<sub>a</sub> and Section II<sub>b</sub>:

We will construct a flat connection on the normalized  $\Theta$ -bundle  $\Theta_{\text{norm}} = \Theta \otimes \omega^{m/2}$ . The construction of a *projectively* flat connection here is standard, and will be reviewed in some detail in language more amenable to our approach. A crucial point for us is to remove the “projectively”, while retaining a connection that can be described explicitly in purely algebraic terms. For this we make essential use of the Gauss–Manin connection. We give a fairly leisurely exposition: Section II<sub>a</sub> is devoted to a review of the abstract question of how to put connections on coherent cohomology of line bundles, and Section II<sub>b</sub> specializes to our case of interest.

- (III)  $\Theta_{\text{norm}}$  is trivial at level  $\Gamma^*(\Delta)$  (as bundle with flat connection), and the  $\underline{\vartheta} \cdot \left(\frac{dq}{q}\right)^{m/2}$  extend to a basis of global flat sections (§III<sub>b</sub>):

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<sup>10</sup>The key point here is that any  $\Gamma$ -invariant section promotes. The action of  $\Gamma$  on the span of  $\underline{\vartheta}$  factors through a finite quotient, and with our choices  $\vartheta_0$  is actually  $\Gamma$ -invariant. We could choose any other  $\vartheta_x$  at the cost of shrinking  $\Gamma$  sometimes.

The key point here is that we can check triviality of  $\Theta_{\text{norm}}$  (as bundle with flat connection) by monodromy arguments; and once we know this, we can check that the locally defined sections  $\underline{\vartheta}$  extend simply because they are flat.

Although the desired assertion is at level  $\Gamma^*(\Delta)$ , it is easier to analyze at lower level: we reduce it to similar questions about a *line* bundle  $\Theta_{\text{line}}$  on the four-punctured projective line  $Y^*$ , where one can check that the global monodromy is trivial merely by checking the monodromy at all the cusps.

To carry out this reduction of level, we use the fact that  $\Theta$  and  $\Theta_{\text{norm}}$  are in fact defined on  $Y^*$ , and canonically admits an action of the Heisenberg-type group scheme  $\mathcal{G}$  to be defined in [Section III<sub>0</sub>](#). We will compare them with another “model” bundle  $\mathbf{M}$  that is also defined on  $Y^*$  and also admits an action of  $\mathcal{G}$ , and constructed via the theory of the finite Weil representation. Then  $\Theta_{\text{line}} = \text{Hom}_{\mathcal{G}}(\mathbf{M}, \Theta_{\text{norm}})$ , is the desired *line* bundle on  $Y^*$ .

(IV) Adding  $\Gamma$ -equivariance ([Section IV](#)):

The fiber of  $\Theta$  at a given point is given by higher cohomology of the line bundle  $\mathcal{L}^B$  on a certain abelian variety. However,  $\Gamma$  acts on the abelian variety and line bundle simultaneously, and we can form a new bundle  $\Theta^\Gamma$  whose fiber is constructed out of  $\Gamma$ -equivariant cohomology.

By (III) we know that  $\vartheta_0 \left(\frac{dz}{z}\right)^{m/2}$  extends to a global section of  $\Theta_{\text{norm}}$ , flat for the connection constructed in (II). We shall promote it to a section  $\tilde{\vartheta}_0$  of the equivariant bundle:

$$(4.4) \quad \tilde{\vartheta}_0 \left(\frac{dz}{z}\right)^{m/2} \in \text{global sections of } \Theta^\Gamma \otimes \omega^{m/2}$$

This is done by first promoting  $\vartheta_0$  near the cusp, using arguments in group cohomology, and then showing the resulting section extends. To write formulas for  $\tilde{\vartheta}_0$  near the cusp, we undertake a study of representation numbers  $r_Q(n)$  for indefinite forms which may be of independent interest ([Section IV<sub>0</sub>](#)).

(V) We conclude in [Section V](#) by computing the  $q$ -expansion coefficients of  $\tilde{\vartheta}_0$ . Let  $\mathcal{O}^{\text{an}}$  be as in (4.3); evaluation of elements of  $\mathcal{O}^{\text{an}}$  at the identity induces

$$(4.5) \quad H^s(\tilde{\Gamma}, \mathcal{O}^{\text{an}}) \longrightarrow H^s(\Gamma) \otimes \mathbf{Z}'((q^{1/w})).$$

We prove that this maps  $\tilde{\vartheta}_0$  to  $\sum Z_m q^m$ , concluding the argument. To carry out this proof, we have to interface our definitions with those of Kudla and Millson.

## I. Explicit Fourier expansions of theta cocycles

### I. CONSTRUCTION OF THE $\theta$ -SECTION NEAR THE CUSP

We will complete step (I) from our introduction [Section 4](#): we construct sections  $\underline{\vartheta} = (\vartheta_x)_{x \in \kappa_B}$  for  $\Theta := R^s \pi_* \mathcal{L}^B$  over a cusp neighbourhood. To construct such objects on a punctured neighborhood of a width- $w$  cusp of a modular curve (whose exact level will

be chosen later), one needs to understand explicitly the universal elliptic curve over such neighborhoods, and then the restriction of the line bundle  $\mathcal{L}^B$  to it. In this section, we first carry this out in the analytic setting, before finally actually constructing the sections  $\vartheta_x$ . Then, we show how the same formulas constructed in the analytic context carry over to the arithmetic moduli, where scheme-theoretic punctured neighborhoods of cusps are given by  $\text{Spec}(\mathbf{Z}'((q^{1/w})))$ -points.

**1.1. The line bundle  $\mathcal{L}$  near the cusp.** We start with the complex-analytic version.

Consider the family of elliptic curves  $E_q := \mathbf{C}^\times / q^{\mathbf{Z}}$  over the *analytic* base  $S = \{q^{1/w} \in \mathbf{C} : 0 < |q^{1/w}| < 1\}$ . This is precisely the universal elliptic curve over any complex-analytic modular curve, when restricted to a neighborhood of any cusp of width  $w$ . There are many possible line bundles on  $E_q$ , but we will consider the three isomorphism classes of symmetric degree-1 line bundles on it: they are given using divisors by  $\mathcal{O}([P])$  for the three nontrivial 2-torsion points  $P = -1, q^{1/2}, -q^{1/2}$ . The latter two choices only globalize over the neighborhood  $S$  if  $w$  is even, so we only consider them in that case.

There is a fourth possible symmetric line bundle on  $E_q$ , namely  $\mathcal{O}([e])$ . It could be included in our theory, but it is not necessary to do so because the corresponding quadratic form on  $E_q[2]$  (see [Section 3.2](#)) has different Arf invariant from the other three, and therefore does not appear in the connected component of the moduli of symmetric rigidified line bundles that we will consider.

We will construct  $(\vartheta_x)_{x \in \kappa_B}$  as *Fourier expansions* in higher cohomology, which will require uniformizing the three line bundles  $\mathcal{O}([P])$  to understand their sections in terms of explicit quasiperiodic functions that admit Fourier expansions. We execute this now.

**Definition I.1.** Let  $\underline{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$  where  $\epsilon_1, \epsilon_2 \in \mathbf{Z}/2$ . Classically, the choices of  $\underline{\epsilon}$  are the “half theta characteristics<sup>11</sup>” that parametrize symmetric line bundles on  $E_q$ . The line bundle  $\mathcal{L}_{\underline{\epsilon}}$  is defined by descending a specific  $q^{\mathbf{Z}} = \langle T \rangle$ -action (depending on  $\underline{\epsilon}$ ) on the holomorphic structure sheaf  $\mathcal{O}_{\mathbf{C}^\times}$  from  $\mathbf{C}^\times$  to the quotient  $E_q = \mathbf{C}^\times / q^{\mathbf{Z}}$ . That action is given by

$$(I.1) \quad (T^y \cdot f)(z) = (-1)^{\epsilon_2 y} q^{(y^2 - \epsilon_1 y)/2} z^y f(qz) = ((-1)^{\epsilon_2} q^{-\epsilon_1/2})^y q^{y^2/2} z^y f(qz).$$

Said differently, local sections of  $\mathcal{L}_{\underline{\epsilon}}$  over an open subset of  $\mathbf{C}^\times / q^{\mathbf{Z}}$  correspond to holomorphic functions on corresponding open subset of  $\mathbf{C}^\times$  that are invariant under  $T$ .

By explicit calculation with cocycles for  $H^1(q^{\mathbf{Z}}, \mathcal{O}(\mathbf{C}^\times)^\times)$ , one sees easily that the four  $\mathcal{L}_{\underline{\epsilon}}$  exhaust all symmetric line bundles on  $E_q$ .

<sup>11</sup>There are different conventions based on whether one wants the numbers  $\epsilon_1$  and  $\epsilon_2$  to label theta functions or 2-torsion points. Since this paper deals explicitly with theta functions, we choose the former.

If  $w$  is odd, then  $\mathcal{L}_\epsilon$  is only defined globally on  $S$  when  $\epsilon_1 = 1$ , since otherwise defining it requires a globally-defined square root of  $q$ .

The bundle  $\mathcal{L}_\epsilon$  has a rigidification  $\mathcal{L}|_1 \cong \mathbf{C}|_1$  given by evaluating quasiperiodic functions at  $z = 1$  (which is also defined in a family over  $q$  in the punctured disk whenever the definition of  $\mathcal{L}_\epsilon$  globalizes).

There is also a symmetric structure on  $\mathcal{L}_\epsilon$  given by inverting  $z$  and multiplying by  $z^{-\epsilon_1}$ , as  $f(q^y z) = (-1)^{\epsilon_2 y} q^{-(y^2 - \epsilon_1 y)/2} z^{-y} f(z)$  implies that  $z \mapsto z^{\epsilon_1} f(z^{-1})$  satisfies exactly the same quasiperiodicity law:

$$(q^y z)^{\epsilon_1} f((q^y z)^{-1}) = q^{\epsilon_1 y} z^{-\epsilon_1} (-1)^{\epsilon_2 y} q^{\frac{-y^2 - \epsilon_1 y}{2}} z^{-y} f(z^{-1}) = (-1)^{\epsilon_2 y} q^{\frac{-y^2 - \epsilon_1 y}{2}} z^y z^{\epsilon_1} f(z^{-1})$$

thus providing the symmetric structure  $[-1]^* \mathcal{L} \rightarrow \mathcal{L}$  given by sending a quasiperiodic function defining a section of  $[-1]^* \mathcal{L}$  to the same function multiplied by  $z^{\epsilon_1}$ .

The specific rigidification and compatible symmetric structure on  $\mathcal{L}_\epsilon$  defined here are considered to be *part of the definition* of the object  $\mathcal{L}_\epsilon$ .

**Lemma I.2.** *The line bundle  $\mathcal{L}_\epsilon$  is isomorphic to  $\mathcal{O}((-1)^{1-\epsilon_2} q^{(1-\epsilon_1)/2})$ .*

*Proof.* We carry out this computation by explicitly writing down a global section. By definition of  $\mathcal{L}_\epsilon$ ,

$$H^0(E_q, \mathcal{L}_\epsilon) \cong \{\vartheta \in \mathcal{O}(\mathbf{C}^\times) : \vartheta(qz) = (-1)^{\epsilon_2} q^{(\epsilon_1-1)/2} z^{-1} \vartheta(z)\}.$$

An analysis of Fourier expansions shows that there is exactly one such global section up to scalars, namely

$$\vartheta_\epsilon := \sum_{n \in \mathbf{Z}} (-1)^{\epsilon_2 n} q^{(n^2 - \epsilon_1 n)/2} z^n$$

which has a zero at  $z = (-1)^{1-\epsilon_2} q^{(1-\epsilon_1)/2}$ . □

We remark here that Jacobi's thetas  $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$  are, sometimes up to a factor of  $q^{1/8}$ , identical to our  $\vartheta_{1,1}, \vartheta_{1,0}, \vartheta_{0,0}, \vartheta_{0,1}$ , respectively.

Given a globally-defined symmetric rigidified line bundle  $\mathcal{L}$  on the universal elliptic curve over a modular curve, we can tell which of the  $\mathcal{L}_\epsilon$  we get by restricting  $\mathcal{L}$  to a neighborhood of a particular cusp, either by looking at Fourier expansions of global sections, or by looking at the zero loci of sections.

**Lemma I.3.** *The  $\{\pm 1\}$ -valued quadratic form on  $E_q[2]$  derived from  $\mathcal{L}_\epsilon$  as in [Section 3.2](#) is*

$$(-1)^a q^{b/2} \mapsto (-1)^{ab + a\epsilon_1 + b\epsilon_2}.$$

*In particular, the Arf invariant is even if and only if  $\epsilon \neq (1, 1)$ .*

*Proof.* An arbitrary 2-torsion point is of the form  $z = (-1)^a q^{b/2}$ , where  $a, b \in \{0, 1\}$ . A cocycle calculation shows that the symmetric structure is given by multiplication by  $z^{\epsilon_1}$  on the level of  $q^{\mathbf{Z}}$ -equivariant structures on the structure sheaf of  $\mathbf{C}^\times$ . Since  $z^{-1} = q^{-b}z$ , that means the symmetric structure at  $z$  is given by multiplication by

$$((-1)^{\epsilon_2 b} q^{(b^2 - \epsilon_1 b)/2} z^{-b}) z^{\epsilon_1} = (-1)^{ab + a\epsilon_1 + b\epsilon_2}.$$

□

As a result, we will only deal with  $\underline{\epsilon} \neq (1, 1)$ , whose corresponding symmetric degree-1 line bundles all have quadratic form of the same Arf invariant. At different cusps, the universal line bundle may restrict to different  $\mathcal{L}_{\underline{\epsilon}}$ .

**Lemma I.4.** *Regard*

$$E_q \otimes \mathbf{Z}^m \simeq E_q^m = (\mathbf{C}^\times)^m / (q^{\mathbf{Z}})^m$$

as the quotient of  $(\mathbf{C}^\times)^m$  by the action of  $\mathbf{Z}^m$  where the  $j$ th generator  $T_j$  of  $\mathbf{Z}^m$  translates the  $j$ th complex coordinate by  $q$ .

Then the line bundle  $\mathcal{L}_{\underline{\epsilon}}^B$  on  $E_q^m$ , obtained by tensoring  $\mathcal{L}_{\underline{\epsilon}}$  with  $B$  as in [Section 3.6](#), arises from<sup>12</sup> the action of  $\mathbf{Z}^m$  on holomorphic functions on  $(\mathbf{C}^\times)^m$  given by

$$(I.2) \quad T_j^y \cdot f(\underline{z}) = f(\underline{z} \cdot \underline{q}_j) \cdot (-1)^{\epsilon_2 b_{jj} y} q^{b_{jj}(y^2 - \epsilon_1 y)/2} \prod_{i=1}^m z_i^{b_{ij} y}.$$

where we write  $\underline{z}$  for  $(z_1, \dots, z_m)$ , and  $\underline{q}_j$  for the element  $(1, \dots, 1, q, 1, \dots, 1)$  where the  $q$  is in position  $j$ .

*Proof.* By property (II) of the tensoring operation from [Section 3.6](#) it is enough to verify in the case of  $B$  of rank one. This amounts to computing the pullback of the line bundle on  $E_q$  by means of the map  $E_q^n \rightarrow E_q$  defined by an  $n$ -tuple  $(\ell_1, \dots, \ell_n)$  of integers. To compute that, we pull back the  $H^1(q^{\mathbf{Z}}, \mathcal{O}(\mathbf{C}^\times)^\times)$  cocycle  $\psi$  defining  $\mathcal{L}_{\underline{\epsilon}}$  under the map  $\underline{z} \mapsto \prod z_i^{\ell_i}$ . That cocycle is

$$T_j \mapsto \psi(T_j^{\ell_j})(z_1^{\ell_1}, \dots, z_m^{\ell_m}) = (-1)^{\epsilon_2 \ell_j} q^{(\ell_j^2 - \epsilon_1 \ell_j)/2} (z_1^{\ell_1} \dots z_m^{\ell_m})^{\ell_j},$$

from which we conclude the claim.

□

The upshot is that given a globally defined line bundle  $\mathcal{L}$  on the universal curve over a modular curve  $Y$ , if we know that  $\mathcal{L}$  restricts over a neighborhood of a specific cusp to  $\mathcal{L}_{\underline{\epsilon}}$ , then we know that  $\mathcal{L}^B$  restricts near the same cusp to the very explicit family  $\mathcal{L}_{\underline{\epsilon}}^B$ .

<sup>12</sup>Again, this means that sections of the bundle correspond to  $\mathbf{Z}^m$ -invariant holomorphic functions, as in the example just given.

In our case,  $\mathcal{L}$  will be the universal symmetric rigidified degree-1 line bundle on the universal elliptic curve over  $Y_0(2)$ . So (for example by [Lemma I.2](#))  $\mathcal{L}$  restricts to a neighborhood of  $\infty$  to  $\mathcal{L}_{1,0}$  and to a neighborhood of 0 to  $\mathcal{L}_{0,0}$ .

Recall that  $\mathcal{L}$  pulls back under  $Y^* \rightarrow Y_0(2)$  to a line bundle that we also abusively call  $\mathcal{L}$  (in fact we no longer have any use for the line bundle  $\mathcal{L}$  on  $Y_0(2)$ ). It is given at a point with moduli interpretation  $(E, \tilde{P}, Q)$  by  $\mathcal{O}_E([Q])$ , hence by  $\mathcal{L}_{1,0}$  near the cusp above  $\infty$  and by  $\mathcal{L}_{0,0}$  at the cusps above 0 (see [Table 1](#)). The twist by  $B$  of  $\mathcal{L}$  is therefore  $\mathcal{L}_{1,0} \otimes B$  near the cusp above  $\infty$ , and  $\mathcal{L}_{0,0} \otimes B$  at the three cusps above 0; those are described exactly by [Lemma I.4](#). The same thing occurs after pullback to  $Y^*(\Delta)$  depending on which cusp of  $Y^*$  the chosen cusp of  $Y^*(\Delta)$  lies over.

**I.2. Sections and higher cohomology.** Let  $x$  denote a fixed coset of  $\mathbf{Z}^m$  within  $B^{-1}\mathbf{Z}^m$ . Fix a  $d$ th root  $q^{1/d}$  of  $q$ . In the case when  $B$  is positive definite, each of the expressions

$$(I.3) \quad \vartheta_x = \sum_{\underline{n} \in x + \mathbf{Z}^m} \underline{z}^{B\underline{n}} q^{Q(\underline{n})}$$

define holomorphic functions on  $(\mathbf{C}^\times)^m$  that are invariant under the action of  $\mathbf{Z}^m$  from [\(I.2\)](#) and so give global sections of  $\mathcal{L}_{(0,0)}^B$ . Indeed these  $d$  functions give a basis of global sections — up to conventions, they are the theta functions with characteristic (see [\[46\]](#)).

In the case that  $B$  is indefinite there is a closely related set of formulas, but now describing the  $s$ -th group cohomology of  $\mathbf{Z}^m$  with coefficients in the holomorphic functions on  $(\mathbf{C}^\times)^m$  under the action [\(I.2\)](#) rather than the invariants. The only difference is that we need to restrict the range of summation — recall from [Section 1.1](#) that restricting the range of summation to certain regions will be our technique of choice for ensuring convergence of theta series for indefinite forms; this time, it is applied to the multivariate theta series with  $m$  elliptic variables.

Namely, let us fix:

- A maximal positive definite rational subspace  $P \subset \mathbf{R}^m$ ,  $\dim_{\mathbf{R}} P = r$ .
- An orientation of  $P$ .
- A base point  $\underline{\alpha} \in \mathbf{R}^m$  such that  $\{1, \alpha_1, \dots, \alpha_m\}$  is a  $\mathbf{Q}$ -linearly independent set in  $\mathbf{R}$ .

Then, for  $\underline{v}$  an  $s+1$ -tuple  $(v_0, \dots, v_s)$  of elements of  $\mathbf{Z}^m$ , let us define  $\mathcal{C}(\underline{v})$  to be the prism

$$(I.4) \quad \mathcal{C}(\underline{v}) = (\text{convex hull of } v_0, \dots, v_s) + P.$$

We will ensure convergence by carrying out the usual construction of theta series, but summing only over lattice points in  $\underline{\alpha} + \mathcal{C}(\underline{v}_i)$ . This way the only unbounded directions in the region of summation are positive-definite for  $Q$  and do not pose convergence problems.

To ensure that the summation procedure we have just described will provide a cocycle, we need to sum with an orientation rule. If  $\Delta^s$  is the standard  $s$ -simplex (with points  $\underline{a} \in \mathbf{R}^{s+1}$

such that  $\sum_1^{s+1} a_i = 1$ ) with the standard orientation, for each  $\underline{v} \in (\mathbf{Z}^m)^{s+1}$  we may consider the map of manifolds with corners

$$\varphi_{\underline{\alpha}, \underline{v}} : \Delta^s \times P \rightarrow \mathbf{R}^m$$

given by

$$(\underline{a}, \underline{x}) \mapsto \underline{\alpha} + \sum_{i=0}^s a_i v_i + \underline{x}.$$

Because of the irrationality property assumed of  $\underline{\alpha}$  combined with the rationality property assumed of  $P$ , no lattice points are in the image of  $\varphi_{\underline{\alpha}, \underline{v}}$  restricted to the boundary of its domain<sup>13</sup>. Therefore, we may write  $\text{sign}(\underline{v}) = 1, -1, 0$  according to whether  $\varphi_{\underline{\alpha}, \underline{v}}$  restricted to the interior of its domain preserves orientation, reverses orientation, or is degenerate (i.e. has image of dimension  $\leq m$ ). The last possibility is irrelevant for us, as we already showed that in that case  $\varphi_{\underline{\alpha}, \underline{v}}$  will not have any lattice points in its image.

Another point of view (which we will take up more systematically in [Section IV<sub>0</sub>](#)) is that we have arranged explicitly for  $\varphi_{\underline{\alpha}, \underline{v}}$  to be *transverse* to the zero-dimensional subset  $\mathbf{Z}^m \subset \mathbf{R}^m$ , in the sense that it is transverse when restricted to the strata of all dimensions in  $\Delta^s \times P$ .

**Lemma I.5.** *Fix a coset  $x$  of  $\mathbf{Z}^m$  inside  $B^{-1}\mathbf{Z}^m$ , and let  $\underline{\epsilon} = (0, 0)$ . The rule*

$$(I.5) \quad \vartheta_x : \underline{v} \mapsto \text{sign}(\underline{v}) \cdot \sum_{\underline{m} \in (\alpha + \mathcal{C}(\underline{v})) \cap \mathbf{Z}^m} \underline{z}^{B(\underline{m} + \underline{x})} q^{Q(\underline{m} + \underline{x})},$$

*the sum being taken over the fixed coset, defines a (homogeneous)  $s$ -cocycle of  $\mathbf{Z}^m$  with values in  $\mathcal{O}((\mathbf{C}^\times)^m) := \text{holomorphic functions on } (\mathbf{C}^\times)^m \text{ with the action defined in (I.2) with half characteristic } \underline{\epsilon}$ .*

*Proof.* Observe that, in this definition, it is irrelevant whether we use all of  $\mathcal{C}$  or just its interior, because all lattice points on  $\mathcal{C}$  lie in the interior, as we already saw.

To prove that the formula above defines a homogeneous  $s$ -cocycle, we must verify:

- (i) The only part actually involving the group action in any way: for all  $y \in \mathbf{Z}^m$ ,

$$\vartheta_x(v_0 + y, \dots, v_s + y) = y \cdot \vartheta_x(v_0, \dots, v_s)$$

where  $\cdot$  refers to the group action of  $\mathbf{Z}^m$  on  $\mathcal{O}((\mathbf{C}^\times)^m)$  that defines the line bundle  $\mathcal{L}_{0,0}^B$ , namely

$$(y \cdot \vartheta_x)(\underline{z}) = q^{Q(y)} \underline{z}^{By} \vartheta_x(q^y \underline{z}).$$

<sup>13</sup>This construction is an explicit version — that works in this specific situation — of the more general transversality theory we will make use of in [Section IV<sub>0</sub>](#). It works because if there was a lattice point  $\lambda \in \mathbf{Z}^m$  in the image of  $\varphi_{\underline{\alpha}, \underline{v}}$  restricted to (without loss of generality) the face not containing  $v_s$ , then  $\lambda - \underline{\alpha} - v_0$  would be in a rational linear subspace of  $\mathbf{R}^m$  of dimension  $m - 1$ , namely the one generated by  $v_i - v_0$  ( $i = 1, \dots, s - 1$ ) and  $P$ . So there would be a *nonzero* linear functional with  $\mathbf{Q}$ -coefficients that kills  $\lambda - \underline{\alpha} - v_0$ , which contradicts the  $\mathbf{Q}$ -linear independence of  $1, \alpha_1, \dots, \alpha_m$ .

(ii) For all  $v_0, \dots, v_{s+1} \in \mathbf{Z}^m$ ,

$$\sum_{i=0}^{s+1} (-1)^i \vartheta_x(v_0, \dots, \widehat{v}_i, \dots, v_{s+1}) = 0.$$

Part (i) follows from a calculation formally equivalent to the one that certifies that Jacobi's  $\vartheta_3(z, \tau)$  is a section of  $\mathcal{L}_{(0,0)}$  :

$$\begin{aligned} \vartheta_x(v_0 + y, \dots, v_s + y) &= \text{sign}(\underline{v} + y) \sum_{\underline{n} \in (\underline{\alpha} + \mathcal{C}(\underline{v} + y)) \cap \mathbf{Z}^m} q^{Q(\underline{n} + \underline{x})} \underline{z}^{B(\underline{n} + \underline{x})} \\ &= \text{sign}(\underline{v}) \sum_{\underline{n} \in (\underline{\alpha} + \mathcal{C}(\underline{v}) + y) \cap \mathbf{Z}^m} q^{Q(\underline{n} + \underline{x})} \underline{z}^{B(\underline{n} + \underline{x})} \\ &= \text{sign}(\underline{v}) \sum_{\underline{n} \in (\underline{\alpha} + \mathcal{C}(\underline{v})) \cap \mathbf{Z}^m} q^{Q(\underline{n} + \underline{x} + y)} \underline{z}^{B(\underline{n} + \underline{x}) + By} \\ &= \text{sign}(\underline{v}) \sum_{\underline{n} \in (\underline{\alpha} + \mathcal{C}(\underline{v})) \cap \mathbf{Z}^m} q^{Q(\underline{n} + \underline{x}) + \langle \underline{n} + \underline{x}, y \rangle + Q(y)} \underline{z}^{B(\underline{n} + \underline{x}) + By} \\ &= \text{sign}(\underline{v}) q^{Q(y)} \underline{z}^{By} \sum_{\underline{n} \in (\underline{\alpha} + \mathcal{C}(\underline{v})) \cap \mathbf{Z}^m} q^{Q(\underline{n} + \underline{x})} (q^y \underline{z})^{B(\underline{n} + \underline{x})} \\ &= \text{sign}(\underline{v}) y \cdot \vartheta_x(v_0, \dots, v_s). \end{aligned}$$

Now let us prove (ii). Each  $\vartheta_j$  here corresponds to a sum over the cone  $\mathcal{C}_j = \mathcal{C}(v_0, \dots, \widehat{v}_j, \dots, v_{s+1})$ . We must show that for all  $w \in \mathbf{Z}^m$ , the alternating sum of characteristic functions

$$(I.6) \quad \sum \text{sign}(\mathcal{C}_j) 1_{\mathcal{C}_j}(w)$$

vanishes.

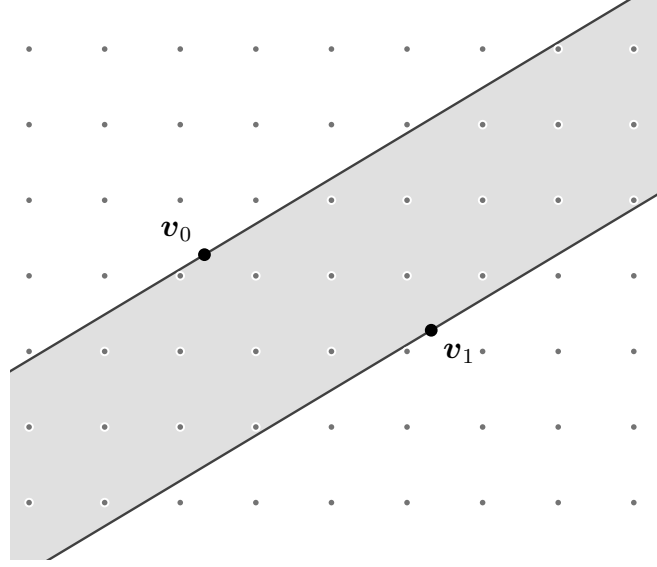
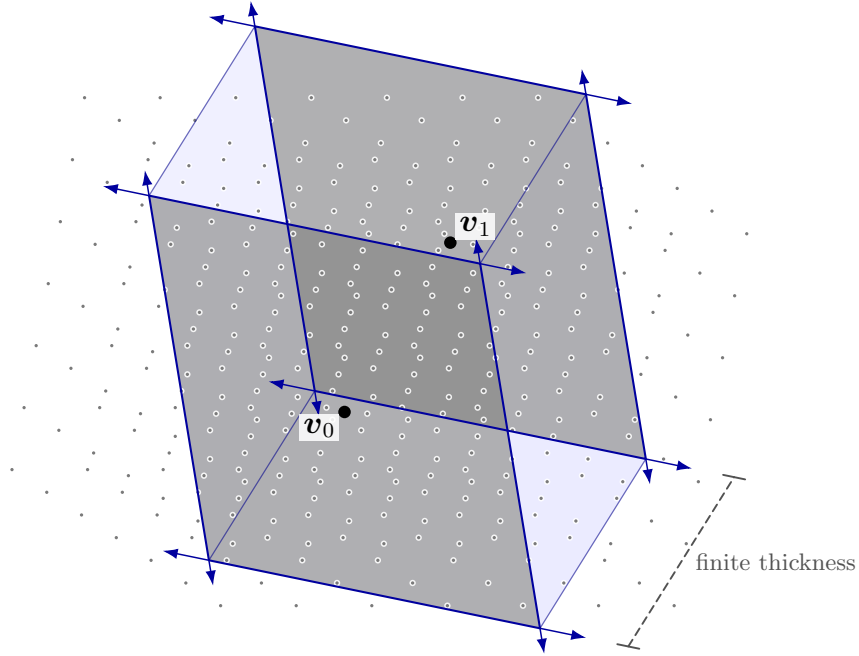
If the  $v_0, \dots, v_{s+1}$  fail to span (in the affine sense) all of  $\mathbf{R}^m/P$ , all of the cones  $\mathcal{C}_j$  are degenerate, and all the terms in (I.6) vanish for the same reason as above. Otherwise consider the affine-linear function

$$(I.7) \quad \text{standard } (s+1)\text{-simplex} \longrightarrow \mathbf{R}^m/P$$

which sends the vertices to  $v_0, v_1, \dots, v_{s+1}$ . The preimage of any  $w \in \mathbf{Z}^m$  is, if nonempty, a line which intersects the boundary of the standard  $s+1$ -simplex in two points. These points are necessarily interior to two faces. These two faces correspond to the exactly two values of  $j$  for which  $w \in \mathcal{C}_j$ , and they contribute to (I.6) with opposite signs.

This suggests, of course, taking a more consistently topological view of the situation, and we will do so in [Section IV<sub>0</sub>](#). For now, we can just say: the vanishing of (I.6) follows from the fact that the coefficient of  $q^{Q(\underline{m})} \underline{z}^{B\underline{m}}$  in  $\vartheta_x(v_0, \dots, v_s)$  is the signed intersection number of  $\{\underline{m}\} \subset \mathbf{R}^m$  with  $\varphi_{\underline{\alpha}, \underline{v}}$ , which is guaranteed to be transverse to it.  $\square$

In [Figure 1](#), [Figure 2](#), and [Figure 3](#) we provide some diagrams showing the regions  $\mathcal{C}$  in different signatures, in the hopes that it may help avoid confusion.

FIGURE 1. An example of  $\alpha + \mathcal{C}(v_0, v_1)$  in the signature  $(1, 1)$  case.FIGURE 2. An example of a  $\alpha + \mathcal{C}(v_0, v_1)$  in signature  $(1, 2)$ 

Step (ii) of the proof of [Lemma I.5](#) is summarized by [Figure 4](#): it amounts to saying that summing over lattice points in two regions is the same as summing over lattice points in their union. This makes intuitive sense, but only actually works because we used the shift by  $\underline{\alpha}$  to ensure that there were no lattice points on the boundaries of those regions. In terms of something already present in the proof of [Lemma I.5](#): we couldn't write down intersection numbers without a guarantee of transversality, which had to be arranged.

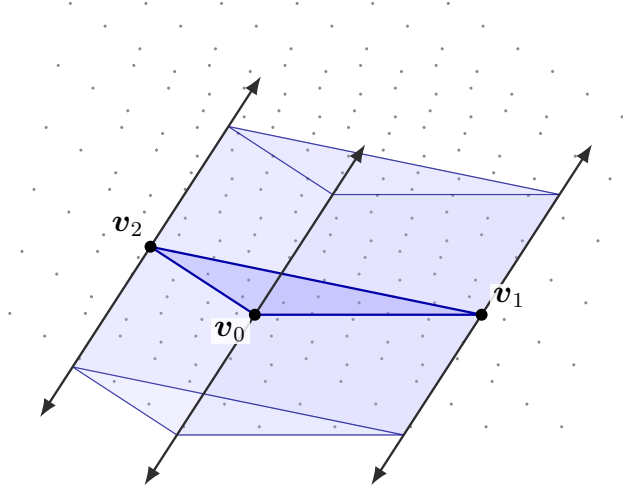
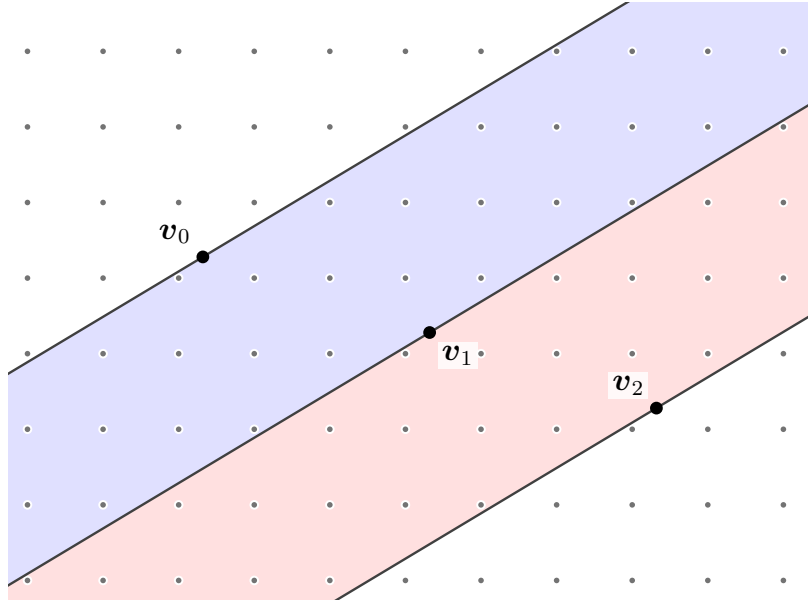
FIGURE 3. An example of  $\mathcal{C}(v_0, v_1, v_2)$  in the case of signature  $(2, 1)$ 

FIGURE 4. Revisiting the signature  $(1, 1)$  cocycle. Here the fact that it is a cocycle amounts to the fact that counting lattice points in the entire shaded region is the same as doing it in the red region followed by the blue region. This only works because there are no lattice points on the boundaries of the regions.



We finally remind the reader of the geometric significance of the cohomology group that the  $\vartheta_x$  live in: it admits an isomorphism

$$(I.8) \quad H^s(\mathbf{Z}^m, \mathcal{O}((\mathbf{C}^\times)^m)) \simeq H^s(E_q^m, \mathcal{L}_B),$$

generalizing the standard one for  $H^0$  of an ample bundle (in fact it holds with  $H^s$  replaced with  $H^i$  for any  $i$ , but only  $H^s$  will be nonzero, and the  $\vartheta_x$  are in  $H^s$ ).

Therefore, the construction of  $\vartheta_x$  is really an explicit construction of families of classes in the higher coherent cohomology of  $\mathcal{L}^B$ .

The analog of (I.8) in the case of the other possible choices of  $\underline{\epsilon}$  is obvious (just change the group action of  $\mathbf{Z}^m$  on  $\mathcal{O}((\mathbf{C}^\times)^m)$  to the one specified by (I.2) depending on  $\underline{\epsilon}$ ). The challenge that is not completely obvious is to make the construction of the sections  $\vartheta_x$  (analog of Lemma I.5) work without using too many roots of  $q$  or roots of 1. The construction that we ended up using will be described in Section III<sub>a</sub>. For the remainder of our description of step (I), we content ourselves with proving the analog of Equation (I.8) in the algebraic setting over  $\mathbf{Z}'$  — and delaying the actual construction of sections in cases other than  $\underline{\epsilon} = (0, 0)$  until Section III<sub>a</sub>.

**I.3. Sections of  $\mathcal{L}$  near the cusps: integral theory.** Our previous discussion constructs a basis of sections for  $\Theta := R^s \pi_* \mathcal{L}^B$  over a small analytic neighbourhood of the three cusps of  $Y^*$  over 0 (at least, after passing to a covering of a neighborhood of the relevant cusp that permits us to take the appropriate roots of  $q$  — so perhaps more accurately, near the cusps of  $Y^*(\mathbf{d})$  over 0).

We can equally well produce a basis of sections of  $\Theta$  over a *formal* neighbourhood of the cusps, even working over  $\mathbf{Z}'$ . (Note that while the sections we are about to write down form a basis, we will not actually prove this, because it is a little tricky when  $s > 0$  and we don't need it.)

The rôle of the degenerating family of elliptic curves  $E_q = \mathbf{C}^\times/q^{\mathbf{Z}}$  as  $q \rightarrow 0$  in a punctured disk is played in the arithmetic theory of moduli of elliptic curves by the Tate curve  $E_{Tate,w}/\mathbf{Z}'((q^{1/w}))$ , which is what the universal elliptic curve over a modular curve restricts to over a punctured formal neighborhood of a cusp of width  $w$ . The existence of  $E_{Tate,w}$  together with a choice of relevant level structure on it certainly provides one with a morphism

$$i : \mathrm{Spec} \mathbf{Z}'((q^{1/w})) \rightarrow Y$$

where  $Y$  is the relevant modular curve, which fits into a Cartesian square

$$\begin{array}{ccc} E_{Tate,w} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathrm{Spec} \mathbf{Z}'((q^{1/w})) & \longrightarrow & Y \end{array}$$

and therefore into

$$\begin{array}{ccc} E_{Tate,w}^m & \longrightarrow & \mathcal{E}^m \\ \downarrow & & \downarrow \\ \mathrm{Spec} \mathbf{Z}'((q^{1/w})) & \longrightarrow & Y \end{array}$$

where the  $m$ -th powers are fibered over their respective bases. The line bundle  $\mathcal{L}^B$  on  $\mathcal{E}^m$  pulls back to a specific line bundle  $i^*\mathcal{L}^B$  on  $E_{Tate,w}$ . The classes in  $H^s(E_{Tate,w}, i^*\mathcal{L}^B)$  are precisely the “restrictions to neighborhoods of cusps” of the algebraic theta functions (sections of  $\Theta$ ) that generalize Jacobi’s  $\vartheta_3(\tau, z)$ . To produce explicit elements of  $H^s(E_{Tate,w}^m, i^*\mathcal{L}^B)$  using the summation recipe from the previous section, one needs an algebraic analog of (I.8) that would give a way to identify  $H^s(E_{Tate,w}^m, i^*\mathcal{L}^B)$  with some group cohomology valued in a ring of power series.

Faltings–Chai [15] (see also [40]) provide a classification of line bundles on  $E_{Tate,w}^m$ , inside of which we may recognize  $i^*\mathcal{L}^B$ . In the case  $s = 0$ , they also provide the exact analog of (I.8) for  $H^s(E_{Tate,w}^m, i^*\mathcal{L}^B)$ , using the technique of Raynaud extensions. Their method does not apply to higher cohomology, but the higher cohomology generalization is proved by different methods in other work of the first-named author [34].

Both the statement of [34] and the statement of Faltings–Chai it generalizes replace the ring of holomorphic functions on  $(\mathbb{C}^\times)^m$  with the following object over  $\mathbf{Z}'((q^{1/w}))$ , whose origin in nonarchimedean geometry is explained in more detail in those works.

**Definition I.6.** The  $\mathcal{O}^{\text{an}}$  consists of Laurent series in variables  $z_1, \dots, z_m$ , whose coefficients lie in  $\mathbf{Z}'((q^{1/w}))$ , and where the  $q$ -valuation increases superlinearly with the exponents of the  $z$ -variables. Symbolically:

$$\mathcal{O}^{\text{an}} := \left\{ \sum a_{\underline{k}} z_1^{k_1} \dots z_m^{k_m} : q\text{-valuation of } a_{\underline{k}} \in \mathbf{Z}'((q^{1/w})) \text{ grows superlinearly} \right\},$$

where the meaning of “superlinear” is that, for all  $C$ , the number of  $\underline{n}$  for which  $\text{val}_q(a_{\underline{n}}) \leq C\|\underline{n}\|$  is bounded. This is a completed version of the ring of functions of  $\mathbf{G}_m^m$  over  $\mathbf{Z}'((q^{1/w}))$  and should be thought of as an analog of the ring of holomorphic functions on the punctured complex plane; but because there is no archimedean convergence condition, there is no map

$$\mathcal{O}^{\text{an}} \longrightarrow \text{holomorphic functions on } (\mathbb{C}^\times)^m$$

With this definition, note that we can produce an action of the abstract group  $\mathbf{Z}^m$  on  $\mathcal{O}^{\text{an}}$  that is spelled the same way as the action of  $\mathbf{Z}^m$  on  $\mathcal{O}((\mathbb{C}^\times)^m)$  from (I.2):

$$\underline{y} \cdot f(\underline{z}) := q^{Q(\underline{y})} \underline{z}^{B\underline{y}} f(q^{\underline{y}} \underline{z}).$$

One of the main results of [34] is that when  $i$  is selected to be the Tate curve neighborhood of one of the cusps above 0 in  $Y_{\mathbf{Z}'}^*$  or  $Y^*(\Delta)$  there is a canonical isomorphism

$$(I.9) \quad H^s(E_{Tate,w}^m, i^*\mathcal{L}^B) \simeq H^s(\mathbf{Z}^m, \mathcal{O}^{\text{an}})$$

generalizing directly the result of Faltings–Chai that applies only when  $s = 0$ .

Similarly, [34] shows that when  $i$  is selected to be a Tate curve neighborhood over a cusp above  $\infty$ , the Fourier expansion isomorphism (I.9) still holds but with the  $\mathbf{Z}^m$ -action replaced with the one described by (I.2) for  $\underline{\epsilon} = (1, 0)$ .

So long as the width  $w$  accommodates the necessary fractional powers of  $q$ , the rule of (I.5) can be regarded, in an evident way, as a class in  $H^s(\mathbf{Z}^m, \mathcal{O}^{\text{an}})$  for  $\underline{\epsilon} = (0, 0)$ , since the  $q$ -valuation of the coefficients *do* grow superlinearly. Therefore, the  $\vartheta_x$  of (I.5) can also be regarded as a collection of sections of  $i^*\Theta$  for  $i : \text{Spec} \mathbf{Z}'((q^{1/(w\text{d}}))) \rightarrow Y$  over a cusp over 0 in  $Y^*(\Delta)$  lying over a cusp of  $Y^*$  of width  $w \in \{2, 4\}$ . As mentioned previously, we postpone the discussion of construction of sections  $\vartheta_x$  near a cusp over  $\infty$  to Section III<sub>a</sub>, since it will be slightly delicate to get the right powers of  $q$  to appear and thus thematically belongs more to the discussion of descent from covers of  $Y^*$ .

## II. Lifting Hitchin's connection to a genuine connection

### II<sub>0</sub>. PRELIMINARIES ON SYMMETRIC POWERS

We will require the use of symmetric powers of tangent sheaves of smooth schemes. In order to make calculations that are exact (so that we may conclude the exact right answer for the weight of the modular forms that we construct), it is necessary for us to fix conventions regarding this concept and its relation to the symmetric powers of the cotangent sheaf. The reader who is not interested in the exact verification of constants can easily skip this without serious loss.

Let  $\pi : X \rightarrow S$  be a smooth morphism of schemes. Then  $T_{X/S}$  and its dual  $\Omega_{X/S}^1$  are vector bundles on  $X$ . Let  $n$  be a nonnegative integer, and suppose that  $n!$  is invertible in  $\mathcal{O}_S$ .

We wish to fix conventions regarding:

- The definition of the vector bundles  $\text{Sym}^n T_{X/S}$  and  $\text{Sym}^n \Omega_{X/S}^1$ .
- The definition of the pairing  $(\text{Sym}^n T_{X/S}) \otimes (\Omega_{X/S}^1)^{\otimes n} \rightarrow \mathcal{O}_X$ .
- The definition of the pairing  $(\text{Sym}^n T_{X/S}) \otimes \Omega_{X/S}^1 \rightarrow \text{Sym}^{n-1} T_{X/S}$ .

In our discussion of differential operators, we will take the convention that  $\text{Sym}^n T_{X/S}$  is the subbundle of symmetric sections of  $T_{X/S}^{\otimes n}$ , i.e. of symmetric linear functionals on  $(\Omega_{X/S}^1)^{\otimes n}$ . This means that it naturally comes with a pairing

$$(\text{Sym}^n T_{X/S}) \otimes (\Omega_{X/S}^1)^{\otimes n} \rightarrow \mathcal{O}_X$$

given on local pure tensors by

$$(II_0.1) \quad \langle Y_1 \otimes \cdots \otimes Y_n, db_1 \otimes \cdots \otimes db_n \rangle := \langle Y_1, db_1 \rangle \cdots \langle Y_n, db_n \rangle.$$

Finally, the pairing with  $\Omega_{X/S}^1$  is given by a single contraction

$$\langle Y_1 \otimes \cdots \otimes Y_n, db \rangle = \langle Y_1, db \rangle Y_2 \otimes \cdots \otimes Y_n.$$

For a local section  $Y$  of  $\mathrm{Sym}^n T_{X/S}$  and a local section  $\eta$  of  $\Omega_{X/S}^1$ , the pairing  $\langle Y, \eta \rangle$  is therefore characterized by the property

$$(II_0.2) \quad \langle \langle Y, \eta \rangle, \mu \rangle = \langle Y, \eta \otimes \mu \rangle.$$

for all local sections  $\mu$  of  $\mathrm{Sym}^{n-1} \Omega_{X/S}^1$ .

So far, none of these definitions have required the hypothesis that  $n!$  is invertible. That assumption comes in useful when we wish to regard sections of  $\mathrm{Sym}^n T_{X/S}$  as homogeneous degree- $n$  functions on the fibers of  $\Omega_{X/S}^1$  (i.e. as a quotient of  $T_{X/S}^{\otimes n}$  instead of a subsheaf). These are equivalent by the technique of averaging

$$Y_1 \odot \cdots \odot Y_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} Y_{\sigma(1)} \otimes \cdots \otimes Y_{\sigma(n)}.$$

Viewed this way, the pairing of  $Y_1 \odot \cdots \odot Y_n$  with a local section  $\eta$  of  $\Omega_{X/S}^1$  can be viewed as taking the derivative in the direction of  $\eta$ :

$$\langle Y_1 \odot \cdots \odot Y_n, \eta \rangle = \frac{1}{n} \sum_{i=1}^n \langle Y_i, \eta \rangle Y_1 \odot \cdots \odot \widehat{Y_i} \odot \cdots \odot Y_n.$$

Let  $n = 2$ , so that we are only assuming that 2 is invertible in  $\mathcal{O}_S$ . Viewing  $\mathrm{Sym}^2 T_{X/S}$  and  $\mathrm{Sym}^2 \Omega_{X/S}^1$  as quotients of the respective tensor squares, we can define a pairing  $\langle -, - \rangle$  between  $\mathrm{Sym}^2 T_{X/S} \subset T_{X/S}^{\otimes 2}$  and  $\mathrm{Sym}^2 \Omega_{X/S}^1 \subset (\Omega_{X/S}^1)^{\otimes 2}$  by defining it on pure tensors as

$$\langle Z_1 \odot Z_2, \eta_1 \odot \eta_2 \rangle = \frac{1}{2} (\langle Z_1, \eta_1 \rangle \langle Z_2, \eta_2 \rangle + \langle Z_1, \eta_2 \rangle \langle Z_2, \eta_1 \rangle).$$

This respects the previous pairing under the identification of symmetric power subspaces with quotients already defined. There is also another pairing which we call the *trace pairing* given abstractly by viewing

$$\mathrm{Sym}^2 T_{X/S} \subset T_{X/S} \otimes T_{X/S} = \underline{\mathrm{Hom}}(\Omega_{X/S}^1, T_{X/S})$$

and

$$\mathrm{Sym}^2 \Omega_{X/S}^1 \subset \Omega_{X/S}^1 \otimes \Omega_{X/S}^1 = \underline{\mathrm{Hom}}(T_{X/S}, \Omega_{X/S}^1);$$

then defining the pairing

$$\begin{aligned} \langle -, - \rangle_{\mathrm{Tr}} : \mathrm{Sym}^2 T_{X/S} \times \mathrm{Sym}^2 \Omega_{X/S}^1 &\subset \underline{\mathrm{Hom}}(\Omega_{X/S}^1, T_{X/S}) \times \underline{\mathrm{Hom}}(T_{X/S}, \Omega_{X/S}^1) \\ &\xrightarrow{(A,B) \mapsto \mathrm{Tr}(AB)} \mathcal{O}_X. \end{aligned}$$

which can in reality be defined in one go by simply contracting all the tensors on pure tensors via

$$\langle Z_1 \otimes Z_2, \eta_1 \otimes \eta_2 \rangle_{\mathrm{Tr}} := \langle Z_2, \eta_1 \rangle \langle Z_1, \eta_2 \rangle$$

and extending linearly. With these definitions, we have

$$(II_0.3) \quad \text{standard pairing} = \text{trace pairing}.$$

II<sub>a</sub>. HEAT OPERATORS FOR A FAMILY OF GENERAL VARIETIES

As a preliminary to step (II) of §4, we investigate a more general question:

Given a proper smooth family  $\pi : X \rightarrow S$  of irreducible algebraic varieties  
(II<sub>a</sub>.1) over the base  $S$  and a line bundle  $\mathcal{L}$  on  $X$ , *how do we compare sections of  $\mathcal{L}_s$  on  $X_s$  for different  $s$ ?*

The results are not novel but we hope that the exposition is useful for the reader who is not familiar with this circle of ideas. The main result is given in Section II<sub>a</sub>.5. We have tried to write in natural language rather than the symbolism of algebraic geometry, but our treatment amounts to nothing more than a translation of the standard one, with some mild improvements in the statement. We refer the reader to the nice papers [50, 55] for versions in the sheaf-theoretic language in a general algebraic geometry context, and to [32] for the original construction of Hitchin in the Kähler geometry setting. We should also remark that these ideas in the case of families of abelian varieties have their origin in work of Welters [60].

**II<sub>a</sub>.1. A naive (failed) attempt to make a connection.** Suppose we are given a family of sections  $f_s$  of  $\mathcal{L}_s$  of  $X_s$ , that is,  $f \in H^0(S, \pi_*\mathcal{L})$ . We would like the ability to compare  $f_s$  with  $f_{s'}$  for  $s'$  “near”  $s$ . To do so, we can ask for a notion of “derivative” of the family of sections, indexed by a vector field  $Z$  on the base  $S$ .

Naively speaking, we might attempt to lift  $Z$  to a vector field  $\tilde{Z}$ . The problem is that such a lifting does not exist in any interesting case. In fact, let us once and for all fix a covering  $X = \cup X_i$  by affine neighbourhoods, such that  $\mathcal{L}$  is trivial on each one; we fix such a trivialization.

Now,  $Z$  can be lifted locally to a vector field  $\tilde{Z}_i$  on each  $X_i$ . Then, fixing a trivialization of  $\mathcal{L}$  on each  $X_i$ , we can certainly regard  $\tilde{Z}_i$  as a first order differential operator on  $\mathcal{L}$  over  $X_i$ ; but these do not glue together. On overlaps,  $\tilde{Z}_i - \tilde{Z}_j$  are vector fields along the fiber direction (meaning: as a local section in  $H^0(X_i \cap X_j, T_X)$  it has image 0 in  $H^0(X_i \cap X_j, \pi^*T_S)$  and therefore is in  $H^0(X_i \cap X_j, T_{X/S})$ ), i.e. sections  $t_{ij}$  of the relative tangent bundle  $T_{X/S}$ . Taken together, they define a Čech class

$$(II_{a.2}) \quad (t_{ij} = \tilde{Z}_i - \tilde{Z}_j) \in H^1(X, T_{X/S})$$

which is precisely the class whose restriction to  $X_s$  classifies the infinitesimal deformation of the family  $X_t$  as one varies  $t$  near  $s$ , that is, the image at  $Z$  of the Kodaira–Spencer map  $T_S|_s \rightarrow H^1(X_s, T_{X_s})$  [29, Ch. 1, §5]. If this class is nontrivial, it is impossible to globally lift  $Z$ .

**II<sub>a</sub>.2. Example.** To see how to fix this, let us consider an example in the analogous complex analytic situation.

We will ultimately only be interested in families of abelian varieties which are fibered powers of families of elliptic curves. The simplest possible such family in the analytic setting is as follows: take  $X$  to be the family of elliptic curves  $E_q := \mathbb{C}^\times/q^{\mathbf{Z}}$  over the *analytic* base  $S$  which is the double cover of punctured unit disk  $\{q \in \mathbb{C} : 0 < |q| < 1\}$  with coordinate  $q^{1/2}$ , and let the line bundle  $\mathcal{L}^\pm$  on  $X$  arise from the action

$$(II_a.3) \quad (T_\pm f)(q, z) = (q^{1/2}z)^{\mp 1} f(q, zq).$$

of  $\mathbf{Z} \simeq q^{\mathbf{Z}}$  on holomorphic functions in  $(q, z)$ . That is, the total space of  $\mathcal{L}$  is the complex manifold

$$(\mathbf{C}^\times \times \mathbf{C})/q^{\mathbf{Z}}, \quad q^y \cdot (z, t) := (q^y z, q^{\pm y^2/2} z^{\pm y} t)$$

so that  $H^0(X, \mathcal{L}^+)$  is identified with the space of holomorphic  $f$  with the property that  $T_+ f = f$ , i.e.  $f(q, qz) = q^{-1/2} z^{-1} f(q, z)$ . With the plus sign,  $\mathcal{L}^+$  is an ample bundle, with a global section  $\sum q^{n^2/2} z^n$ ; with the negative sign,  $\mathcal{L}^-$  is the inverse bundle, which has no global section.

Write  $D_q = q \frac{\partial}{\partial q}$  and similarly write  $D_z = z \frac{\partial}{\partial z}$ . We will take  $Z = D_q$ , and try to lift to a differential operator that acts on local sections of  $\mathcal{L}^\pm$ ; such an operator can be described by giving a differential operator in  $q$  and  $z$  on  $\mathbf{C}^\times \times \mathbf{C}$  that commutes with the  $T$ -action above and therefore descends to  $\mathcal{L}^\pm$ . Any such differential operator acts on  $H^0(X, \mathcal{L}^+)$  viewed as a space of quasiperiodic functions on  $\mathbf{C}^\times$  since it preserves  $T$ -invariance. The most naive attempt is to take  $D_q$  itself; but it does not commute with  $T$ . This corresponds to the failure described in [Section II<sub>a</sub>.1](#).

We shall attempt to modify it by a polynomial in  $D_z$  so that it *does* commute with  $T$ , taking instead

$$D = D_q + P(D_z),$$

for a polynomial  $P$ . We know that taking  $P$  to be of degree 1 will only work if the deformations induced by  $X/S$  are trivial, which is not true complex-analytically. Therefore, we abandon the idea of lifting  $D_q$  to a vector field – a differential operator of order one – and allow ourselves a differential operator of higher order.

The simplest formal way to check whether  $D$  commutes with  $T$  is to write functions in Laurent expansions  $f = \sum c_n(q) z^n$ . With respect to this Laurent expansion, the differential operator  $D$  acts on each  $c_n$  individually by means of  $D_q + P(n)$ . On the other hand,  $T$  has the effect  $c_n \leftarrow c_{n-1} q^{n-1/2}$  in the ample case, or  $c_n \leftarrow c_{n+1} q^{n+1/2}$  in the antiample case. So the condition of  $D$  commuting with  $T$  amounts to asking

$$P(n-1) = P(n) + (n-1/2) \text{ or } P(n+1) = P(n) + (n+1/2)$$

which has solution  $P(n) = \mp n^2/2$ , unique up to constants.

We conclude that there is one possibility for  $D$  of this general form up to addition of constant scalars, namely

$$(II_a.4) \quad D = (q\partial_q) \mp \frac{1}{2}(z\partial_z)^2,$$

where the  $-$  sign corresponds to the ample choice of bundle  $\mathcal{L}^+$ .

Note, too, that this kills the section  $\sum q^{n^2/2}z^n$  in the  $+$  case, and it also formally kills the (non-convergent) expression  $\sum q^{-n^2/2}z^n$  in the antiample case. The former is precisely the fact that the theta function  $\vartheta_3(q, z) := \sum q^{n^2/2}z^n$  is a solution to a certain steady-state heat equation, dating back to [16].

**II<sub>a</sub>.3. The notion of heat operator.** Let us return to the general case, but with the prior example as motivation.

We want to assign to each vector field  $Z$  on  $S$  a differential operator  $\mathcal{D}_Z$  on  $\mathcal{L}$  over  $X$ . Here, we recall from [25, §16.8] the following standard definition of differential operator:

**Definition II<sub>a</sub>.1.** A differential operator  $\mathcal{D}$  on  $\mathcal{L}$  of order  $\leq n$  is an endomorphism of  $\mathcal{L}$  (*not*  $\mathcal{O}_X$ -linear unless  $n = 0$ ) which satisfies either of the following equivalent conditions (at least when  $n!$  is invertible in  $\mathcal{O}_S$  to avoid complications with the symmetric powers of the tangent bundle):

- It is given in local coordinates trivializing  $\mathcal{L}$  by an algebraic differential operator of order  $\leq n$ , i.e., a sum of operators, each of which is a product of at most  $n$  vector fields.
- Let  $f$  be a regular function on any open set; the commutator operator  $[\mathcal{D}, f]$ , sending  $\lambda$  to  $\mathcal{D}(f\lambda) - f(\mathcal{D}\lambda)$ , is a differential operator of order  $\leq n - 1$  (for  $n = 0$ , we understand this to mean that the commutator is zero, i.e. that  $\mathcal{D}$  is  $\mathcal{O}_X$ -linear).

The idea will be to modify the naive construction of §II<sub>a</sub>.2 by, locally, taking

$$\mathcal{D}_Z = \text{a lift of } Z + \text{a fiber-wise differential operator},$$

where a “fiber-wise differential operator” is one that can be locally expressed by sums and products of vector fields that lie along the fibers of  $X \rightarrow S$ . Any operator of this sort will satisfy the Leibniz rule

$$(II_a.5) \quad \mathcal{D}_Z(f \cdot s) = (Zf) \cdot s + f(\mathcal{D}_Z s)$$

for all  $s$  a local section of  $\mathcal{L}$ , and  $f$  a local section of  $\mathcal{O}_X$  pulled back from a local section of  $\mathcal{O}_S$  (as opposed to an arbitrary local section of  $\mathcal{O}_X$ ). The condition [Equation \(II<sub>a</sub>.5\)](#) is enforcing the condition that  $\mathcal{D}_Z$  is locally a lift of  $Z$  up to a fiber-wise differential operator, rather than having a higher-order operator in the base direction.

We will call such a linear assignment  $Z \mapsto \mathcal{D}_Z$ , from vector fields on the base to differential operators on the line bundle  $\mathcal{L}$ , a *generalized heat operator*. In the case where the fiberwise differential operators have order  $\leq 2$  (which is what we will ultimately use in [Section II<sub>b</sub>](#)),

this goes by the name *Heat operator* [55]. What is very important for us is the principle that generalized heat operators indeed give an answer to (II<sub>a</sub>.1):

A generalized heat operator induces a connection on the family of higher cohomology group  $H^i(X_s, \mathcal{L}_s)$

or, more formally, a connection on the higher direct images  $R^q\pi_*\mathcal{L}$ . This is the principle of [55, §2.3.3] and [61, §2.2]. At an abstract level, this is nothing more than the fact that  $\mathcal{D}_Z$  defines an endomorphism of the sheaf  $\mathcal{L}$  (albeit not one that is linear over functions) and thereby induces a self-map on  $R^s\pi_*\mathcal{L}$  that satisfies the Leibniz rule due to (II<sub>a</sub>.5); explicitly speaking, we choose Čech cocycle representatives and apply  $\mathcal{D}_Z$  to the coefficients.

We say that a generalized heat operator is *flat* if  $Z \mapsto \mathcal{D}_Z$  respects commutators, and that it is *projectively flat* if it respects commutators up to zeroth order operators. Flatness means that the induced connection on the cohomology groups  $H^q(X_s, \mathcal{L}_s)$  has trivial local holonomy; projective flatness means that the induced connection has scalar local holonomy.

In the remainder of the section, we will explain how to carry this out; the final results are given in §II<sub>a</sub>.5. We shall suppose that the base  $S$  is affine over a field of characteristic zero, and its tangent bundle is trivial. (We will discuss lifting this assumption in Section II<sub>b</sub>, but in our eventual application we will do the globalization “by hand.”)

**II<sub>a</sub>.4. Differential operators on line bundles.** Begin, as in §II<sub>a</sub>.1, by constructing lifts  $\tilde{Z}_i$  locally of  $Z$  to  $X = \bigcup X_i$ , where  $\mathcal{L}$  is trivialized on each  $X_i$ . We seek to construct  $\mathcal{D}_Z$  by modifying the  $\tilde{Z}_i$  by an “along-the-fibers” differential operator  $D_i^*$  of *second order*, just as we did in the example of §II<sub>a</sub>.2. The point is to arrange that in the overlaps  $X_i \cap X_j$ , the difference  $D_i^* - D_j^*$  be a first order operator that exactly matches  $\tilde{Z}_j - \tilde{Z}_i$  so that they cancel out and the  $\tilde{Z}_i + D_i^*$  glue together to make a well-defined operator  $\mathcal{D}_Z$ .

To carry this out, we must recall a little about how differential operators change under the changes of coordinates involved in passing between charts and trivializations of  $\mathcal{L}$ . This section is devoted to that general purpose, and we return to the construction of  $\mathcal{D}_Z$  in the next section. Since the foundational discussion of gluing local differential operators on trivializing charts is orthogonal to the complication added by the relative nature of our situation, in this section we will consider only the situation of a single smooth variety  $X/k$ . When considering differential operators of order  $n$ , we always assume unless otherwise indicated that  $n!$  is invertible in  $k$ . We return to the relative context  $X/S$  in the next section.

Following the definition of a differential operator from the previous section, fixing  $X = \bigcup X_i$  and a trivialization of  $\mathcal{L}$  on each  $X_i$ , to specify  $\mathcal{D}_Z$  in local coordinates as we proposed above, we must give

- an ordinary differential operator  $D_i$  on each open set  $X_i$ , such that

- $D_j = D_i^{\alpha_{ij}}$ , where  $\alpha_{ij} \in H^0(X_i \cap X_j, \mathcal{O}_X^\times)$  is the transition function between set  $i$  and set  $j$  and we write  $\mathcal{D}^{\alpha_{ij}}$  for the operator  $f \mapsto \alpha_{ij}^{-1} \mathcal{D}(\alpha_{ij} f)$ .

**Example II<sub>a</sub>.2.** Let us analyze, for example, what it means to specify, by means of gluing local expressions, a differential operator of order  $\leq 1$  on a line bundle  $\mathcal{L}$  on the smooth variety  $X/k$ .

On  $X_i$ , after trivializing  $\mathcal{L}|_{X_i}$ , the operator  $D$  has the form  $f \mapsto Y_i f + a_i f$ , where  $Y_i$  is a vector field and  $a_i$  is a regular function. For all regular functions  $b$  on an arbitrary Zariski open subset of  $X$ , the definition of “order-1” says that there is a unique local function  $y_b$  on  $X$  such that

$$(D \circ b - b \circ D)s = y_b \cdot s \quad \text{for all local sections } s \text{ of } \mathcal{L}.$$

Moreover,  $b \mapsto y_b$  is a derivation, so in fact there is a global vector field  $Y$ , called the *symbol* of  $D$ , with the property that  $y_b = \langle Y, db \rangle$  for all local functions  $b$ . Restricting to  $X_i$ , one observes that  $Y_i$  has an intrinsic characterization via

$$(D \circ b - b \circ D)|_{X_i} = \langle Y_i, db \rangle \quad \text{for all local functions } b.$$

In particular, the local vector field  $Y_i$  must be the restriction to  $X_i$  of the global vector field  $Y =: \text{Symb}^1(D)$ , which satisfies

$$(II_a.6) \quad Y^b - Y = \langle Y, d \log b \rangle.$$

and therefore

$$(II_a.7) \quad D^b - D = \langle Y, d \log b \rangle$$

as well. For the  $D_i$  to glue together to give a differential operator on  $\mathcal{L}$ , we would need

$$(II_a.8) \quad D_i^{\alpha_{ij}} - D_j = 0 \text{ on } X_i \cap X_j \text{ for all } i, j.$$

The difference between the two local expressions when restricted to  $X_i \cap X_j$  using one of the trivializations is

$$\begin{aligned} (D_i^{\alpha_{ij}} - D_j)(f) &= (Y^{\alpha_{ij}} - Y)f + (a_i - a_j)f. \\ &= \langle Y, (d \log \alpha_{ij}) \rangle f + (a_i - a_j)f. \end{aligned}$$

Therefore, to give a first order differential operator on  $\mathcal{L}$  by local expressions is the same as giving a global vector field  $Y$  – specifying the symbol of the operator – together with a collection of regular functions  $a_i$  on each open set of a trivializing covering for  $\mathcal{L}$  that satisfy

$$(II_a.9) \quad a_i - a_j = \langle Y, d \log \alpha_{ij} \rangle.$$

Not every vector field  $Y$  is the symbol of a differential operator on  $\mathcal{L}$ . The right-hand side of (II<sub>a</sub>.9), which a priori defines a Čech 1-cocycle for  $H^1(X, \mathcal{O}_X)$ , must be trivial.

We want to generalize this observation to higher order operators in a moment; let us first rephrase it in cohomological language. The collection  $\frac{d\alpha_{ij}}{\alpha_{ij}}$  of 1-forms define a cocycle in  $H^1(\Omega^1)$  which is an incarnation of the Chern class of the bundle  $\mathcal{L}$ , and we denote it by  $[\mathcal{L}]$ . Manifestly, it is the image of the defining class of  $\mathcal{L}$  in  $H^1(X, \mathcal{O}_X^\times)$  under the map induced by  $\text{dlog} : \mathcal{O}_X^\times \rightarrow \Omega_{X/k}^1$ . Cup product with  $[\mathcal{L}]$  defines a morphism

$$(II_a.10) \quad H^0(X, T_{X/k}) \rightarrow H^1(X, \mathcal{O}_X)$$

from global sections of the tangent bundle, to the first cohomology of the structure sheaf. In the context of [Example II\\_a.2](#), the symbol  $Y$  is in  $H^0(X, T_{X/k})$ , and the Čech cocycle given by  $\langle Y, \text{dlog} \alpha_{ij} \rangle$  is precisely the image of  $Y$  under the map of [\(II\\_a.10\)](#). In this language, we can restate the observation of [Example II\\_a.2](#): the vector field  $Y$  is the symbol of a differential operator on the line bundle  $\mathcal{L}$  if (and only if)  $Y$  lies in the kernel of this map [\(II\\_a.10\)](#).

The above is nothing but an explication of the common manoeuvre (present in all of the references we have mentioned on this subject) of setting up the “symbol exact sequence”

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{D}\text{iff}_{X/k}^1(\mathcal{L}) \xrightarrow{\text{Symb}^1} T_{X/k} \rightarrow 0$$

( $\mathcal{D}\text{iff}_{X/k}^1$  being the sheaf of order-1 differential operators on  $\mathcal{L}$  defined by sheafifying the definition of differential operator — again see [\[25, §16.8\]](#)) and using the fact that the obstruction to a global vector field  $Y \in H^0(X, T_{X/k})$  being in the image of  $\text{Symb}^1$  is precisely its image under the connecting homomorphism to  $H^1(X, \mathcal{O}_X)$ , which is just the negation of the map [\(II\\_a.10\)](#).

It is useful to note that the map [\(II\\_a.10\)](#) has the following informal interpretation: it is the derivative of the map

$$\text{automorphisms of } V \longrightarrow \text{line bundles on } V$$

which is given<sup>14</sup> by sending  $g$  to  $g^*\mathcal{L}$ . From this interpretation it follows that [\(II\\_a.10\)](#) vanishes when the line bundle  $\mathcal{L}$  is the canonical bundle  $\omega := \det \Omega_{X/k}^1$ , or any power thereof; for, as the name indicates, the canonical bundle is preserved under pullbacks by an automorphism.

Finally, for a differential operator  $D$  of order  $\leq n$ , there is a standard notion of *symbol* of  $D$  as well [\[25, §16.8\]](#), which is a global section of  $\text{Sym}^n T_{X/k}$  and retains its interpretation as being the global product of vector fields which always gives the highest-order term of the local expressions for  $D$ :

**Definition II\_a.3.** Let  $D$  be a differential operator of order  $\leq n$  on  $\mathcal{L}$ . Its *symbol* is a global section  $\text{Symb}^n(D)$  of  $\text{Sym}^n T_X$  characterized by the property

$$\sum_{I \subset \{1, \dots, n\}} (-1)^{|I|} \left( \prod_{i \in I} b_i \right) \circ D \circ \left( \prod_{i \notin I} b_i \right) = \langle \text{Symb}^n(D), db_1 \otimes \cdots \otimes db_n \rangle$$

<sup>14</sup>The example of this principle that will be relevant to us is when  $X$  is an abelian variety, and  $X \rightarrow X^\vee$  the usual polarization map given on points by  $x \mapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$ . The derivative of this map is the map  $\text{Lie}(X) = H^0(X, T_{X/k}) \rightarrow \text{Lie}(X^\vee) = H^1(X, \mathcal{O}_X)$  given by precisely the map  $-\cup [\mathcal{L}]$ .

for all local regular functions  $b_1, \dots, b_n$ . Here the convention on the pairing between  $\text{Sym}^n T_{X/S}$  and  $(\Omega_{X/S}^1)^{\otimes n}$  is that of (II<sub>0</sub>.1).

And the symbol generally satisfies a change-of-coordinate formula generalizing (II<sub>a</sub>.7):

**Lemma II<sub>a</sub>.4.** *Let  $D$  be a differential operator of order  $\leq n$  on  $\mathcal{L}$ , and let  $\alpha$  be an invertible regular function on  $X$ . Then,*

$$\text{Symb}^n(D^\alpha - D) = 0$$

(so  $D^\alpha - D$  is of order  $\leq n - 1$ ) and

$$\text{Symb}^{n-1}(D^\alpha - D) = \langle \text{Symb}^n D, d\log \alpha \rangle.$$

*Proof.* The first part follows from the definition of order- $n$  differential operator. To prove the second part, we start with the characterization from (II<sub>0</sub>.2) of the section of  $\langle \text{Symb}^n(D), d\log \alpha \rangle$  of  $T_{X/k}^{\otimes(n-1)}$  by defining its value on  $db_1 \otimes \dots \otimes db_{n-1}$  to be given by

$$\langle \text{Symb}^n(D), d\log \alpha \rangle, db_1 \otimes \dots \otimes db_{n-1} \rangle = \langle \text{Symb}^n(D), (d\log \alpha) \otimes db_1 \otimes \dots \otimes db_{n-1} \rangle.$$

All we have to do is verify that  $\text{Symb}^{n-1}(D^\alpha - D)$  satisfies the same property.

By applying  $D^\alpha - D = \alpha^{-1}(D\alpha - \alpha D)$  and using the fact that the expression appearing in Definition II<sub>a</sub>.3 is just repeated application of commutator with  $b_i$ ,

$$\begin{aligned} \langle \text{Symb}^{n-1}(D^\alpha - D), db_1 \otimes \dots \otimes db_{n-1} \rangle &= \alpha^{-1} \sum_{I \subset \{1, \dots, n-1\}} (-1)^{|I|} \left( \prod_{i \in I} b_i \right) \circ (D\alpha - \alpha D) \circ \left( \prod_{i \notin I} b_i \right) \\ &= \alpha^{-1} \langle \text{Symb}^n D, d\alpha \otimes db_1 \otimes \dots \otimes db_{n-1} \rangle \\ &= \langle \text{Symb}^n D, d\log \alpha \otimes db_1 \otimes \dots \otimes db_{n-1} \rangle \end{aligned}$$

as desired. □

Having appropriately generalized the tools used in its proof, here is the promised generalization, then, of Example II<sub>a</sub>.2 and especially the remark after (II<sub>a</sub>.9) to higher degree operators.

**Lemma II<sub>a</sub>.5.** *In order for a function  $\sigma$  on the cotangent bundle of  $X$ , homogeneous of degree  $n$  along fibers, to be the symbol of a differential operator of order  $n$  on  $\mathcal{L} \otimes \omega^{1/2}$  it is necessary that  $\sigma$  lie in the kernel of the map*

$$(II_{a.11}) \quad [\mathcal{L}]_n : H^0(\text{Sym}^n T) \rightarrow H^1(\text{Sym}^{n-1} T)$$

*obtained by taking the cup product with the class of  $\mathcal{L}$  in  $H^1(\Omega_{X/k}^1)$ .*

In order to understand the meaning of this lemma, we must first discuss the meaning of  $\omega^{1/2}$  here. But at a first reading, we might suggest the reader simply pretend that  $\omega$  has a square root and skip over this technicality.

The crucial point to observe is that for a differential operator  $D$  on a sheaf that is well-defined such as the structure sheaf,  $D^\alpha$  can be computed given knowledge of  $D$  in local coordinates on  $X_i$  and  $\frac{d\alpha}{\alpha}$  – we do not need  $\alpha$  itself. For example, an order-0 differential operator on  $\mathcal{L}$  is still just a section of  $\mathcal{O}_X$ ; and we already saw that an order-1 differential operator  $D$  of symbol  $\text{Symb}^1(D) = Y$  can be defined to be a collection of regular functions  $a_i$  on  $X_i$  such that on  $X_i \cap X_j$ ,

$$a_i - a_j = Y^{\alpha_{ij}^{1/2}} - Y = \langle Y, \text{dlog}(\alpha_{ij}^{1/2}) \rangle = \frac{1}{2} \langle Y, \text{dlog} \alpha_{ij} \rangle,$$

where the last quantity is well-defined as long as  $1/2$  is, salvaging the fact that the others might not be well-defined if  $\alpha_{ij}^{1/2}$  is not. We can define operators of higher order  $n$  by similarly gluing local expressions, using the fact that the change-of-coordinates formula from  $D$  to  $D^\alpha$  only uses  $\text{dlog} \alpha$  instead of actually needing  $\alpha$ . More precisely, if  $X = \cup X_i$  is a covering of  $X$  that trivializes  $\mathcal{L}$ , then it also trivializes “ $\mathcal{L}^{1/2}$ ” with transition functions “ $\alpha_{ij}^{1/2}$ .” Given differential operators  $D_i$  on  $\mathcal{L}^{1/2}|_{X_i} \simeq \mathcal{O}_{X_i}$  (a well-defined concept since  $\mathcal{O}_{X_i}$  is a well-defined concept), analogously to (IIa.8), the condition that is required for the collection  $(D_i)_i$  to glue together to give an order- $n$  operator on  $\mathcal{L}$  is that

$$(II_a.12) \quad D_i^{\alpha_{ij}^{1/2}} - D_j = 0.$$

We assert that  $D_i$  acts on sections  $f$  of  $\mathcal{O}_{X_i}$  by

$$(II_a.13) \quad D_i(f) := Y_i^{(n)} f + Y_i^{(n-1)} f + \cdots + Y_i^{(0)} f,$$

where  $Y_i^{(n)} \in H^0(X_i, T_{X/k}^{\otimes n})$  are viewed as acting on functions by sequential differentiation. To prove this assertion, by induction, it suffices to show that we can find  $Y_i^{(n)} \in H^0(X_i, T_{X/k}^{\otimes n})$  such that  $\text{Symb}^n(Y_i^{(n)}) = \text{Symb}^n(D_i)$  where the  $Y_i^{(n)}$  is interpreted as an operator  $f \mapsto Y_i^{(n)} f$  in the previous sentence, by successive differentiation. If we consider any local section  $Y^{(n)}$  of  $T_{X/k}^{\otimes n}$ , then the symbol of the operator  $f \mapsto Y^{(n)} \cdot f$  is  $(n!)Y^n$  viewed in the quotient  $\text{Sym}^n T_{X/S}$  using the convention of Section II<sub>0</sub>, as one can see on pure tensors  $Y^{(n)} = Z_1 \cdots Z_n$  (with  $Z_i$  vector fields) from the computation

$$\begin{aligned} \langle \text{Symb}^n(Y^{(n)}), db_1 \cdots db_n \rangle &= \sum_{I \subset \{1, \dots, n\}} (-1)^I \left( \prod_{i \in I} b_i \right) \circ Z_1 \cdots Z_n \circ \left( \prod_{i \notin I} b_i \right) \\ &= \sum_{\sigma \in S_n} \prod_{i=1}^n \langle Z_{\sigma(i)}, db_i \rangle \\ &= \left\langle \sum_{\sigma \in S_n} Z_{\sigma(1)} \cdots Z_{\sigma(n)}, db_1 \otimes \cdots \otimes db_n \right\rangle. \end{aligned}$$

The second line comes from the fact that each commutator of  $Z_1 \cdots Z_n$  with a  $b_i$  is a sum over  $\sigma$  of terms  $\langle Z_{\sigma(i)}, db_i \rangle Z_1 \cdots \widehat{Z}_i \cdots Z_n$ .

Since  $n!$  is invertible in  $k$ , we can take  $Y_i^{(n)} = \frac{1}{n!} \text{Symb}^n(D_i)$ , thus proving that  $D_i$  can be written in the form (IIa.13)

But this means we can express the gluing condition (II<sub>a</sub>.12) *without* needing an explicit square root of  $\alpha_{ij}$  to exist:

$$D_i^{\alpha_{ij}^{1/2}} - D_j = \sum_{\ell=0}^n (Y_i^{(\ell)})^{\alpha_{ij}^{1/2}} - Y_j^{(\ell)},$$

and by writing  $Y_i^{(\ell)}$  as a linear combination of pure tensors  $Z_1 \cdots Z_\ell$ , we already know from (II<sub>a</sub>.6) that

$$(Z_1 \cdots Z_\ell)^{\alpha_{ij}^{1/2}} = Z_1^{\alpha_{ij}^{1/2}} \cdots Z_\ell^{\alpha_{ij}^{1/2}} = (Z_1 + \langle Z_1, \frac{1}{2} \text{dlog} \alpha_{ij} \rangle) \cdots (Z_\ell + \langle Z_\ell, \frac{1}{2} \text{dlog} \alpha_{ij} \rangle).$$

Thus, so long as  $2(n!)$  is invertible in  $k$ , we can rewrite the gluing condition (II<sub>a</sub>.12) without assuming the  $\alpha_{ij}^{1/2}$  exist, and therefore define the notion of differential operator of order  $n$  on  $\mathcal{L}^{1/2}$  with symbol  $Y \in H^0(X, \text{Sym}^n(T_{X/k}))$ , even if  $\mathcal{L}^{1/2}$  is not yet defined. This gives a meaning to Lemma II<sub>a</sub>.5: if  $\alpha_{ij}$  are the transition functions for  $\mathcal{L}$  and  $\beta_{ij}$  are the transition functions for  $\omega$  on a common trivializing cover, then the definition of a differential operator of order  $n$  on  $\mathcal{L} \otimes \omega^{1/2}$  does not require us to define  $\alpha_{ij}\beta_{ij}^{1/2}$ , only  $\text{dlog} \alpha_{ij} + \frac{1}{2} \text{dlog} \beta_{ij}$ .

We remark that we will only really need the definition of a differential operator of order  $n \geq 2$ , so we only ever need to invert 2. The order-3 operators that are relevant for studying the curvature are obtained from order-2 operators by taking commutators and are therefore well-defined objects even if the notion of order-3 operator is not in general. So the only prime that we genuinely need to invert for the purpose of considering differential operators is 2. We have to invert 2 anyway to have a smooth integral model of the curve  $Y^*$  and to have a notion of differential operators on  $\mathcal{L} \otimes \omega^{1/2}$ . Therefore we need not further optimize the characteristic in the discussion of differential operators.

We need one more concept to prove the Lemma. On a real manifold, the adjoint of a differential operator  $D$  is the differential operator  $D^*$  satisfying

$$\int (Df) \cdot g = \int f \cdot (D^*g)$$

For this to make sense without further choices, we should regard  $g$  as a volume form; thus the adjoint operator naturally acts on volume forms. Pondering this, one arrives at a similar theory in the algebraic context (which is standard in the theory of algebraic  $\mathcal{D}$ -modules [3, 4], though we need not use that language and in fact prefer to work in local charts anyway in order to best accommodate the definitions we have taken for differential operators on non-integral powers of line bundles): for a differential operator  $D$  on the line bundle  $\mathcal{L}$  we can form an adjoint differential operator  $D^\dagger$  on the line bundle  $\omega \otimes \mathcal{L}^{-1}$ . The adjoint is determined by the properties

- $(D_1 D_2)^\dagger = D_2^\dagger D_1^\dagger$
- For a differential operator  $D$  on  $\mathcal{O}_X$  given by vector field  $Y$ ,  $D^\dagger$  is given by the action of the Lie derivative by  $-Y$  on  $\omega$ .
- $a^\dagger = a$  whenever  $a$  is a regular function.

which comes directly from integration by parts in the analytic setting, but is defined completely algebraically. We can check purely algebraically that this is a well-defined concept: since  $D$  is always locally a polynomial of vector fields, this uniquely determines  $D^\dagger$  on a trivializing cover for  $\mathcal{L}$ . One can verify that it agrees on overlaps. For example, if  $D_i = Y_i + a_i$  as in [Example II<sub>a</sub>.2](#), then

$$\begin{aligned} (D_i^\dagger)^{\alpha_{ij}} - D_j^\dagger &= \alpha_{ij}^{-1} \mathcal{L}_{-Y_i} \alpha_{ij} - \mathcal{L}_{-Y_i} + a_i - a_j \\ &= \langle \text{dlog} \alpha_{ij}, -Y_i \rangle + a_i - a_j \\ &= 0 \end{aligned}$$

using the fact that  $\omega$  consists of top forms, so Cartan's formula defining  $\mathcal{L}_{-Y_i}$  simplifies to only one term. We apologize for the abusive use of the symbol  $\mathcal{L}$  here — this is the only time it will appear as referring to the Lie derivative. Since all of this is a local calculation involving ultimately only  $\text{dlog}$  of transition functions, we have no problem applying it to half-integral powers of line bundles. So if  $D$  is a differential operator on the line bundle  $\omega^{1/2}$ , its adjoint  $D^\dagger$  is also a differential operator on  $\omega^{1/2}$ . If  $D$  has degree  $n$ , the local formulas imply

$$(II_{a.14}) \quad \text{Symb}^n(D^\dagger) = (-1)^n \text{Symb}^n(D).$$

We say that  $D$  is *self-adjoint* if  $D^\dagger = (-1)^n D$ , which only makes sense if  $D$  is a differential operator on  $\omega^{1/2}$ . The adjoint also satisfies the property

$$(II_{a.15}) \quad (D^\dagger)^\dagger = D,$$

as can be checked in the local coordinates. The possibility of choosing self-adjoint operators locally on  $\omega^{1/2}$  is very useful, and we can use it to give the proof of [Lemma II<sub>a</sub>.5](#).

*Proof.* (of [Lemma II<sub>a</sub>.5](#)):

Take  $\sigma$  a global section of  $\text{Sym}^n T_{X/k}$  on  $X$ . On a cover  $X = \cup_i X_i$  by open affines trivializing  $\mathcal{L}$ , we can lift  $\sigma$  to an order- $n$  differential operator  $D'_i$  acting on  $\omega^{1/2}|_{X_i}$ . By taking

$$D_i := \frac{1}{2}(D'_i + (-1)^n (D'_i)^\dagger),$$

[\(II<sub>a</sub>.14\)](#) and [\(II<sub>a</sub>.15\)](#) imply that  $D_i$  is a *self-adjoint* differential operator  $D_i$  acting on  $\omega^{1/2}$  with the property that  $\text{Symb}^n(D_i) = \sigma$ . The advantage of *self-adjoint* here is that the operators  $D_i$  and  $D_j$  on  $X_i \cap X_j$  differ only by an operator of degree  $\leq n-2$ , for the difference  $D_i - D_j$  has order  $\leq n-1$  (as the order- $n$  symbol of  $D_i - D_j$  is  $\text{Symb}^n(D_i) - \text{Symb}^n(D_j) = \sigma - \sigma = 0$ ) and is also self-adjoint: so if  $D_i - D_j$  actually did have order  $n-1$ , then we would have by [\(II<sub>a</sub>.14\)](#) and then [\(II<sub>a</sub>.15\)](#)

$$\text{Symb}^{n-1}(D_i - D_j) = (-1)^{n-1} \text{Symb}^{n-1}((D_i - D_j)^\dagger) = (-1)^{2n-1} \text{Symb}^{n-1}(D_i - D_j),$$

and thus  $\text{Symb}^{n-1}(D_i - D_j) = 0$  as well.

The condition for the operators  $D_i$  to glue together into a global operator on  $\mathcal{L} \otimes \omega^{1/2}$  (which necessarily would have symbol  $\sigma$ ) is that

$$D_i^{\alpha_{ij}} - D_j = 0 \text{ on } X_i \cap X_j$$

for all  $i, j$  where  $\alpha_{ij}$  are the transition functions of  $\mathcal{L}$  induced by our choices of trivializations.

In order that there exist an order- $n$  differential operator on  $\mathcal{L} \otimes \omega^{1/2}$  with the symbol  $\sigma$ , we must be able to modify the  $D_i$  by lower order terms  $E_i$  of order  $\leq n-1$  so that

$$(II_a.16) \quad (D_i + E_i)^{\alpha_{ij}} - D_j - E_j = 0 \text{ on } X_i \cap X_j.$$

Our goal is just to derive a consequence of this, so we don't mind losing information by applying  $\text{Symb}^{n-1}$  to (II\_a.16) (this is valid because the order of the left hand side of (II\_a.16) is  $\leq n-1$  since  $E_i, E_i^{\alpha_{ij}}$  are of order  $\leq n-1$  and  $D_i^{\alpha_{ij}} - D_j$  is of order  $\leq n-1$  as  $\text{Symb}^n(D_i) = \text{Symb}^n(D_i^{\alpha_{ij}}) = \sigma$ ). Doing that yields: if  $\sigma$  is the symbol of an operator of order  $n$  on  $\mathcal{L} \otimes \omega^{1/2}$  which agrees after trivializing  $\mathcal{L}|_{X_i}$  with self-adjoint  $D_i$ , then there are operators  $E_i$  on  $\omega^{1/2}|_{X_i}$  of order  $\leq n-1$  with the property that

$$\begin{aligned} 0 &= \text{Symb}^{n-1}(D_i^{\alpha_{ij}} + E_i^{\alpha_{ij}} - D_j - E_j) \\ &= \text{Symb}^{n-1}(D_i^{\alpha_{ij}} - D_j) + \text{Symb}^{n-1}(E_i^{\alpha_{ij}}) - \text{Symb}^{n-1}(E_j) \\ &= \text{Symb}^{n-1}(D_i^{\alpha_{ij}} - D_i) + \text{Symb}^{n-1}(D_i - D_j) + \text{Symb}^{n-1}(E_i) - \text{Symb}^{n-1}(E_j) \\ &= \langle \sigma, \text{dlog} \alpha_{ij} \rangle + \text{Symb}^{n-1}(E_i) - \text{Symb}^{n-1}(E_j). \end{aligned}$$

where the last line follows from Lemma II\_a.4. In other words, it must be the case that the Čech 1-cocycle  $X_i \cap X_j \mapsto -\langle \sigma, \text{dlog} \alpha_{ij} \rangle$ , which by definition represents the class of  $-[\mathcal{L}]_n \sigma$ , is split (by the cochain  $(\text{Symb}^{n-1}(E_i))_i$ ). Hence the class  $[\mathcal{L}]_n \sigma$  vanishes, as desired.  $\square$

We note that our Lemma II\_a.5 can be phrased more abstractly as the calculation of the induced map on cohomology of  $\text{Symb}^{n-1}$  applied to the obstruction in  $H^1(X, \mathcal{D}iff^{n-1}(\mathcal{L} \otimes \omega^{1/2}))$  of the existence of a lift of  $\sigma \in H^0(X, \text{Sym}^n T_{X/k})$  coming from the first connecting homomorphism of the long exact sequence in cohomology coming from the short exact sequence

$$0 \rightarrow \mathcal{D}iff_{X/k}^{n-1}(\mathcal{L} \otimes \omega^{1/2}) \rightarrow \mathcal{D}iff_{X/k}^n(\mathcal{L} \otimes \omega^{1/2}) \rightarrow \text{Sym}^n T_{X/k} \rightarrow 0.$$

As such, it is a descendant of (and essentially equivalent to) the explicit cocycle calculations in [60, Lemma 1.16] and especially [50, Proposition 3.6.1]. The only major difference is that we have modified the role of the bundle  $\omega^{1/2}$  to always have it be attached to  $\mathcal{L}$  as opposed to being an artifact of calculations involving  $\mathcal{L}$ . This is psychologically relevant in our paper because we will use the relative version of Lemma II\_a.5 to upgrade the *projective* connection on  $\mathcal{L}$  constructed by [32, 50, 55, 60] to a genuine connection on  $\mathcal{L} \otimes \omega^{1/2}$ .

**II\_a.5. The main statement.** We return now to the question of constructing a heat operator, in the sense of §II\_a.3, for  $X/S$ , i.e. to associate to each vector field  $Z$  on the base a differential

operator  $\mathcal{D}_Z$  on  $X$  that lifts  $Z$ . In this setting we get a morphism

$$T_s S \rightarrow H^1(X_s, T_{X/S})$$

from the tangent space on the base, to the first cohomology of the fibers. Let us call the *deformation subspace* for  $X/S$  the subspace of  $H^1(X_s, T_{X/S})$  that is the image of  $T_s$ , i.e. the subspace of the full deformation space that is realized by actual deformations in the family  $S$ .

We also need to repeat some of our previous discussion “relative to  $S$ ,” and we might as well assume that  $S$  is affine. Associated to the line bundle  $\mathcal{L}$  is a class in the first cohomology  $R^1\pi_*(\Omega_{X/S}^1) = H^1(X, \Omega_{X/S}^1)$ . which is dlog of the corresponding class in  $H^1(X, \mathcal{O}_X^\times)$ ; and using the projection  $\Omega^1 \rightarrow \Omega_{X/S}^1$  to the *relative* cotangent bundle, we get a class  $[\mathcal{L}] \in H^1(\Omega_{X/S}^1) = R^1\pi_*(\Omega_{X/S}^1)$  (by abuse of notation we will use the same notation for it as before). Now  $\Omega_{X/S}^1$  has a pairing with  $\text{Sym}^n T_{X/S}$  valued in  $\text{Sym}^{n-1} T_{X/S}$  (see [Section II<sub>0</sub>](#)) giving rise to a relative version of our prior connecting maps:

$$[\mathcal{L}]_n : \pi_* \text{Sym}^n T_{X/S} \rightarrow R^1\pi_* \text{Sym}^{n-1} T_{X/S}.$$

which specializes at fibers at all points  $s \in S$  to the map

$$[\mathcal{L}]_n : H^0(X_s, \text{Sym}^n T_{X/S}) \rightarrow H^1(X_s, \text{Sym}^{n-1} T_{X/S})$$

from the previous section. Recall that for every statement involving  $\text{Sym}^n$ , we are assuming that  $n!$  is invertible in  $\mathcal{O}_S$ .

We define the line bundle  $\omega_{X/S} := \det \Omega_{X/S}^1$  on  $X$ .

**Proposition II<sub>a</sub>.6.** *Suppose  $\pi : X \rightarrow S$  is smooth projective,  $S$  is affine with trivial tangent bundle, and that 2 is invertible in  $\mathcal{O}_S$ . Assume, additionally, that<sup>15</sup>*

(I)  $[\mathcal{L}]_1$  is an isomorphism.

(II)  $[\mathcal{L}]_2$  is injective and its image contains the deformation subspace for  $X/S$

Then there exists a heat operator  $Z \mapsto \mathcal{D}_Z$  on the half-form twist  $\mathcal{L} \otimes \omega_{X/S}^{1/2}$  of  $\mathcal{L}$  where  $\mathcal{D}_Z$  differs from a local lift of  $Z$  by an operator of order  $\leq 2$  along fibers. This heat operator is unique up to the adjustment  $\mathcal{D}_Z \leftarrow \mathcal{D}_Z + \langle \nu, Z \rangle$  where  $\nu$  is a 1-form on the base.

If we assume, moreover that 3 is invertible in  $\mathcal{O}_S$  and that

(III)  $[\mathcal{L}]_3$  is injective

then the resulting heat operator is also projectively flat, that is to say,  $[\mathcal{D}_Y, \mathcal{D}_Z] - \mathcal{D}_{[Y, Z]}$  is a zeroth order operator for any two vector fields  $Y, Z$ .

In the statement, we have used the analogue of our prior discussion after the statement of [Lemma II<sub>a</sub>.5](#) about twisting for heat operators. Namely, we can in an evident way generalize the notion of “heat operator on a line bundle  $\mathcal{L}$  on  $X/S$ ” to “heat operator on  $\mathcal{L} \otimes \eta^{1/2}$ ” for any line bundle  $\eta$ .

<sup>15</sup>In words: these conditions say that (I) fiberwise, any infinitesimal deformation of  $\mathcal{L}$  arises uniquely from a vector field (II) the class of any deformation of a fiber  $X_s$  arises from a section of  $\text{Sym}^2 T_{X/S}$ .

If  $\eta$  is pulled back from  $S$  (which is not necessarily the case in the general situation considered in [Proposition II<sub>a</sub>.6](#), but will be the case when we specialize to the case of abelian schemes), then the notion of heat operator becomes simpler. Indeed, cover  $S$  by small enough affine opens  $S_i$  where  $\eta$  is trivialized. Following the discussion after the statement of [Lemma II<sub>a</sub>.5](#), to give a heat operator on  $\mathcal{L} \otimes \eta$  amounts, after trivializing  $\eta$  on each  $S_i$ , to giving

- a heat operator on  $\mathcal{L}$  on each  $S_i$
- which differ, on overlaps, by the 1-form associated to  $d \log g$ , where  $g$  is the transition function on  $S_i \cap S_j$  relating the trivialization on  $S_i$  and the trivialization on  $S_j$

And to make sense of the same for (e.g.)  $\mathcal{L} \otimes \sqrt{\eta}$  we just replace  $\frac{dg}{g}$  by  $\frac{1}{2} \frac{dg}{g}$ . We will frequently make use of this notion throughout the document – a connection or heat operator on a bundle that is twisted through the square root of an auxiliary bundle. For our purposes, the naive definition above – working coordinate chart by coordinate chart on the base – will always be sufficient, and seems preferable to a more abstract one for our purposes. Actually, it will almost always be harmless to pretend that the auxiliary bundle actually has a square root, and to work with this concrete square root. We encourage the reader to indulge in this pretense.

*Proof.* For simplicity of writing, we shall suppose that  $\omega_{X/S}$  admits a square root  $\omega_{X/S}^{1/2}$ . The general case, however, works exactly the same way, according to the general principles discussed in [§II<sub>a</sub>.4](#).

Let us note, first of all, that  $[\mathcal{L}]_n = [\mathcal{L} \otimes \omega_{X/S}^{1/2}]_n$  for all  $n$ ; indeed,  $[\omega_{X/S}^{1/2}]_n = 0$  for all  $n$ . We saw this for  $n = 1$  after [\(II<sub>a</sub>.10\)](#), but it follows in general by checking fiberwise, where it is an application of [Lemma II<sub>a</sub>.5](#) to the trivial line bundle.

Assumption (II) in the statement of [Proposition II<sub>a</sub>.6](#) means that there is a unique section  $\sigma_2$  of  $\text{Sym}^2 T_{X/S}$  so that  $-\sigma_2$  maps, under  $[\mathcal{L}]_2$ , to the deformation class [\(II<sub>a</sub>.2\)](#). Choose on each  $X_i$  a *self-adjoint* second order fibral differential operator  $D_i$  on  $\omega_{X/S}^{1/2}$  with the symbol  $\sigma_2$ . This is possible on the *affine* set  $X_i$ .<sup>16</sup> On the intersections  $X_i \cap X_j$ , then,  $D_i$  and  $D_j$  agree up to a zeroth order operator.

Following our proof of [Lemma II<sub>a</sub>.5](#), let us fix a trivialization of  $\mathcal{L}$  on each  $X_i$ . This permits us to transport the operators  $D_i$  from half-forms  $\omega^{1/2}$  to  $\mathcal{L} \otimes \omega^{1/2}$ . On the overlap  $X_{ij} = X_i \cap X_j$ , the obstruction to gluing the operators  $D_i$  and  $D_j$  on  $\mathcal{L} \otimes \omega^{1/2}$  is the first order differential operator  $D_i^{\alpha_{ij}} - D_j$ , where  $\alpha_{ij}$  is the transition function between the trivializations of  $\mathcal{L}$  on  $X_i$  and  $X_j$ . Each of these first order “errors” has a symbol, and collectively those symbols define a Čech class within in  $H^1(X, T_{X/S})$ , which the proof of [Lemma II<sub>a</sub>.5](#) shows is given by  $\langle \sigma_2, d \log \alpha_{ij} \rangle$ , i.e.  $[\mathcal{L}]_2 \sigma_2$ . By choice, this is the *negative* of the deformation class

<sup>16</sup>The space of differential operators forms a coherent sheaf, and the symbol map exhibits a surjective map from this sheaf to  $\text{Sym}^n T_{X/S}$ , and over the affine set  $X_i$  we can lift due to vanishing of coherent cohomology.

corresponding to  $Z$ . We will cancel out this obstruction to gluing the  $D_i$  together by adding a lift of  $Z$ .

Begin by fixing an arbitrary lift  $\tilde{Z}_i$  of  $Z$  to a vector field on each  $X_i$  acting on  $\mathcal{L} \otimes \omega^{1/2}$ . These  $\tilde{Z}_i$  do not agree on overlaps  $X_i \cap X_j$ ; rather the  $\tilde{Z}_i - \tilde{Z}_j$  define a cohomology class in  $H^1(X, T_{X/S})$  which agrees with the deformation class corresponding to  $Z$ .

Therefore, the  $D_i + \tilde{Z}_i$ , considered as a differential operator on  $\mathcal{L} \otimes \omega^{1/2}$  on  $X_i$ , differ on overlaps  $X_{ij}$  by first order errors, whose symbols collectively define the trivial class in  $H^1(X, T_{X/S})$ ; that means that, changing the choice of  $\tilde{Z}_i$ , we can arrange that

the operators  $D_i + \tilde{Z}_i$  differ, on overlaps  $X_{ij}$  by a zeroth order differential operator, i.e. by multiplication by a regular function  $a_{ij}$  on  $X_{ij}$ .

The collection of functions  $\{a_{ij}\}$  collectively define a Čech class in  $H^1(X, \mathcal{O})$ . Now by assumption (I),  $[\mathcal{L} \otimes \omega^{1/2}]_1$  induces an isomorphism

$$H^0(X, T_{X/S}) \rightarrow H^1(X, \mathcal{O}).$$

Therefore, the class defined by  $\{a_{ij}\}$  arises from a unique global section of the relative tangent sheaf  $T_{X/S}$ . Modify the  $D_i$  by any associated first order differential operators. Then  $D_i + \tilde{Z}_i$  agree strictly on overlaps, and glue to the desired second order differential operator on the line bundle  $\mathcal{L} \otimes \omega^{1/2}$ .

This  $\mathcal{D}_Z$  is unique up to the addition of a zeroth order operator. Indeed, our construction prescribes its symbol uniquely on each  $X_i$ , so a different choice  $\mathcal{D}'_Z$  would have the property that  $\mathcal{D}'_Z - \mathcal{D}_Z$  is a first-order differential operator on sections of the line bundle  $\mathcal{L} \otimes \omega^{1/2}$ . Now, equation (II<sub>a</sub>.5) implies that  $\mathcal{D}_Z - \mathcal{D}'_Z$  commutes with multiplication by functions on the base  $S$ , which means that the symbol of this operator – a vector field on  $X$  – must actually point along fibers, i.e., defines a section of  $H^0(T_{X/S})$ ; but the injectivity of  $[\mathcal{L} \otimes \omega^{1/2}]_1$  and Lemma II<sub>a</sub>.5 then implies that this symbol is zero.

We now verify the assertion concerning curvature. Fix a linear assignment  $Z \mapsto \mathcal{D}_Z$  as above. Now take vector fields  $Z_1, Z_2$  with associated differential operators  $\mathcal{D}_1, \mathcal{D}_2$ ; the difference

$$[\mathcal{D}_1, \mathcal{D}_2] - \mathcal{D}_{[Z_1, Z_2]}$$

is a third order differential operator on  $\mathcal{L} \otimes \omega^{1/2}$ . It is again a “fibrar” differential operator, that is, it commutes with multiplication by functions on the base. We now iteratively apply Lemma II<sub>a</sub>.5 to check that this operator is of order 0.  $\square$

As with its fiberwise version Lemma II<sub>a</sub>.5, Proposition II<sub>a</sub>.6 is only a small psychological change away from the general relative framework of [50, 55].

*Remark.* If we drop the “smallness” assumption on  $S$  – namely, that  $S$  is affine with trivial tangent bundle – there is no guarantee that the construction globalizes. The obstruction

to doing so is a class in  $H^1(S, \Omega^1)$  which can be computed by means of the Riemann-Roch formula, but we have no need for this; we will “explicitly” globalize it in the case when  $X/S$  is a family of abelian varieties and  $\mathcal{L}$  is a nondegenerate line bundle.

## II<sub>b</sub>. HEAT OPERATORS FOR A FAMILY OF ABELIAN VARIETIES

We now complete step (II) of §4, using the results of §II<sub>a</sub>. Namely, we specialize the discussion to the case of §3.7: when  $X/S$  is the abelian scheme over  $Y^*$  or  $Y^*(\Delta)$  obtained by tensoring the universal elliptic curve with  $B$ , and the line bundle  $\mathcal{L}^B$  is obtained by tensoring the universal line bundle  $\mathcal{L}$  on the universal elliptic curve with the nondegenerate quadratic form  $B$ .

In fact, although in this paper we will only apply it in the case of  $(\mathcal{E}, \mathcal{L}) \otimes B$ , we actually will achieve the refinement of Hitchin’s projective connection to a genuine connection (as promised in step (II)) in the following general situation:

- $\pi : X \rightarrow S$  is an abelian scheme of relative dimension  $g$ . We assume that  $S$  is connected.
- $\mathcal{L}$  is a nondegenerate line bundle on  $X$  with the property that the induced map  $X \rightarrow X^\vee$  induces an isomorphism  $\mathrm{Lie}(X) \rightarrow \mathrm{Lie}(X^\vee)$ .

These conditions are satisfied by  $(\mathcal{E}, \mathcal{L}) \otimes B$ , thanks to Lemma 3.1.

We can let the nonnegative integer  $s$  be the *index* of  $\mathcal{L}$ , namely, the unique integer such that  $\Theta := R^s \pi_* \mathcal{L} \neq 0$  (the proof of Lemma 3.2 shows that this is well-defined, and that  $R^s \pi_* \mathcal{L}$  is a vector bundle). We let  $\Theta_{\mathrm{norm}} := R^s \pi_* \mathcal{L} \otimes (\det e^* \Omega_{X/S}^1)^{1/2}$ , and we write  $\omega = \det e^* \Omega_{X/S}$  for short. Note that this is a line bundle on  $S$ : the bundle that we used to call  $\omega$  is now  $\pi^* \omega$ .

In the specific case of  $(\mathcal{E}, \mathcal{L}) \otimes B$ , Lemma 3.2 further guarantees that  $R^s \pi_* \mathcal{L}$  is a vector bundle of rank  $\mathbf{d}$  and that  $s$  is the negative-definite part of the signature of  $Q$ . Also,  $(\det e^* \Omega_{\mathcal{E}_m/Y}^1)^{1/2} = \omega^{m/2}$ , where  $Y = Y_{\mathbf{Z}^*}$  or  $Y^*(\Delta)$  as usual. This is how one goes from the “ $\omega^{1/2}$ ” in the general statement of where our flat connection lives to the “weight  $m/2$ ” in the consequence about modularity of theta functions.

In the general situation, we will construct

a flat connection on the normalized  $\Theta$ -bundle  $\Theta_{\mathrm{norm}}$ ,

which of course specializes to the desired object in the case of  $(\mathcal{E}, \mathcal{L}) \otimes B$ .

Without great difficulty, we will verify (see §II<sub>b</sub>.3) that the conditions (I) – (III) of Section II<sub>a</sub>.4 are satisfied; thus there is, locally on  $S$ , a projectively flat heat operator on  $\mathcal{L} \otimes (\pi^* \omega)^{1/2}$ , uniquely specified up to modification by a 1-form on  $S$ . Our main goal here is

to *normalize* this, i.e. we uniquely specify it, not just up to a 1-form on the base. It will follow immediately from this that it can be defined globally on  $S$ . Such an object induces a connection on  $R^s\pi_*\mathcal{L} \otimes \omega^{1/2}$ .

**II<sub>b</sub>.1. Setup and notation.** We will introduce some notation to simplify the typography. Introduce the following three vector bundles:

- We denote by  $\Omega^1$  the pushforward  $\pi_*\Omega_{X/S}^1$ , i.e. global 1-forms along the fiber; it is a rank- $g$  vector bundle on the base  $S$  and is identified (by extending to invariant differentials) with the bundle  $Lie(X)^\vee$ .
- $\mathfrak{t}$  for the global sections  $\pi_*T_{X/S}$ , i.e. for the global vector fields along the fiber, or equivalently by technique of extending to invariant vector fields,  $\mathfrak{t} := Lie(X)$ ; it is a rank- $g$  vector bundle on  $S$  dual to  $\Omega^1$ .
- $R^1\pi_*\mathcal{O}$  for the first relative cohomology  $R^1\pi_*\mathcal{O}_X$  of the structure sheaf along  $X/S$ .

We have isomorphisms of vector bundles

$$(II_{b.1}) \quad R^1\pi_*T_{X/S} = R^1\pi_*\pi^*\mathfrak{t} = \mathfrak{t} \otimes R^1\pi_*\mathcal{O}_X = \text{Hom}(\mathfrak{t}^\vee, R^1\pi_*\mathcal{O})$$

(see [9, Ch. 4, Proposition 2], which we have already been using implicitly) and similarly

$$(II_{b.2}) \quad R^1\pi_*\Omega_{A/S}^1 = \text{Hom}(\mathfrak{t}, R^1\pi_*\mathcal{O})$$

.

Fix a vector field  $Z$  defined locally on an affine open  $U \subset S$ . Our goal is to lift it to a differential operator on  $\mathcal{L} \otimes (\det \Omega^1)^{1/2}$  which is locally of the form (lift of  $Z$ ) + (relative operator of order  $\leq 2$ ) in a way that is unique and canonical, so that the construction globalizes.

As a result of (II<sub>b</sub>.1), the Kodaira-Spencer class  $\kappa_{X/S}(Z) \in H^0(U, R^1\pi_*T_{X/S})$  can be viewed as an  $\mathcal{O}_U$ -linear map  $\mathfrak{t}^\vee \rightarrow R^1\pi_*\mathcal{O}$ . As a result of (II<sub>b</sub>.2), the relative fundamental class of the line bundle  $\mathcal{L}$  (defined by applying  $\text{dlog}$  to the class in  $R^1\pi_*\mathcal{O}_X^\times$  that defines  $\mathcal{L}$ ) in  $R^1\pi_*\Omega_{A/S}^1$  can be viewed as an  $\mathcal{O}_S$ -linear map  $[\mathcal{L}] : \mathfrak{t} \rightarrow R^1\pi_*\mathcal{O}$ . It will be very convenient in what follows to always regard  $\kappa_{X/S}(Z), [\mathcal{L}]$  as morphisms in this way, armed with the following mathematical facts about them.

**Lemma II<sub>b</sub>.1.** *The map  $[\mathcal{L}]$  has the following properties:*

- (1) *Under the identification of  $\mathfrak{t}$  with  $\pi_*T_{X/S}$  by invariant vector fields, the map  $[\mathcal{L}]$  agrees with the map  $[\mathcal{L}]_1 : \pi_*T_{X/S} \rightarrow R^1\pi_*\mathcal{O}_X$ .*
- (2) *Under the natural identification of  $R^1\pi_*\mathcal{O}_X$  with  $Lie(X^\vee)$ , the map  $[\mathcal{L}]$  is precisely the map on Lie algebra sheaves induced by the standard map*

$$\phi_{\mathcal{L}} : X \rightarrow X^\vee$$

*given on points by  $x \mapsto T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ .*

*Proof.* (1) is true by the hands-on definition of  $[\mathcal{L}]$ , which is to contract with the 1-cocycle  $d\log \alpha_{ij}$  — it is just a matter of restricting to the identity section. (2) is a standard computation: see for example [40, Proposition 2.1.5.13].  $\square$

Similarly, we will require one further piece of mathematical knowledge about  $\kappa_{X/S}(Z)$ :

**Lemma II<sub>b</sub>.2.** *For all vector fields  $Z$  over the affine  $U \subset S$ , the class  $-\kappa_{X/S}(Z) : \mathfrak{t}^\vee \rightarrow R^1\pi_*\mathcal{O}_X$  is equivalently obtained by considering the Hodge filtration*

$$0 \rightarrow \mathfrak{t}^\vee \xrightarrow{i} H_{dR}^1(X/S) \xrightarrow{p} \mathfrak{t} \stackrel{[\mathcal{L}]}{=} R^1\pi_*\mathcal{O} \rightarrow 0$$

and composing

$$\mathfrak{t}^\vee \xrightarrow{i} H_{dR}^1(X/S) \xrightarrow{\nabla_{GM}(Z)} H_{dR}^1(X/S) \xrightarrow{p} R^1\pi_*\mathcal{O}.$$

Here  $H_{dR}^1(X/S)$  is the rank- $2g$  vector bundle on  $S$  obtained by algebraic de Rham cohomology, and  $\nabla_{GM}$  is the Gauss–Manin connection on  $H_{dR}^1(X/S)$  defined as in [37].

*Proof.* We give a standard reference: combine [35, Proposition 1.4.1.7] and [5, Proposition 2.5.2]. The sign  $-\kappa_{X/S}$  is there due to differing conventions. This is somewhat difficult to track down, because the convention for the cocycle defining the Kodaira–Spencer class in our (II<sub>a</sub>.2) matches exactly with [35, (1.1.3)]. The sign is switched because we have differing conventions about the connecting map on Čech cocycles compared to [35].  $\square$

**II<sub>b</sub>.2. The fundamental torsor.** The goal is to uniquely determine  $D_Z = (\text{lift of } Z) + (\text{section of } \mathcal{D}iff_{X/S}^2(\mathcal{L} \otimes (\pi^*\omega)^{1/2}))$  rather than only up to scalars. We will accomplish this by *normalizing* its values when applied to sections of  $\mathcal{L} \otimes \omega^{1/2}$  near the identity of  $X$ .

Let us consider a section of  $\mathcal{L}$  in some neighbourhood of the identity section of  $A/S$ . We say that such a section is *normalized symmetric* if it is:

- symmetric (for the symmetric structure on  $\mathcal{L}$ ) and
- takes value 1 at the identity (for the rigidification on  $\mathcal{L}$ ).

In fact, this notion makes sense if we identify two sections if they agree up to second order along the identity section of  $X/S$ . Equivalently, we can look only at sections on the second order infinitesimal neighbourhood of the identity section of  $X/S$ : the closed subscheme of  $X/S$  cut out by the square of the ideal defining that identity-section.

**Definition II<sub>b</sub>.3.** Let  $\mathcal{F}$  denote the sheaf on  $S$  of normalized symmetric sections of  $\mathcal{L}$  over the second-order infinitesimal neighborhood of 1. Then  $\mathcal{F}$  forms a torsor under the normalized symmetric sections of the structure sheaf of the second-order infinitesimal neighborhood of the identity section, i.e. the sheaf  $\text{Sym}^2\mathfrak{t}^\vee$  where sections  $\eta_1 \odot \eta_2$  act by multiplication by the corresponding section  $1 + \eta_1 \odot \eta_2$  of  $1 + \text{Sym}^2\mathfrak{t}^\vee \subset \mathcal{O}_{\text{second-order nbhd of } 1}$ . Since it is important for constant factors, we note that here it is convenient to view  $\text{Sym}^2\Omega_{X/S}^1$  as a quotient of the tensor square, and use the isomorphism  $\mathcal{F}^2/\mathcal{F}^3 \simeq \text{Sym}^2\Omega_{X/S}^1$  given by  $f \cdot g \mapsto df \odot dg$ , where

$\mathcal{J}$  is the ideal sheaf in  $\mathcal{O}_X$  defining the identity section, to define the inclusion  $1 + \text{Sym}^2 \mathfrak{t}^\vee$  into the structure sheaf of the order-2 infinitesimal neighborhood of the identity,  $\mathcal{O}_X/\mathcal{J}^3$ .

We denote the action of a local section  $\eta_1 \odot \eta_2$  of  $\text{Sym}^2 \mathfrak{t}^\vee$  (or any linear combination thereof) on a local section  $s$  of  $\mathcal{F} \subset \pi_*(\mathcal{L}|_{\text{second-order nbhd of } 1})$  by

$$(\eta_1 \odot \eta_2) \cdot s := (1 + \eta_1 \odot \eta_2)s.$$

The question of explicitly normalizing a heat operator on  $(A/S, \mathcal{L})$  as we have proposed is closely related to the question of “identifying” the  $\text{Sym}^2 \mathfrak{t}^\vee$  torsor  $\mathcal{F}$  in some explicit way.

Now, the  $S$ -vector bundle  $\text{Sym}^2 \mathfrak{t}^*$  of quadratic forms on  $\mathfrak{t}$  comes with another canonical torsor, namely, the set of Lagrangian splittings of the Hodge exact sequence of  $S$ -vector bundles

$$(II_b.3) \quad 0 \rightarrow \mathfrak{t}^* \rightarrow H_{dR}^1(X/S) \rightarrow (R^1 \pi_* \mathcal{O}^{[\mathcal{L}]^{-1}}) \rightarrow \mathfrak{t} \rightarrow 0$$

where the Lagrangian condition is with respect to the alternating pairing on  $H_{dR}^1(X/S)$  induced by the choice of  $\mathcal{L}$  as in [12].

Indeed, any two such splittings differ by a map  $\mathfrak{t} \rightarrow \mathfrak{t}^*$ , i.e. a class in  $\mathfrak{t}^* \otimes \mathfrak{t}^*$ , and the condition that the splitting remain Lagrangian forces this to be symmetric.

A crucial fact for us is that this torsor is equivalent to the previous one by a map  $\varphi : \mathcal{F} \rightarrow \text{Hom}(\mathfrak{t}, H_{dR}^1(X/S))$ , which is defined according to the following principle: A normalized symmetric section  $s$  of  $\mathcal{L}$  over the second-order infinitesimal neighborhood of the identity determines a Lagrangian splitting of (II\_b.3) in the following way. For  $x$  a point of  $X$ , the quotient  $T_x^*(s)s^{-1}$  gives a section of  $T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$ . The latter bundle is in  $\text{Pic}_e^0(X/S)$  and therefore admits a flat connection, which is unique up to addition of a global section  $\eta$  of  $\Omega_{X/S}^1$ . The form  $\eta$  has to be translation-invariant and is therefore determined by its restriction to the identity, the section  $e^* \eta$  of  $\mathfrak{t}^\vee$ . It is thus possible to *uniquely determine* the choice of flat connection on  $T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$  by requiring  $T_x s \otimes s^{-1}$  be annihilated by it. By associating to  $T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$  a flat connection in this way, and using the fact that  $x \mapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$  is an isomorphism when restricted to a first order infinitesimal neighborhood of the identity, this means that we have associated to  $s$  a section of the universal vector extension (see [42])

$$0 \rightarrow \mathfrak{t}^\vee \rightarrow E^\sharp \rightarrow X^\vee \rightarrow 1$$

on a first order infinitesimal neighborhood of 1 in  $X^\vee$ . Hence after applying the functor  $\text{Lie}$ , we have associated to  $s$  a section  $\varphi(s)$  of (II\_b.3).

The right-section

$$0 \longrightarrow \mathfrak{t}^\vee \longrightarrow H_{dR}^1(X/S) \longrightarrow \mathfrak{t} \longrightarrow 0$$

$\swarrow \varphi(s)$

is equivalent to the data of the left-section

$$(II_b.4) \quad \text{proj}_s : H_{\text{dR}}^1 \rightarrow \mathfrak{t}^\vee, \quad \text{proj}_s(w) := w - (\varphi(s) \circ [\mathcal{L}]^{-1} \circ p)(w).$$

**Lemma II<sub>b</sub>.4.** *The association  $\varphi$  from normalized symmetric sections  $s$  of  $\mathcal{L}$  on a second-order neighborhood of 1 to Lagrangian splittings of (II<sub>b</sub>.3) commutes with the action of  $\text{Sym}^2 \mathfrak{t}^*$  on both sides, in the sense that*

$$(II_b.5) \quad \varphi((1 + \eta_1 \odot \eta_2)s) = \varphi(s) - 2(\eta_1 \odot \eta_2) \text{ in } \text{Hom}(\mathfrak{t}, \mathfrak{t}^\vee)$$

In other words,  $\varphi : \mathcal{F} \rightarrow \{\text{Lagrangian sections of (II<sub>b</sub>.3)}\}$  is an isomorphism of  $\text{Sym}^2 \mathfrak{t}^\vee$ -torsors, so long as we regard the  $\text{Sym}^2 \mathfrak{t}^\vee$  action on the codomain to be by subtracting the corresponding map  $\mathfrak{t} \rightarrow \mathfrak{t}^\vee$ .

*Proof.* It is enough to consider the sections of  $\text{Sym}^2 \mathfrak{t}^\vee$  that arise as squares of linear functionals on  $\mathfrak{t}$ , i.e. pure tensors  $\eta_1 \odot \eta_2$  with  $\eta_1 = \eta_2$ . So let  $\eta$  be a local section of  $\mathfrak{t}^\vee$ , and consider the section  $\eta^2 := \eta \odot \eta$  of  $\text{Sym}^2 \mathfrak{t}^\vee$ . If we change  $s$  to  $(1 + \eta^2)s$ , then  $T_x^*(s)s^{-1}$  is modified for infinitesimal  $x$  (that is, considered as a section of<sup>17</sup>  $\mathfrak{t}$ ) by adding  $2\eta\langle\eta, x\rangle$ , as

$$\begin{aligned} \left. \frac{T_x^*((1 + \eta^2)s)}{(1 + \eta^2)s} \right|_{\text{first-order nbhd of 1}} &= \left. \frac{x\text{-directional derivative of } ((1 + \eta^2)s)}{(1 + \eta^2)s} \right|_{\text{first-order nbhd of 1}} \\ &= \left. \frac{(x\text{-directional derivative of } ((1 + \eta^2)s))}{s} \right|_{\text{first-order nbhd of 1}} \\ &= (x\text{-derivative of } (1 + \eta^2)) + \left. \frac{(1 + \eta^2)(x\text{-derivative of } s)}{s} \right|_{\text{first-order nbhd of 1}} \\ &= 2\langle\eta, x\rangle\eta + \frac{T_x^*s}{s}. \end{aligned}$$

The connection that renders this section flat is correspondingly modified by adding the 1-form  $-2\langle\eta, x\rangle\eta$ . Therefore, for all local sections  $x$  of  $\mathfrak{t}$ ,

$$(II_b.6) \quad \varphi((1 + \eta^2)s)(x) = \varphi(s)(x) - 2\eta\langle\eta, x\rangle.$$

The map  $\mathfrak{t} \rightarrow \mathfrak{t}^\vee$  that takes  $x$  to  $-2\eta\langle\eta, x\rangle$  goes by the name  $-\eta^2 = -\eta \odot \eta \in \text{Sym}^2 \mathfrak{t}^\vee$  (recall the conventions from Section II<sub>0</sub>).  $\square$

**II<sub>b</sub>.3. Applying the results of §II<sub>a</sub>.** We now verify that all of the conditions (I)-(III) of Proposition II<sub>a</sub>.6 apply to  $(X/S, \mathcal{L})$ :

- (I) Both parts of Lemma II<sub>b</sub>.1 and the assumption that  $\phi_{\mathcal{L}}$  induces an isomorphism on Lie algebra sheaves implies that  $[\mathcal{L}]_1$  is an isomorphism.

<sup>17</sup>Believing that it would not make the logic easier to follow, we do not carry out the full detail in which  $x$  is of course viewed as a  $S[\epsilon]$ -point of  $X$  where  $\epsilon^2 = 0$  and everything is calculated taking  $\epsilon$  into account. The reader is invited to verify that such a calculation yields the same result as the slightly more informal one presented here.

- (II) Under the usual isomorphisms using technique of invariant vector fields, the map  $[\mathcal{L}]_2 : \pi_* \text{Sym}^2 T_{X/S} \rightarrow R^1 \pi_* T_{X/S}$  is equivalent to the composite map

$$\text{Sym}^2 \mathfrak{t} \rightarrow \mathfrak{t} \otimes \mathfrak{t} \xrightarrow{1 \otimes [\mathcal{L}]} \mathfrak{t} \otimes R^1 \pi_* \mathcal{O}.$$

This is injective because  $[\mathcal{L}] = [\mathcal{L}]_1$  is injective. That the image of  $[\mathcal{L}]_2$  contains the deformation space of  $X/S$  is a well-known fact, according to [55, §2.3.8].

- (III) The map  $[\mathcal{L}]_3$  is equivalent to the composition

$$\text{Sym}^3 \mathfrak{t} \rightarrow (\text{Sym}^2 \mathfrak{t}) \otimes \mathfrak{t} \xrightarrow{1 \otimes [\mathcal{L}]} \text{Sym}^2 \mathfrak{t} \otimes R^1 \pi_* \mathcal{O}$$

which is again manifestly injective.

Therefore **Proposition II<sub>a</sub>.6** assures that we can construct, locally on  $S$ , a projectively flat heat operator  $Z \mapsto D_Z$ . The symbol  $\Sigma_Z$  of this flat connection lies in

$$(II_{b.7}) \quad \text{Sym}^2 \mathfrak{t} \subset \mathfrak{t} \otimes \mathfrak{t} = \text{Hom}(\mathfrak{t}^\vee, \mathfrak{t}),$$

since  $D_Z$  is given locally by (lift of  $Z$ ) + (section of  $\mathcal{D}iff_{X/S}^2(\mathcal{L} \otimes \omega^{1/2})$ ). Then we have  $[\mathcal{L}] \circ \Sigma_Z = -\kappa_{X/S}(Z)$  as maps  $\mathfrak{t}^\vee \rightarrow R^1 \pi_* \mathcal{O}$  or equivalently

$$(II_{b.8}) \quad \Sigma_Z = -[\mathcal{L}]^{-1} \circ \kappa_{X/S}(Z),$$

because that is how  $\Sigma_Z$  was constructed in the proof of **Proposition II<sub>a</sub>.6** (which also shows that there was no other choice for its symbol).

**II<sub>b</sub>.4. Formula for the heat operator.** Suppose we choose, locally on  $S$ , a normalized section  $s$  as above. This induces a connection on  $\omega$  in the following way: We begin with the  $g$ -th wedge power of the Gauss–Manin connection on  $H_{\text{dR}}^1$  and use the top wedge power of the projection  $\text{proj}_s$  to project to  $\omega = \det \mathfrak{t}^\vee$ . That is to say: we take a local section of  $\omega$ , differentiate it according to Gauss–Manin to get a section of  $H_{\text{dR}}^1$ , and apply  $\text{proj}_s$  to get back to a section of  $\omega$ . We call the resulting connection on  $\omega$  by the name  $\nabla^s$ . In symbols,

$$\nabla_Z^s = \bigwedge^g (\text{proj}_s \circ \nabla_{\text{GM}, Z}).$$

It induces a connection  $\nabla^{s, 1/2}$  on  $\omega^{1/2}$  by sending local tangent vectors  $Z$  on  $S$  to the first-order differential operators on  $\omega^{1/2}$  given locally in horizontal trivializing charts for  $\omega$  by  $\frac{1}{2} \nabla_Z^s$ . This defines a first-order differential operator on  $\omega^{1/2}$  because to define one is to define first-order operators on the structure sheaf of trivializing opens that agree up to  $\frac{1}{2} \text{dlog}(\text{transition functions for } \omega)$  on overlaps.

**Proposition II<sub>b</sub>.5.** *There is a unique heat operator on  $\mathcal{L} \otimes \pi^* \omega^{1/2}$  such that, locally on  $S$  in a trivializing chart  $U$  for  $\omega$ , and for each normalized section  $s$  of  $\mathcal{L}$ , each section  $\nu$  of  $\omega^{1/2}$ , and each vector field  $Z$  (all defined on  $U$ ), we have*

$$(II_{b.9}) \quad D_Z(s \otimes \nu) \text{ evaluated at the identity section} = \nabla_Z^{s, 1/2} \nu$$

*Proof.* Fix the vector field  $Z$  over  $U$ . Given the choice of  $s, \nu$ , the normalization (II<sub>b</sub>.9) uniquely determines  $D_Z|_U$ , because of the uniqueness up to 1-forms guaranteed by Proposition II<sub>a</sub>.6. It therefore suffices to show that the choice of  $D_Z$  determined by (II<sub>b</sub>.9) does not depend on the choice of  $s$  (non-dependence on the choice of  $\nu$  is obvious), as then the  $D_Z$  that are defined over different opens in  $S$  are forced to agree on overlaps. We call the  $D_Z$  defined by Proposition II<sub>a</sub>.6 and (II<sub>b</sub>.9) using the choice of normalized symmetric section  $s$  by the name  $D_Z^s$ .

By the uniqueness of  $Z \mapsto D_Z$  up to 1-forms and the fact that the symmetric normalized sections of  $\mathcal{L}$  form a  $\text{Sym}^2 \mathfrak{t}^\vee$  torsor according to Definition II<sub>b</sub>.3, it suffices to show that

$$(II_b.10) \quad D_Z^s((1 + \eta^2)s \otimes \nu)|_{\text{identity section}} = D_Z^{(1+\eta^2)s}((1 + \eta^2)s \otimes \nu)|_{\text{identity section}}$$

for all local sections  $\eta$  of  $\mathfrak{t}^\vee$ .

We just calculate both sides of Equation (II<sub>b</sub>.10) in terms of the knowledge we have built about these objects. For the left hand side, it follows from Definition II<sub>a</sub>.3 (with  $b_1 = b_2 = \eta$ ) after evaluating at the identity and from (II<sub>b</sub>.8) that

$$\begin{aligned} D_Z^s((1 + \eta^2)s \otimes \nu)|_{\text{identity}} &= D_Z^s(s \otimes \nu)|_{\text{identity}} + D_Z^s(\eta^2 s \otimes \nu)|_{\text{identity}} \\ &= D_Z^s(s \otimes \nu)|_{\text{identity}} + \langle \text{Symb}^2 D_Z^s, \eta \odot \eta \rangle \nu \\ &= \nabla_Z^{s,1/2}(\nu) - \langle [\mathcal{L}]^{-1} \circ \kappa_{X/S}(Z), \eta \odot \eta \rangle \nu \\ &= \nabla_Z^{s,1/2}(\nu) - \langle [\mathcal{L}]^{-1} \circ \kappa_{X/S}(Z), \eta \odot \eta \rangle_{\text{Tr}} \nu \end{aligned}$$

where the last line comes from the careful convention-wrangling from Section II<sub>0</sub>.

Meanwhile, the right hand side of (II<sub>b</sub>.10) is defined by (II<sub>b</sub>.9) to be

$$\begin{aligned} D_Z^{(1+\eta^2)s}((1 + \eta^2)s \otimes \nu)|_{\text{identity}} &= \nabla_Z^{(1+\eta^2)s,1/2}(\nu) \\ &= \frac{1}{2} \left( \bigwedge^g \text{proj}_{(1+\eta^2)s} \circ \nabla_{GM,Z} \right) (\nu) \\ &= \frac{1}{2} \left( \bigwedge^g (\text{proj}_s + 2(\eta \odot \eta) \circ [\mathcal{L}]^{-1} \circ p) \circ \nabla_{GM,Z} \right) (\nu) \\ &= \nabla_Z^{s,1/2}(\nu) - \langle [\mathcal{L}]^{-1} \circ \kappa_{X/S}(Z), \eta \odot \eta \rangle_{\text{Tr}} \nu \end{aligned}$$

where Lemma II<sub>b</sub>.4 and Lemma II<sub>b</sub>.2 and (II<sub>b</sub>.4) have been used.

□

**II<sub>b</sub>.5. Explicit calculation in the analytic case.** We revisit §II<sub>a</sub>.2. Following the notations of that example, a projective connection has the form

$$(II_b.11) \quad D = (q\partial_q) \mp (z\partial_z)^2/2 + A(q).$$

Now any trivialization of  $\omega$  gives rise, by means of our normalization (II<sub>b</sub>.9), to a genuine connection in this class, i.e. a choice of  $A$ ; we will show that in fact

$$A \equiv 0$$

when  $\omega$  is trivialized by means of  $\frac{dz}{z}$ .

We check the case of the ample bundle, which corresponds to the  $-$  sign in (II<sub>b</sub>.11). It is more convenient in this computation to work in the coordinates  $q \leftarrow e^{2\pi i \tau}$ ,  $z \leftarrow e^{2\pi i w}$ , with respect to which the operator becomes after rescaling

$$D \sim \partial_\tau - \frac{1}{4\pi i} \partial_w^2 \pi i + A.$$

Our goal is still to prove that  $A = 0$ . Equivalently, we want to prove that the heat operator  $\partial_\tau - \frac{1}{4\pi i} \partial_w^2 \pi i$  admits a symmetric normalized solution  $s$  near  $z = 0$  which is simultaneously flat for Gauss–Manin in the sense that  $\varphi(s)$  (the section of (II<sub>b</sub>.3) induced by the choice of  $s$ ) is flat for Gauss–Manin.

For  $s$ , we take the Gaussian  $\tau^{-1/2} e^{-i\pi w^2/\tau}$ .

$$s^2 \otimes \nu = [\tau^{-1/2} e^{-i\pi w^2/\tau}]^2 \otimes dw = e^{-2\pi i w^2/\tau} \otimes \tau^{-1} dw.$$

We must prove that the Gauss–Manin derivative of  $\omega$  lies in the Lagrangian direction determined by  $\varphi(s)$ .

Let  $e_1, e_2$  be the generators for  $H_1(E_\tau, \mathbf{Z})$  with  $dw$ -periods given by 1 and  $\tau$ , and thus with  $\tau^{-1}dw$ -periods  $1/\tau$  and 1. The Gauss–Manin derivative  $\nabla_{GM, \tau}(\tau^{-1}dw)$  lies in the direction of  $e_1^*$ , since  $e_1 \mapsto 1/\tau$ ,  $e_2 \mapsto 0$ .

On the other hand, the  $t$ -differenced line bundle  $\mathcal{L}_t^+$  has for sections those functions satisfying

$$f(w+1) = f(z), f(w+\tau) = e^{2\pi i t} f(w),$$

and the differenced section  $s(w+t)/s(w)$  is given locally by  $e^{-2\pi i w t/\tau}$  up to a constant; this is flat for the connection given by  $\frac{d}{dw} + 2\pi i t/\tau$  and for infinitesimally small  $t$  the monodromy is given by

$$e_1 \mapsto \frac{2\pi i t}{\tau}, e_2 \mapsto 0,$$

which is a multiple of  $e_1^*$  as required.

We propose that this Gauss–Manin flatness is the algebraic significance of the Gaussian  $\tau^{-1/2} e^{-i\pi z^2/\tau}$ , which we already saw in Section 2 is killed by the correct heat operator. Recall that in Section 2 we noted that if you write  $\Theta_{\text{norm}}$  as being descended from a congruence-subgroup-equivariant line bundle on the upper half plane, the fact that you can descend  $\nabla^{\text{Heat}}$  from the upper half plane to the modular curve boils down to the fact that it kills this Gaussian.

II<sub>b</sub>.6. **Example.** In the case of the family of §3.6 the appropriate heat operator associates to the vector field  $q\partial_q$  the following operator:

$$(II_b.12) \quad D = (q\partial_q) - \frac{1}{2} \sum (B^{-1})_{ij} D_{z_i} D_{z_j},$$

with  $D_{z_i} = z_i \partial_{z_i}$ . In particular, the family of sections  $\underline{\theta} \otimes (\frac{dq}{q})^{m/2}$  is flat: it is enough to check that we can compute the flat connection by means of simply applying the differential operator to a representing cocycle. We leave this exercise in the definitions to the reader.

### III. Trivializing the monodromy of the connection: relationship to the Weil representation

#### III<sub>0</sub>. BACKGROUND ON HEISENBERG GROUPS AND THE WEIL REPRESENTATION

In this section, we collect some facts about Heisenberg groups and Weil representations, but taken with  $\mathbf{Z}'$ -coefficients rather than over a field. They are scattered around the literature and all “well-known” to someone, but since this someone was not always us, we will give direct definitions and proofs. The reader might like to skip this section at a first reading; the case of even  $d$  is relegated to the appendix [Section B](#). In fact, for the rest of this paper we assume that  $d$  is odd — the necessary changes are handled in [Section B](#).

We continue with the notation that  $B$  is a bilinear form on  $\mathbf{Z}^m$  of discriminant  $d$ .

III<sub>0</sub>.1. **The Heisenberg group.** Let  $\kappa_B$  be the kernel of  $B$  on  $d^{-1}\mathbf{Z}^m/\mathbf{Z}^m$ , or, to put it differently,

$$\kappa_B = (B^{-1}\mathbf{Z}^m)/\mathbf{Z}^m$$

is the discriminant group of  $B$ . Its order is precisely the discriminant  $d$ . It comes with an orthogonal self-pairing  $\kappa_B \times \kappa_B \rightarrow \frac{1}{d}\mathbf{Z}/\mathbf{Z}$  given by the rule

$$\langle x, y \rangle = \cdot x^t B y.$$

The *Heisenberg group* is the algebraic group scheme  $H_B := \mathbb{G}_m \times \kappa_B \times \kappa_B$  over the spectrum of  $\mathbf{Z}'$ , whose group law is defined by imposing that the two copies  $\kappa_B^{(1)}, \kappa_B^{(2)}$  are copies of the (constant group scheme)  $\kappa_B$ , and satisfy the commutation relation

$$(III_{0.1}) \quad [\kappa_1, \kappa_2] = \zeta_d^{d\langle \kappa_1, \kappa_2 \rangle}.$$

Then  $H_B$  “has a unique representation over  $\mathbf{Z}'$  by which  $\mathbb{G}_m$  acts by scaling.” By “representation over  $\mathbf{Z}'$ ” we mean either of the following equivalent things:

- a free  $\mathbf{Z}'$ -module with two actions of  $\kappa_B$ , satisfying the commutation relation [\(III<sub>0</sub>.1\)](#),  
or

- a vector bundle over the spectrum of  $\mathbf{Z}'$  with an action of the group scheme  $H_B$ . In general, the action of an affine  $S$ -group scheme  $G$  on a vector bundle  $V$  on  $S$  is defined, following [43, Ch. V, §1] (see [14, Exposé I, §4.7.2] for a more detailed exposition) to be a morphism of sheaves of groups

$$G \rightarrow \underline{\text{Aut}}(V)$$

in the flat-fppf site, whose objects are flat morphisms  $T \rightarrow S$  with the coverings of  $T$  consisting of fppf coverings. Equivalently, if  $G$  is additionally assumed to be flat over  $S$ , it is a quasicoherent sheaf morphism (satisfying the axioms of a group coaction)

$$V \rightarrow V \otimes \mathcal{A}_G,$$

where  $\mathcal{A}_G$  is the quasicoherent  $\mathcal{O}_S$ -algebra such that  $G = \underline{\text{Spec}}_S \mathcal{A}_G$ .

We will move freely between the two languages, but will generally prefer (a) because it feels more like the standard theory of group representations. We will use the word “dimension” in place of “rank,” to align better with that picture. Later on when we consider Heisenberg groups in families parametrized by modular curves, the second viewpoint will be more profitable.

**III<sub>0</sub>.2. The Stone–von Neumann “representation”.** The most important representation of  $H_B$  is on the free  $\mathbf{Z}'$ -module spanned by  $\kappa_B$ . Here,  $\mathbb{G}_m$  acts by scaling, the first copy of  $\kappa_B$  acts by translation, and  $y \in \kappa_B$  considered in the second copy acts by multiplication by the function  $\zeta^{(x,y)}$ . We call this representation **SVN**. We can regard it as an induced representation, induced from the action of  $\mathbb{G}_m \times \kappa_B^{(2)}$  on the  $\mathbf{Z}'$ -span of the zero element of  $\kappa_B$ ; here  $\mathbb{G}_m$  acts by scaling and  $\kappa_B^{(2)}$  acts trivially.

**Lemma III<sub>0</sub>.1.** *Let  $\mathbf{Z}' \rightarrow k$  be a ring homomorphism to a field. The induced representation on  $\text{SVN} \otimes k$  is irreducible; it is the unique irreducible representation of  $H_B$  in  $k$ -vector spaces on which  $\mathbb{G}_m$  is acting by scaling.*

*Moreover, any representation over  $\mathbf{Z}'$  (in the sense above) of  $H_B$ , of dimension  $D$ , where  $\mathbb{G}_m$  acts by scaling, is given by tensoring **SVN** with an ideal of  $\mathbf{Z}'$ , i.e. is the natural representation of  $H_B$  on a sum of copies of  $I$  indexed by  $\kappa_B$ , where  $\kappa_B^{(1)}$  acts by translation and  $\kappa_B^{(2)}$  acts by the same multiplication operator as above.*

*Proof.* We leave irreducibility in (I) to the reader. To check uniqueness take any other such representation  $W$  and locate an eigenvector  $\nu$  under  $\kappa_B^{(1)}$ ; modifying  $\nu$  by  $\kappa_B^{(2)}$ , we can arrange that  $\nu$  is actually  $\kappa_B^{(1)}$ -fixed. This shows there is a nontrivial homomorphism from **SVN** to  $W$ .

We proceed to (II). Consider any action of  $H_B$  on a locally free  $\mathbf{Z}'$ -module  $\Lambda$  of rank  $D$ . By (I), we know that  $\Lambda \otimes \mathbf{Q}(\zeta)$  is isomorphic to  $\text{SVN} \otimes \mathbf{Q}(\zeta)$ . Rescaling by a suitable element of  $\mathbf{Q}(\zeta)$  we can suppose that  $\Lambda \subset \text{SVN}$ . The result then follows from (I) in a routine way.  $\square$

**III<sub>0</sub>.3. Weil representation for  $d$  odd.** We assume throughout this section that  $d$  is odd, postponing the more technical discussion of the even case for the appendix. The standard action of  $\mathrm{SL}_2(\mathbf{Z}/d)$  on  $\kappa_B \times \kappa_B$  by

$$\gamma \cdot (x, y) := (ax + by, cx + dy), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

can then be uniquely lifted to an action on  $H_B$  that is trivial on  $\mathbf{G}_m$  by following [57, §5]. Indeed, the squaring map is an automorphism of the finite group  $\kappa_B = \prod_i \mathbf{Z}/d_i$  since the  $d_i$  are all odd thanks to the oddness of  $d$ . For the reader's convenience, we repeat the definition here.

**Definition III<sub>0</sub>.2.** The unique action of  $\mathrm{SL}_2(\mathbf{Z}/d)$  on  $H_B$  that is trivial on  $\mathbf{G}_m \subset H_B$  must be of the form

$$\gamma \cdot (\lambda, x, y) = (\gamma \cdot \lambda)(\gamma \cdot (1, x, y)) = (\lambda f(x, y), \gamma(x, y)),$$

where  $f(x, y)$  is a function from  $\kappa_B \times \kappa_B$  to  $\mathbf{G}_m$  that must satisfy

$$\frac{f(\underline{v} + \underline{w})}{f(\underline{v})f(\underline{w})} = \frac{\exp(2\pi i \langle (\gamma \underline{v})_1, (\gamma \underline{w})_2 \rangle)}{\exp(2\pi i \langle v_1, w_2 \rangle)}$$

for all  $\underline{v}, \underline{w} \in \kappa_B \times \kappa_B$  where the right hand side is interpreted algebraically as being a  $d$ -th root of unity in  $\mu_d \subset \mathbf{G}_{m, \mathbf{Z}'}$ . The fact that  $d$  is odd implies that there is a unique way to refine that symmetric pairing (it is symmetric because  $\gamma$  respects the natural alternating pairing induced by  $\langle -, - \rangle$  viewed as being valued in  $\mathbf{Q}/\mathbf{Z}$ ), namely

$$f(x, y) := \frac{\exp((d+1)\pi i \langle (\gamma(x, y))_1, (\gamma(x, y))_2 \rangle)}{\exp((d+1)\pi i \langle x_1, y_2 \rangle)}$$

which is again a  $d$ -th root of unity since  $d$  is odd, and therefore still valued in  $\mathbf{G}_m$  over  $\mathbf{Z}'$ .

**Lemma III<sub>0</sub>.3.** For  $d$  odd, the representation of  $H_B$  on  $\mathrm{SVN}$  extends uniquely to the larger group  $H_B \rtimes \mathrm{SL}_2(\mathbf{Z}/d)$  in such a way that the one-dimensional fixed space for  $\kappa_B^{(1)}$  is also fixed by the pointwise stabilizer of  $\kappa_B^{(1)}$  inside  $\mathrm{SL}_2(\mathbf{Z}/d\mathbf{Z})$ , and the same is true for  $\kappa_B^{(2)}$ .

We sketch a direct proof because the result is straightforward and we were not able to find an entirely convenient reference. Rather than write explicit formulas, we have isolated in the statement the precise property that will be relevant for us.

*Proof.* We first note that the pullback of  $\mathrm{SVN}$  by any element of  $\mathrm{SL}_2(\mathbf{Z}/d)$  is in fact isomorphic to  $\mathrm{SVN}$ . This can be obtained abstractly from Lemma III<sub>0</sub>.1, but this requires a little effort, and it is more convenient to proceed explicitly. It is enough to check the statement for the generating elements  $T, U, S$ , and we define them to act on  $\mathrm{SVN}$  in the following way:

- $T$  acts by  $\zeta_{1/2}^{d\langle x, x \rangle}$ .
- The Weyl element  $S$  acts by a scalar multiple of the Fourier transformation, that is to say, the map

$$\mathbf{Z}'[\kappa_B] \longrightarrow \mathbf{Z}'[\kappa_B]$$

which sends the basis element indexed by  $x \in \kappa_B$  to  $y \mapsto \zeta^{d\langle x, y \rangle}$ .

- $U^{-1} = STS^{-1}$  acts by multiplication on the Fourier transform by  $\zeta_{1/2}^{d\langle x, x \rangle}$ .

By Schur's lemma after change of base to  $\mathbf{C}$  (and the routine verification that this action intertwines the action of  $h \in H_B$  with the action of  $T \cdot h$ , resp.  $S \cdot h$ , resp.  $U \cdot h$ ), and because (any two of)  $T, S, U$  generate  $\mathrm{SL}_2(\mathbf{Z}/d)$  [6, (1.1)], this defines a projective representation of  $\mathrm{SL}_2(\mathbf{Z}/d)$ ,

$$\mathrm{SL}_2(\mathbf{Z}/d) \rightarrow \mathrm{PGL}(\mathbf{C}[\kappa_B]),$$

the same one constructed in this setting by Weil [57] that now goes by the name of the *Weil representation*.

Since we are supposing that  $d$  is odd, the Schur multiplier of  $\mathrm{SL}_2(\mathbf{Z}/d)$  is trivial [6] which implies that there is a lift to a genuine representation, at least with entries in  $\mathbf{C}$ . By definition, there is a constant  $c \in \mathbf{C}^\times$  such that in this lift,  $T$  acts by  $c\zeta_{1/2}^{d\langle x, x \rangle}$ . Already, we know that  $c$  must be a  $d$ -th root of unity, since  $T^d = 1$  must act by  $c^d$ .

On any such lift with  $T$  acting by  $c\zeta_{1/2}^{d\langle x, x \rangle}$ ,  $U^{-1}$  acts by multiplication on the Fourier transform by  $c\zeta_{1/2}^{d\langle x, x \rangle/2}$ , since conjugation by any scalar multiple of the Fourier transform is the same as conjugation by the Fourier transform.

It remains to prove:

There exists such a lift with  $c = 1$ .

Such a lift with  $c = 1$  is unique, since  $\mathrm{SL}_2(\mathbf{Z}/d)$  is generated by  $T, U$  [6, (1.1)]. We just need to show that the lift whose existence (with some  $d$ -th root of unity  $c$ ) is guaranteed by the vanishing of the Schur multiplier can be chosen to have  $c = 1$ .

The diagonal matrix  $\delta_\alpha$  with entries  $\alpha, \alpha^{-1}$  acts (up to scalar) by “dilation by  $\alpha$ ”, that is, it sends the basis element for  $\mathrm{SVN}$  indexed by  $x \in \kappa_B$  to the basis element indexed by  $\alpha \cdot x$ . Therefore,

$$\delta_\alpha T \delta_\alpha^{-1}$$

acts by

$$\mathbf{1}_x \mapsto c\zeta_{1/2}^{\alpha^2 d\langle x, x \rangle} \mathbf{1}_x.$$

Comparing with the formula  $\delta_\alpha T \delta_\alpha^{-1} = T^{\alpha^2}$ , which acts by

$$\mathbf{1}_x \mapsto c^{\alpha^2} \zeta_{1/2}^{\alpha^2 d\langle x, x \rangle} \mathbf{1}_x$$

we see that  $c^{\alpha^2-1} = 1$  for all  $\alpha$  prime to  $d$ , and so  $c$  is a 3rd root of unity (by using the fact that  $d$  is odd and taking  $\alpha = 2$ ). If  $c \neq 1$ , then we must have  $3|d$ , but then we can use the homomorphism  $\mathrm{SL}_2(\mathbf{Z}/d) \rightarrow \mathrm{SL}_2(\mathbb{F}_3) \rightarrow A_4 \rightarrow A_3$  to adjust  $c$  to be trivial.  $\square$

**Definition III<sub>0</sub>.4.** Let  $d$  be odd. The representation  $\mathrm{SL}_2(\mathbf{Z}/d) \rightarrow \mathrm{GL}(\mathbf{Z}'[\kappa_B])$  such that

$$T \cdot \mathbf{1}_x = \zeta_{1/2}^{d\langle x, x \rangle} \mathbf{1}_x$$

$STS^{-1}$  acts through the Fourier transform by multiplication by  $\zeta_{1/2}^{d\langle x, x \rangle}$ .

whose existence and uniqueness is ensured by [Lemma III<sub>0</sub>.3](#) is denoted by  $\gamma \mapsto \mathbf{r}_\gamma$ . It is (by definition) guaranteed to have the property that

$$(III_0.2) \quad \mathbf{r}_\gamma \cdot \rho(h) \cdot \mathbf{r}_\gamma^{-1} = \rho(\gamma \cdot h)$$

as operators on SVN.

### III<sub>a</sub>. REDUCING THE $\Theta$ -BUNDLE FROM A VECTOR BUNDLE TO A LINE BUNDLE

In §3.6 we have constructed a “theta” vector bundle  $\Theta$  of rank  $D$  on  $Y^*$ . Our goal here is to complete step (III<sub>a</sub>) of §4: to cut down  $\Theta$  to a *line bundle*  $\Theta_{\text{line}}$  that captures most of its essential properties. To do this, we use the natural “theta-group” scheme over  $Y^*$  or  $Y^*(\mathfrak{d})$  that acts on  $\Theta$  and exploit its representation theory and the comparison between it and the finite Heisenberg group constructed in the previous section.

**III<sub>a</sub>.1. The theta-group scheme.** Recall from [Section 3](#) that the fibers of the abelian schemes that are relevant to us are abelian varieties of the specific form  $E^m$ , where  $E$  is an elliptic curve. The  $E^m$  come with a line bundle  $\mathcal{L}|_E \otimes B$ , where  $\mathcal{L}|_E$  is a symmetric rigidified degree-1 (ample) line bundle<sup>18</sup>. It is known that over an algebraically closed field  $k$  of characteristic not dividing  $\mathfrak{d}$  (see for instance [\[44, §1\]](#) which considers a situation more general than the powers of elliptic curves that suffice for Kudla–Millson theory), the Heisenberg group  $H_B \times_{\mathbf{Z}} k$  constructed in [Section III<sub>0</sub>](#) acquires geometric significance as the “theta-group” of  $\mathcal{L}|_E \otimes B$ , that is, the group of all automorphisms of the total space of the line bundle that cover a translation.

The object is ultimately to study the monodromy of the heat equation, so we are required to take into account all of this structure in families: as in [Section 3](#), we have a universal symmetric rigidified (ample) degree-1 line bundle  $\mathcal{L}$  on the universal elliptic curve  $\mathcal{E}$  over  $Y^*$ , and we consider the nondegenerate line bundle  $\mathcal{L} \otimes B$  on  $Y^*$ . As such, one can make the definition of the theta-group in families (and as before all of this works for arbitrary nondegenerate line bundles on abelian schemes<sup>19</sup>, but this is the only case that is necessary for us).

For the line bundle  $\mathcal{L}^B$  pulled back to various bases, we will need to use the group schemes  $K(\mathcal{L}^B)$  and  $\mathcal{G}(\mathcal{L}^B)$ , whose definitions and basic properties are recalled in [Section A](#). We should explain how that structure specializes in the case of our line bundle  $\mathcal{L}^B$  on  $\mathcal{E}^m$  over  $Y^*$ . We already saw in [Lemma 3.1](#) that the induced map  $\phi_{\mathcal{L}^B} : \mathcal{E}^m \rightarrow (\mathcal{E}^m)^\vee = (\mathcal{E}^\vee)^m \cong \mathcal{E}^m$  (where the last identification is just  $m$  copies of  $\phi_{\mathcal{L}}$ ) is precisely the map  $B : \mathcal{E}^m \rightarrow \mathcal{E}^m$ ,

<sup>18</sup>We denote this by  $\mathcal{L}|_E$  because in a moment we will of course be considering this whole situation in the larger family.

<sup>19</sup>the requirement of inverting enough primes is subsumed into the “nondegenerate” hypothesis

making  $\mathcal{L}^B$  always a nondegenerate line bundle on  $\mathcal{E}^m$ . Hence  $K(\mathcal{L}^B)$  is a finite flat  $Y^*$ -group scheme, and the exact sequence

$$1 \rightarrow \mathbf{G}_m \rightarrow \mathcal{G}(\mathcal{L}^B) \rightarrow K(\mathcal{L}^B) \rightarrow 0$$

induces a nondegenerate alternating commutator pairing on  $K(\mathcal{L}^B)$ . The same remains true after arbitrary base change over  $Y^*$ .

**Lemma III<sub>a</sub>.1.** *The finite flat  $Y^*$ -group scheme  $K(\mathcal{L}^B)$  becomes étale after base-change to  $Y_{\mathbf{Z}'}^*$ .*

*Proof.* The finite flat commutative  $Y^*$ -group scheme  $K(\mathcal{L}^B)$  is annihilated by  $\mathbf{d}$  (use Lemma 3.1 and the Smith normal form of  $B$ ). So it is étale as long as  $\delta$  is invertible in the base.  $\square$

We wish to access a theta structure (that is, an identification between the theta group scheme  $\mathcal{G}(\mathcal{L}^B)$  and the Heisenberg group  $H_B$ ) for  $\mathcal{L}^B$ . This can be accomplished only after base change to an *explicit* modular curve, and we will be explicit about the level and base ring that are required. From [39, Lemma 3.8], we know that theta structures exist étale locally; in order to produce a global one, it is extremely convenient to use extra structure coming from the symmetric structure of  $\mathcal{L}^B$  that rigidifies the situation, thus allowing for étale descent. This concept appeared at least in [45, p. 78], but we summarize it below in the case of  $\mathcal{L}^B$  (or any base change thereof).

**Definition III<sub>a</sub>.2.** Let  $\mathcal{F}$  be a symmetric rigidified line bundle on an abelian scheme  $\pi : X \rightarrow S$ . The symmetric rigidified structure of  $\mathcal{F}$  means that there is a homomorphism  $\delta_{-1} : \mathcal{G}(\mathcal{F}) \rightarrow \mathcal{G}(\mathcal{F})$  given functorially on scheme-theoretic points  $(x, \varphi)$  of  $\mathcal{G}(\mathcal{L})$  by

$$(x, \varphi : \mathcal{L} \xrightarrow{\sim} T_x^* \mathcal{L}) \mapsto (-x, (T_{-x}^* \iota)^{-1} \circ ([ -1 ]^* \varphi) \circ \iota).$$

The correct étale local model for this structure on the  $H_B$  side is readily deduced from the abstract properties of  $\delta_{-1}$ : it induces the negation automorphism on  $\mathcal{K}(\mathcal{L})$ , whose local model is  $\kappa_B \times \kappa_B$ , and it is the identity on the center. Therefore, one defines the group automorphism

$$D_{-1} : H_B \rightarrow H_B$$

by

$$D_{-1}(\lambda, x, y) := (\lambda, -x, -y)$$

(and verifies that it is a group automorphism) A *symmetric theta structure* for  $\mathcal{L}^B$  is then defined to be a group scheme isomorphism

$$\beta : \mathcal{G}(\mathcal{L}^B) \longrightarrow (H_B)_{Y^*(\delta)}$$

such that  $D_{-1} \circ \beta = \beta \circ \delta_{-1}$ .

As mentioned earlier, the notion of symmetric theta structure is useful for us because it *rigidifies* the situation. To study the rigidity properties of the moduli of theta structures is to study the automorphisms of those structures and try to impose enough extra structure (in

our case, the symmetric structure) to make there be no automorphisms besides the trivial one.

**Proposition III<sub>a</sub>.3.** *The line bundle  $\mathcal{L}^B$  on  $\mathcal{E}^m$  admits a unique symmetric theta structure after base change to the  $\mathbf{Z}'$ -scheme  $Y^*(\mathbf{d})$ .*

*Proof.* After base-change to  $Y^*(\mathbf{d})$ , one has a global isomorphism of  $Y^*(\mathbf{d})$ -group schemes

$$\varphi : \mathcal{E}^m[\mathbf{d}] = \mathcal{E}[\mathbf{d}]^m \simeq (\mathbf{Z}/\mathbf{d})^m \times (\mathbf{Z}/\mathbf{d})^m,$$

which preserves the Weil pairing coming from  $\mathcal{L}^B$ . Since  $K(\mathcal{L}^B)$  is annihilated by  $\mathbf{d}$ , we have

$$K(\mathcal{L}^B) \subset \mathcal{E}^m[\mathbf{d}] = \mathcal{E}[\mathbf{d}]^m,$$

and is (by Lemma 3.1) identified with

$$\ker(B : \mathcal{E}[\mathbf{d}]^m \rightarrow \mathcal{E}[\mathbf{d}]^m),$$

which is identified under  $\varphi$  with

$$\ker(B : (\mathbf{Z}/\mathbf{d})^m \times (\mathbf{Z}/\mathbf{d})^m \rightarrow (\mathbf{Z}/\mathbf{d})^m \times (\mathbf{Z}/\mathbf{d})^m)$$

which is precisely the constant group scheme  $\kappa_B \times \kappa_B$ .

So the universal object over  $Y^*(\mathbf{d})$  provides us with a uniquely determined trivialization of  $K(\mathcal{L}^B)$  that we also denote by  $\varphi$ . It remains to lift  $\varphi$  to a  $Y^*(\mathbf{d})$ -group scheme isomorphism  $\tilde{\varphi} : \mathcal{G}(\mathcal{L}^B) \simeq H_B$  such that the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathcal{G}(\mathcal{L}^B) & \longrightarrow & K(\mathcal{L}^B) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow \tilde{\varphi} & & \downarrow \varphi \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & H_B & \longrightarrow & \kappa_B \times \kappa_B \longrightarrow 0 \end{array}$$

[39, Lemma 3.8] (itself generalizing the argument of [44, Corollary of Theorem 1]) is enough to deduce the existence of that isomorphism étale locally on  $Y^*(\mathbf{d})$ . We see in Section A that one can modify that étale local theta structure by an inner automorphism of  $H_B$  so that it is symmetric, and that this symmetric theta structure is *unique* given that it lifts  $\varphi$ ; therefore descent in the étale topology says that a global theta structure  $\tilde{\varphi}$  exists and is unique.  $\square$

The group  $\mathcal{G}(\mathcal{L}^B)$  and the group  $H_B$  both admit natural actions of  $\mathrm{SL}_2(\mathbf{Z}/\mathbf{d})$ . The former comes from the fact that  $\mathcal{L}^B$  is pulled back from  $Y_{\mathbf{Z}'}^*$  under the Galois (surjective étale) cover  $Y^*(\mathbf{d}) \rightarrow Y_{\mathbf{Z}'}^*$  with group  $\mathrm{SL}_2(\mathbf{Z}/\mathbf{d})$ , so the  $Y^*(\mathbf{d})$ -group scheme  $\mathcal{G}(\mathcal{L}^B)$  is pulled back from the group scheme on  $Y_{\mathbf{Z}'}^*$  that goes by the same name. So  $\mathcal{G}(\mathcal{L}^B)_{Y^*(\mathbf{d})}$  acquires a datum that descends it to  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$ , which is a compatible collection of isomorphisms of  $Y^*(\mathbf{d})$ -group schemes

$$\gamma^* \mathcal{G}(\mathcal{L}^B) \simeq \mathcal{G}(\mathcal{L}^B)$$

for all  $\gamma \in \mathrm{SL}_2(\mathbf{Z}/d)$ . Since the automorphism  $\delta_{-1}$  of  $\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$  is pulled back from the one of the same name for  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$ , the descent datum is also a descent datum for the extra structure of  $\delta_{-1}$ , that is to say, the diagram

$$\begin{array}{ccc} \gamma^* \mathcal{G}(\mathcal{L}^B) & \longrightarrow & \mathcal{G}(\mathcal{L}^B) \\ \downarrow \gamma^* \delta_{-1} & & \downarrow \delta_{-1} \\ \gamma^* \mathcal{G}(\mathcal{L}^B) & \longrightarrow & \mathcal{G}(\mathcal{L}^B) \end{array}$$

commutes.

The group scheme  $(H_B)_{Y^*(d)}$  is base-changed from the group scheme  $(H_B)_{Y^*}$ , so it automatically admits a  $\mathrm{SL}_2(\mathbf{Z}/d)$ -equivariant structure

$$\gamma^* H_B = H_B.$$

The equals sign is used here because this structure is the canonical one: it comes from the isomorphism  $\gamma^* \kappa_B = \kappa_B$  coming from the fact that  $\kappa_B$  is constant, and the canonical isomorphism  $\gamma^* \mathbf{G}_m = \mathbf{G}_m$ . We will put a different  $\mathrm{SL}_2(\mathbf{Z}/d)$ -equivariant structure on it:

**Definition III<sub>a</sub>.4.** For  $\gamma \in \mathrm{SL}_2(\mathbf{Z}/d)$ , take the isomorphism

$$\gamma^* H_B = H_B \rightarrow H_B$$

to be given by the action of  $\gamma$  on  $H_B$  given by the isomorphism between symmetric automorphisms of  $H_B$  and  $Sp(\kappa_B \times \kappa_B)$ , which is a constant group scheme whose  $Y^*(d)$ -points admit a map from  $\mathrm{SL}_2(\mathbf{Z}/d)$ . This forms an  $\mathrm{SL}_2(\mathbf{Z}/d)$ -equivariant structure on  $H_B$ .

**Lemma III<sub>a</sub>.5.** *Under the unique symmetric theta-structure  $\varphi$  on  $\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$ , the  $\mathrm{SL}_2(\mathbf{Z}/d)$ -equivariant structure on  $\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$  agrees with the  $\mathrm{SL}_2(\mathbf{Z}/d)$ -equivariant structure on  $(H_B)_{Y^*(d)}$  that we just defined.*

*Proof.* The statement of the lemma amounts to the commutativity of the diagram of  $Y^*(d)$ -group schemes

$$\begin{array}{ccc} \gamma^* \mathcal{G}(\mathcal{L}^B) & \longrightarrow & \mathcal{G}(\mathcal{L}^B) \\ \downarrow \gamma^* \tilde{\varphi} & & \downarrow \tilde{\varphi} \\ \gamma^* H_B & \longrightarrow & H_B \end{array}$$

which is because of the fact that the two alternative ways to go from  $\mathcal{G}(\mathcal{L}^B)$  to  $H_B$  are a symmetric theta structure inducing the same isomorphism  $\varphi : K(\mathcal{L}^B) \rightarrow \kappa_B \times \kappa_B$ .  $\square$

III<sub>a</sub>.2. **Descent to a line bundle.** The previous section shows that situation after base-change to  $Y^*(\mathbf{d})$  consists of:

- An isomorphism  $\mathcal{G}(\mathcal{L}^B)_{Y^*(\mathbf{d})} \simeq (H_B)_{Y^*(\mathbf{d})}$
- An action of  $\mathcal{G}(\mathcal{L}^B)_{Y^*(\mathbf{d})}$  on  $^{20}R^s\pi_*\mathcal{L}^B \otimes \omega^{m/2} = (\Theta_{\text{norm}})_{Y^*(\mathbf{d})}$ .
- An action of  $(H_B)_{Y^*(\mathbf{d})}$  on the trivial rank- $\mathbf{d}$  vector bundle  $\text{SVN}_{Y^*(\mathbf{d})}$ .

The forgetful map  $p_{\mathbf{d}} : Y^*(\mathbf{d}) \rightarrow Y_{\mathbf{Z}'}^*$  is a finite surjective étale cover with Galois group  $\text{SL}_2(\mathbf{Z}/\mathbf{d})$ , in the sense of [26, Exposé V, Definition 2.8]. If a vector bundle or quasicoherent sheaf  $\mathcal{F}$  (resp. morphism  $\alpha : \mathcal{F} \rightarrow \mathcal{G}$  thereof) descends from  $Y^*(\mathbf{d})$  to  $Y^*$ , it admits a system of compatible isomorphisms  $\gamma^*\mathcal{F} \simeq \mathcal{F}$  for each  $\gamma \in \text{SL}_2(\mathbf{Z}/\mathbf{d})$  (resp. the diagram

$$\begin{array}{ccc} \gamma^*\mathcal{F} & \xrightarrow{\gamma^*\alpha} & \gamma^*\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} \end{array}$$

commutes). Such a system is equivalent to a descent datum for the desired structure because of the isomorphism  $Y^*(\mathbf{d}) \times_{Y^*} Y^*(\mathbf{d}) \simeq Y^*(\mathbf{d}) \times_{Y^*} \text{SL}_2(\mathbf{Z}/\mathbf{d})_{Y^*}$  (see [26, Exposé V, §2]), under which a descent datum

$$p_1^*\mathcal{F} \rightarrow p_2^*\mathcal{F}$$

becomes equivalent to the aforementioned compatible systems of isomorphisms  $\gamma^*\mathcal{F} \simeq \mathcal{F}$ . We will use without further comment the following basic facts about descent:

- By fpqc descent [26, Exposé VIII, Théorème 1.1 and corollaries], one can descend quasicoherent sheaves and morphisms between them on  $Y^*(\mathbf{d})$  with descent data to  $Y^*$ .
- fpqc descent preserves the property of being a vector bundle (and therefore the rank of the vector bundle): [26, Exposé VIII, Théorème 1.10].
- Flat connections can also be descended from vector bundles  $\mathcal{F}$  on  $Y^*(\mathbf{d})$  to  $Y^*$ , despite their lack of  $\mathcal{O}$ -linearity. This is because they can be viewed as  $\mathcal{O}$ -linear sections of the Atiyah sequence. The condition for descending a connection  $\nabla$  is that the section of  $\text{Hom}(\gamma^*\mathcal{F}, \mathcal{F})$  provided by the descent datum should be horizontal for the connection induced by  $\nabla$  and  $\gamma^*\nabla$ . The property of flatness also descends under this procedure: see for example [48, 2.2.15] for a stack-theoretic version, or [1, §4.2] for a language more in line with ours.

We will perform the following instances of descent:

- (i)  $\mathcal{G}(\mathcal{L}^B)_{Y^*(\delta)}$  is by definition the pullback to  $Y^*(\mathbf{d})$  of  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$ . Since  $\mathcal{G}(\mathcal{L}^B)_{Y^*(\mathbf{d})} \simeq (H_B)_{Y^*(\mathbf{d})}$ , there is a descent datum on  $(H_B)_{Y^*(\mathbf{d})}$  for the action of  $\text{SL}_2(\mathbf{Z}/\mathbf{d})$  that descends it to the  $Y^*$ -group scheme  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$ . We will prove in Lemma III<sub>a</sub>.6 that

<sup>20</sup>We will use without comment the fact that  $R^s\pi_*\mathcal{L}^B \otimes \omega^{m/2} = \Theta_{\text{norm}}$  doesn't depend on whether it is computed over  $Y^*(\mathbf{d})$  or computed over  $Y_{\mathbf{Z}'}^*$ , and then base-changed from  $Y_{\mathbf{Z}'}^*$  to  $Y^*(\mathbf{d})$ , as a consequence of the flatness of  $p_{\delta} : Y^*(\mathbf{d}) \rightarrow Y_{\mathbf{Z}'}^*$ , the fact that  $\pi : \mathcal{E}^m \rightarrow Y^*$  and its pullback  $p^*\pi : \mathcal{E}_{Y^*(\mathbf{d})}^m \rightarrow Y^*(\mathbf{d})$  are abelian schemes and [24, Proposition (1.4.15)], [24, Ch. 0, Proposition (12.2.3)]

this descent datum is in some sense precisely the action of  $\mathrm{SL}_2(\mathbf{Z}/d)$  on the finite Heisenberg group scheme  $(H_B)_{Y^*(d)}$ .

- (ii) The  $Y^*$ -vector bundle  $(\Theta_{\mathrm{norm}})_{Y^*(d)}$  descends to the vector bundle  $(\Theta_{\mathrm{norm}})_{Y^*}$ , which admits an action of  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$  descending the upstairs  $\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$ -action.
- (iii) The trivial rank- $d$   $Y^*(d)$ -vector bundle  $\mathrm{SVN}_{Y^*(d)}$  descends via the Weil action of  $\mathrm{SL}_2(\mathbf{Z}/d)$  (in a sense that we will make precise) to a rank- $d$  vector bundle  $\mathbf{M}$  on  $Y^*$ .
- (iv) The action of  $(H_B)_{Y^*(d)}$  on  $\mathrm{SVN}_{Y^*(d)}$ , after the descent of (i) and (iii), descends to an action of  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$  on  $\mathbf{M}_{Y^*}$ .
- (v) Being now equipped by (ii) and (iv) with rank- $d$  vector bundles with  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$ -actions,  $\mathbf{M}_{Y^*}$  and  $(\Theta_{\mathrm{norm}})_{Y^*}$ , we form the vector bundle

$$\underline{\mathrm{Hom}}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*}),$$

which is descended from  $\underline{\mathrm{Hom}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})$ . We will prove that the sheaf of  $\mathcal{G}(\mathcal{L}^B)$ -fixed vectors,

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{L}^B)}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*}),$$

turns out to be a *line bundle* on  $Y^*$ , which is descended from

$$\mathrm{Hom}_{(H_B)_{Y^*(d)}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)}).$$

- (vi) The flat connection  $\nabla$  on  $(\Theta_{\mathrm{norm}})_{Y^*(d)}$  together with the trivial connection  $d$  on  $\mathrm{SVN}_{Y^*(d)}$  induce a flat connection on the  $Y^*(d)$ -line bundle  $\mathrm{Hom}_{(H_B)_{Y^*(d)}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})$ , which descends to a flat connection on the  $Y^*$ -line bundle  $\mathrm{Hom}_{\mathcal{G}(\mathcal{L}^B)}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*})$ .
- (vii) We will construct a horizontal section  $\underline{\theta}_{\mathrm{line}}$  of

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{L}^B)}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*})$$

near a cusp of  $Y^*$  by descent from a horizontal section of

$$\mathrm{Hom}_{(H_B)_{Y^*(d)}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})$$

near a cusp of  $Y^*(d)$  over that cusp.

Let us now execute all of this in full detail, following the same order. The reader already convinced of the correctness of any of these steps is encouraged to skip it.

(i): Descent of  $(H_B)_{Y^*(d)}$  to  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$ .

Base-changing  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$  by  $p_d : Y^*(d) \rightarrow Y^*$  gives  $\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$ , which admits a uniquely determined symmetric theta structure

$$\varphi : \mathcal{G}(\mathcal{L}^B)_{Y^*(d)} \simeq (H_B)_{Y^*(d)}.$$

This induces a uniquely determined descent datum,

$$\gamma^*(H_B)_{Y^*(d)} \simeq (H_B)_{Y^*(d)} \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbf{Z}/d)$$

which descends  $(H_B)_{Y^*(d)}$  to  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$  (use fpqc descent for the quasicoherent sheaf of  $\mathcal{O}_{Y^*(d)}$ -algebras defining the  $Y^*$ -group scheme  $(H_B)_{Y^*(d)}$  — in general one can descend affine morphisms by fpqc base change this way, as in [26, Exposé VIII, Théorème 2.1]).

**Lemma III<sub>a</sub>.6.** *The descent datum  $\gamma^*(H_B)_{Y^*(d)} \simeq (H_B)_{Y^*(d)}$  defined by the isomorphism  $(H_B)_{Y^*(d)} = \mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$  as above is given precisely by the composition*

$$\gamma^*(H_B)_{Y^*(d)} = (H_B)_{Y^*(d)} \rightarrow (H_B)_{Y^*(d)}.$$

*Proof.* Follows directly from (III<sub>0</sub>.2). □

(ii): Descent of  $(\Theta_{\text{norm}})_{Y^*(d)}$  to  $(\Theta_{\text{norm}})_{Y^*}$

The vector bundle  $(\Theta_{\text{norm}})_{Y^*}$  is defined to be  $R^s\pi_*\mathcal{L}^B \otimes \omega^{m/2}$ , where  $\pi$  fits into the Cartesian square

$$\begin{array}{ccc} \mathcal{E}_{Y^*(d)}^m & \xrightarrow{p_{\mathcal{E}}} & \mathcal{E}_{Y^*}^m \\ \downarrow \pi' & & \downarrow \pi \\ Y^*(d) & \xrightarrow{p_d} & Y^* \end{array}$$

On the other hand,

$$(\Theta_{\text{norm}})_{Y^*(d)} := p_{\mathcal{E}}^*(\Theta_{\text{norm}})_{Y^*} = R^s\pi'_*\mathcal{L}^B \otimes \omega^{m/2}$$

So we know that we may consider  $(\Theta_{\text{norm}})_{Y^*(d)}$  by its other name  $R^s\pi'_*\mathcal{L}^B \otimes \omega^{m/2}$  and that it descends to  $(\Theta_{\text{norm}})_{Y^*}$ .

(iii)-(iv): Descent of  $\text{SVN}_{Y^*(d)}$  to the Schrödinger model vector bundle on  $Y^*$ .

Recall that  $\text{SVN}$  is defined to be a trivial rank- $d$  vector bundle over  $\text{Spec}(\mathbf{Z}')$ , with the  $\mathbf{Z}'$ -group scheme  $H_B$  acting by the Schrödinger model of the Stone–von Neumann representation. Pull back  $\text{SVN}$  to a vector bundle over  $Y^*(d)$ , which we denote by  $\text{SVN}_{Y^*(d)}$ . The vector bundle  $\text{SVN}_{Y^*(d)}$  is equipped with an action of the group scheme  $(H_B)_{Y^*(d)}$  pulled back from the action of  $H_B$  on  $\text{SVN}$ . More precisely, the coaction

$$\text{SVN} \rightarrow \text{SVN} \otimes_{\mathbf{Z}'} \mathcal{A}_{H_B}$$

after base-change to  $Y^*(d)$  becomes

$$\text{SVN}_{Y^*(d)} \rightarrow \text{SVN}_{Y^*(d)} \otimes_{Y^*(d)} \mathcal{A}_{(H_B)_{Y^*(d)}};$$

alternatively, it is the morphism of flat-fppf sheaves (see [43, Ch. V, §1], though it is harmless to just consider these to be sheaves on the small fppf site of  $Y^*(d)$ )

$$(H_B)_{Y^*(d)} \rightarrow \underline{\text{Aut}}(\text{SVN}_{Y^*(d)})$$

given by base-changing the one over  $\mathbf{Z}'$  that was defined earlier.

The action of  $\text{SL}_2(\mathbf{Z}/d)$  on  $\text{SVN}$  can be interpreted as a descent datum for the rank- $d$  vector bundle  $\text{SVN}_{Y^*(d)}$ : there is a canonical isomorphism of  $Y^*(d)$ -vector bundles  $\gamma^*\text{SVN}_{Y^*(d)} = \text{SVN}_{Y^*(d)}$  coming from the standard basis of  $\text{SVN}_{Y^*(d)}$  identifying it with  $\mathcal{O}_{Y^*(d)}^{\oplus d}$ , so composing

that with the Weil-representation  $\gamma$ -action on  $\text{SVN}$  from the previous section gives a descent datum

$$\gamma^* \text{SVN}_{Y^*(d)} = \text{SVN}_{Y^*(d)} \xrightarrow{\mathbf{r}(\gamma)} \text{SVN}_{Y^*(d)} \quad \text{for all } \gamma \in \text{SL}_2(\mathbf{Z}/d).$$

**Definition III<sub>a</sub>.7.** The rank- $d$  vector bundle on  $Y_{\mathbf{Z}}^*$  defined by having pullback under  $p_d : Y^*(d) \rightarrow Y^*$  isomorphic to  $\text{SVN}_{Y^*(d)}$  with  $\text{SL}_2(\mathbf{Z}/d)$ -descent datum as defined by the Weil representation above is denoted  $\mathbf{M}$ . We denote it by  $\mathbf{M}_{Y^*}$  when we wish to emphasize that the base is  $Y^*$  (the default option), and we denote its base-change to  $Y^*(d)$  by  $\mathbf{M}_{Y^*(d)}$ . By definition,  $\mathbf{M}_{Y^*(d)} \simeq \text{SVN}_{Y^*(d)}$ .

That  $\mathbf{M}$  is the correct global version over  $Y^*$  of the Schrödinger model of the Heisenberg group for  $B$  (whose correct global model over  $Y^*$  we already know to be  $\mathcal{G}(\mathcal{L}^B)$  by (i)) follows from the characterizing property of the Weil representation of  $\text{SL}_2(\mathbf{Z}/d)$ , as explained by the following lemma.

**Lemma III<sub>a</sub>.8.** *The action  $(H_B)_{Y^*(d)} \rightarrow \underline{\text{Aut}}(\text{SVN}_{Y^*(d)})$  descends via the  $\text{SL}_2(\mathbf{Z}/d)$ -descent structure on the domain and codomain viewed as affine  $Y^*(d)$ -schemes (the former coming from the theta structure that we constructed on  $\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$  and the latter induced from the construction we just explained) to an action*

$$\mathcal{G}(\mathcal{L}^B)_{Y^*} \rightarrow \underline{\text{Aut}}(\mathbf{M}).$$

*Proof.* To descend this morphism of affine  $Y^*$ -schemes is to prove that the diagram

$$\begin{array}{ccc} \gamma^* H_B & \longrightarrow & \underline{\text{Aut}}(\gamma^* \text{SVN}_{Y^*(d)}) \\ \downarrow & & \downarrow \\ H_B & \longrightarrow & \underline{\text{Aut}}(\text{SVN}_{Y^*(d)}) \end{array}$$

commutes for all  $\gamma \in \text{SL}_2(\mathbf{Z}/d)$ . Writing down the two paths across the diagram shows that its commutativity is equivalent to

$$\mathbf{r}_\gamma^{-1} h \mathbf{r}_\gamma|_{\text{SVN}} = (\gamma \cdot h)|_{\text{SVN}}$$

which is precisely the characterizing property of the Weil representation.  $\square$

To summarize: as a vector bundle with  $G(\mathcal{L}^B)_{Y^*(d)} = (H_B)_{Y^*(d)}$ -action,  $\text{SVN}_{Y^*(d)}$  can be descended to  $Y^*$  via the Weil representation of  $\text{SL}_2(\mathbf{Z}/d)$ , where it acquires an action of  $\mathcal{G}(\mathcal{L}^B)$ .

(v): Descending the  $\mathcal{G}(\mathcal{L}^B)$ -equivariant Hom sheaf Knowing that  $\text{SVN}_{Y^*(d)}$  descends to a rank- $d$  vector bundle  $\mathbf{M}$  on  $Y^*$  with  $\mathcal{G}(\mathcal{L}^B)$ -action by (iv), and that  $(\Theta_{\text{norm}})_{Y^*}$  is also a rank- $d$  vector bundle on  $Y^*$  with  $\mathcal{G}(\mathcal{L}^B)$ -action by definition of  $\mathcal{G}(\mathcal{L}^B)$ , we can form the rank- $d^2$  vector bundle

$$V = \underline{\text{Hom}}(\mathbf{M}_{Y^*}, (\Theta_{\text{norm}})_{Y^*}) \text{ on } Y^*$$

which admits an  $\mathcal{O}_{Y^*}$ -linear action of the  $Y^*$ -group scheme  $\mathcal{G}(\mathcal{L}^B)$ . The action of  $\mathbf{G}_m \subset \mathcal{G}(\mathcal{L}^B)$  is trivial on  $V$  (as it is by the identity character on both  $\mathbf{M}$  and  $\Theta_{\text{norm}}$ ), so the action factors through the finite étale  $Y^*$ -group scheme  $K(\mathcal{L}^B)$ . The same applies to the action of  $\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}$  on the pullback

$$p_d^*V = \underline{\text{Hom}}(\text{SVN}_{Y^*(d)}, (\Theta_{\text{norm}})_{Y^*(d)}) \text{ on } Y^*(d).$$

What we have done so far provides a descent datum for the action of  $H_B$  on  $p_d^*V$  that descends it to the action of  $\mathcal{G}(\mathcal{L}^B)_{Y^*}$  on  $V$ .

The fixed-point sheaves

$$\underline{\text{Hom}}_{\mathcal{G}(\mathcal{L}^B)}(\mathbf{M}_{Y^*}, (\Theta_{\text{norm}})_{Y^*}) := V^{\mathcal{G}(\mathcal{L}^B)}$$

and

$$\underline{\text{Hom}}_{(H_B)_{Y^*(d)}}(\text{SVN}_{Y^*(d)}, (\Theta_{\text{norm}})_{Y^*(d)}) := (p_d^*V)^{\mathcal{G}(\mathcal{L}^B)_{Y^*(d)}}$$

are *defined* as flat-fppf sheaves on  $Y^*$  and  $Y^*(d)$ , following [14, Exposé 1, Définition 2.3.2] to have points

$$V^{\mathcal{G}(\mathcal{L}^B)}(T \rightarrow S) := \{v \in \Gamma(T, V_T) : g \cdot v_{T'} = v_{T'} \text{ for all } g \in \mathcal{G}(\mathcal{L}^B)(T'), \text{ fppf } T' \rightarrow T\}$$

for all flat  $S$ -schemes  $T$ , and similarly for  $p_d^*V$ .

**Lemma III<sub>a</sub>.9.**  $V^{\mathcal{G}(\mathcal{L}^B)}$  and  $(p_d^*V)^{(H_B)_{Y^*(d)}}$  are quasicoherent sheaves on their respective bases  $Y^*$  and  $Y^*(d)$ .

*Proof.* This is true for *all* fixed-point sheaves of flat affine  $S$ -group schemes  $G$  acting on quasicoherent  $\mathcal{O}_S$ -modules  $\mathcal{F}$ , where  $S$  is an arbitrary scheme, as shown in [43, Ch. V, §1.2].

Indeed, the flatness of  $G/S$  allows one to package the action of  $G$  on a quasicoherent  $\mathcal{O}_S$ -module  $\mathcal{F}$  as an  $\mathcal{O}_S$ -module coaction

$$\rho : \mathcal{F} \xrightarrow{1 \otimes -} \mathcal{A}_G \otimes_S \mathcal{F} \rightarrow \mathcal{A}_G \otimes_S \mathcal{F}.$$

To unpack the definition of this from [14, Exposé I, §4]: over an affine  $U = \text{Spec} A \subset S$ , where  $\mathcal{F}|_U = \widetilde{M}$  for an  $A$ -module  $M$ , the universal point

$$1 \in G_U(G_U)$$

induces an action  $\mathcal{F}(G_U) \rightarrow \mathcal{F}(G_U)$ , which is equivalent to a map  $\mathcal{A}_{G_U} \otimes M \rightarrow \mathcal{A}_{G_U} \otimes M$ , which can be composed with the pullback map  $M \rightarrow \mathcal{A}_{G_U} \otimes M$  to get the coaction

$$\rho|_U : M \rightarrow \mathcal{A}_{G_U} \otimes M \rightarrow \mathcal{A}_{G_U} \otimes M,$$

the formation of which is compatible with passing from  $A$  to  $A_f$  for any nonzero  $f \in A$ , hence producing the quasicoherent sheaf homomorphism  $\rho$ .

Following [43, Ch. V, §1.2], we will prove:

- (1) The restriction  $(\mathcal{F}^G)_{\text{Zar}}$  of  $F^G$  to the small Zariski site of  $S$  is quasicoherent.
- (2) The flat-fppf sheaf  $\mathcal{F}^G$  is represented by  $(\mathcal{F}^G)_{\text{Zar}}$ .

Both of these follow from the fact that for a flat  $S$ -scheme  $T$ , a vector  $v \in \mathcal{F}(T) = \Gamma(T, \mathcal{F}_T)$  is in  $\mathcal{F}^G(T)$  if and only if  $1_{G_T} \cdot v_{G_T} = v_{G_T}$ . This fact is true because  $G/S$  is flat and because for all fppf  $T$ -morphisms  $g : T' \rightarrow G_T$ , the condition  $1_{G_T} \cdot v_{G_T} = v_{G_T}$  pulls back under  $g$  to  $g \cdot v_{T'} = v_{T'}$ .

Indeed, this implies that for an affine Zariski open  $U = \text{Spec} A \subset S$ ,

$$(\mathcal{F}^G)_{\text{Zar}}(U) = \ker(\rho|_U - 1 \otimes (-)) \subset M = \Gamma(U, \mathcal{F}).$$

This kernel being compatible with localizations (since they are flat), we obtain

$$(\mathcal{F}^G)_{\text{Zar}}(U_f) = \ker((\rho|_U)_f - 1 \otimes (-) : M_f \rightarrow \mathcal{A}_{G_{U_f}} \otimes M_f) \subset M_f = \Gamma(U_f, \mathcal{F})$$

for all nonzero  $f \in A$ , which implies that  $(\mathcal{F}^G)_{\text{Zar}}$  is the quasicoherent sheaf with values  $\ker(\rho|_U - 1 \otimes (-))$  on affine patches  $U$ , that is,

$$(\mathcal{F}^G)_{\text{Zar}} = \ker(\rho - 1 \otimes (-) : \mathcal{F} \rightarrow \mathcal{A}_G \otimes \mathcal{F})$$

as a quasicoherent  $\mathcal{O}_S$ -module.

(2) follows from the exact same argument with localization replaced with arbitrary flat base change: for flat  $T \rightarrow S$ , we have

$$\begin{aligned} (\mathcal{F}_{\text{Zar}}^G)(T) &:= \Gamma(T, \mathcal{F}_{\text{Zar}, T}^G) \\ &= \Gamma(T, \ker((\rho_T - 1 \otimes (-)) : \mathcal{F}_T \rightarrow \mathcal{A}_{G_T} \otimes_T \mathcal{F}_T)) \\ &= \ker(\rho_T - 1 \otimes (-) : \Gamma(T, \mathcal{F}_T) \rightarrow \Gamma(T, \mathcal{A}_{G_T} \otimes \mathcal{F}_T)) \\ &= \mathcal{F}^G(T) \end{aligned}$$

where the second line uses flatness of  $T \rightarrow S$  to pass the base-change to  $T$  past the kernel, and the last line is from the aforementioned principle that a section in  $\Gamma(T, \mathcal{F}_T)$  is fixed after all fppf base changes by all  $G_T(T')$  if and only if it is fixed after base change to  $G_T$  by  $1 \in G_T(G_T)$ .  $\square$

As a corollary of the lemma and the base change principle in its proof, we have:

**Lemma III<sub>a</sub>.10.** *The quasicoherent sheaf*

$$\underline{\text{Hom}}_{\mathcal{G}(\mathcal{L}^B)_{Y^*}}(\mathbf{M}_{Y^*}, (\Theta_{\text{norm}})_{Y^*}) \subset \underline{\text{Hom}}(\mathbf{M}_{Y^*}, (\Theta_{\text{norm}})_{Y^*})$$

on  $Y^*$  is descended from the quasicoherent sheaf

$$\underline{\text{Hom}}_{(H_B)_{Y^*(\mathbf{d})}}(\text{SVN}_{Y^*(\mathbf{d})}, (\Theta_{\text{norm}})_{Y^*(\mathbf{d})}) \subset \underline{\text{Hom}}(\text{SVN}_{Y^*(\mathbf{d})}, (\Theta_{\text{norm}})_{Y^*(\mathbf{d})})$$

on  $Y^*(\mathbf{d})$  with descent datum for the action of  $\text{SL}_2(\mathbf{Z}/\mathbf{d})$  that is restricted from the one for the ambient Hom-sheaves.

*Proof.* This is equivalent to the statement that there is an isomorphism

$$p_{\mathbf{d}}^* \underline{\text{Hom}}_{\mathcal{G}(\mathcal{L}^B)_{Y^*}}(\mathbf{M}_{Y^*}, (\Theta_{\text{norm}})_{Y^*}) \simeq \underline{\text{Hom}}_{(H_B)_{Y^*(\mathbf{d})}}(\text{SVN}_{Y^*(\mathbf{d})}, (\Theta_{\text{norm}})_{Y^*(\mathbf{d})})$$

for which the diagram

$$\begin{array}{ccc}
 p_d^* \underline{\mathrm{Hom}}_{\mathcal{G}(\mathcal{L}^B)_{Y^*}}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*}) & \longrightarrow & \underline{\mathrm{Hom}}_{(H_B)_{Y^*(d)}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)}) \\
 \downarrow & & \downarrow \\
 p_d^* \underline{\mathrm{Hom}}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})
 \end{array}$$

commutes. The existence of such an isomorphism follows from the characterization from the previous lemma of  $\mathcal{F}^G$  as the kernel of the group coaction minus the pullback map.  $\square$

Finally, we promised to prove

**Lemma III<sub>a</sub>.11.**  $\underline{\mathrm{Hom}}_{\mathcal{G}(\mathcal{L}^B)_{Y^*}}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*})$  (and its pullback to  $Y^*(d)$ ) is a line bundle.

*Proof.* The sheaf  $\underline{\mathrm{Hom}}_{(H_B)_{Y^*(d)}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})$  is quasicoherent, and it is a submodule of the coherent  $\mathcal{O}_{Y^*(d)}$ -module  $\underline{\mathrm{Hom}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})$ , so it is coherent.

More relevantly, both  $\mathrm{SVN}_{Y^*(d)}$  and  $(\Theta_{\mathrm{norm}})_{Y^*(d)}$  are  $(H_B)_{Y^*(d)}$ -representations on rank- $d$  vector bundles of weight 1 by definition. So the desired lemma follows from [52, Proposition 2.12], as [52, Proposition 2.12(ii)] implies that the line bundle  $\mathcal{M}$  of [52, Proposition 2.12(iii)] is recovered from  $\mathcal{F}, \mathcal{F}'$  by taking  $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')$ .  $\square$

(vi): Descending the flat connection to  $\Theta_{\mathrm{line}}$ . The vector bundle  $\mathrm{SVN}_{Y^*(d)} \simeq \mathcal{O}^{\oplus d}$  is equipped with the trivial flat connection  $d$ . Since the matrix entries of the Weil action of  $\mathrm{SL}_2(\mathbf{Z}/d)$  on  $\mathrm{SVN}_{Y^*(d)}$  are constants valued in  $\mathbf{Z}'$ , and that action is used to descend  $\mathrm{SVN}_{Y^*(d)}$  to  $\mathbf{M}_{Y^*(d)}$  (see (iii)), the connection  $d$  descends to a flat connection denoted  $\nabla_{\mathbf{M}}$  on  $\mathbf{M}_{Y^*}$ .

Similarly, the flat connection  $\nabla$  on  $(\Theta_{\mathrm{norm}})_{Y^*(d)}$  is by definition pulled back from the flat connection on  $(\Theta_{\mathrm{norm}})_{Y^*}$  that has the same name.

The two connections  $d$  and  $\nabla$  induce a flat connection  $\nabla_{\mathrm{Hom}}$  on the rank- $d^2$  vector bundle  $\underline{\mathrm{Hom}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})$ , which descends to a flat connection on the rank- $d^2$   $Y^*$ -vector bundle  $\underline{\mathrm{Hom}}(\mathbf{M}_{Y^*}, (\Theta_{\mathrm{norm}})_{Y^*})$ .

**Lemma III<sub>a</sub>.12.** *The flat connection  $\nabla_{\mathrm{Hom}}$  restricts to a flat connection on the fixed-point sheaf  $\underline{\mathrm{Hom}}_{(H_B)_{Y^*(d)}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})$ .*

*Proof.* The action of  $(H_B)_{Y^*(d)}$  restricts to the identity character of  $\mathbf{G}_m$  on both the domain and codomain of the Hom sheaf; therefore, the action factors through the quotient to the constant group scheme  $K(\mathcal{L}^B)_{Y^*(d)} \simeq (\kappa_B \times \kappa_B)_{Y^*(d)}$ , and it suffices to prove that  $\nabla_{\mathrm{Hom}}$  restricts to a connection on

$$\underline{\mathrm{Hom}}(\mathrm{SVN}_{Y^*(d)}, (\Theta_{\mathrm{norm}})_{Y^*(d)})^{\kappa_B \times \kappa_B}.$$

It suffices to prove that the connection commutes with the action of  $\kappa_B^{(1)}$  and  $\kappa_B^{(2)}$ . To do that, it suffices to prove:

- (1)  $(H_B)_{Y^*(\mathbf{d})}$  commutes with  $d$  on  $(\text{SVN})_{Y^*(\mathbf{d})}$ .
- (2)  $(H_B)_{Y^*(\mathbf{d})}$  commutes with  $\nabla$  on  $(\Theta_{\text{norm}})_{Y^*(\mathbf{d})}$ .

(1) is obvious, as the action of  $H_B$  on  $\text{SVN}$  is already defined over  $\mathbf{Z}'$ : it is just by matrices whose entries are in  $\mathbf{Z}'$  and therefore constant from the point of view of  $d$ .

(2) is slightly less obvious. It follows from the fact that the action of  $(H_B)_{Y^*(\mathbf{d})}$  on  $(\Theta_{\text{norm}})_{Y^*(\mathbf{d})}$  is defined to be the natural action of  $\mathcal{G}(\mathcal{L}^B)_{Y^*(\mathbf{d})} \simeq (H_B)_{Y^*(\mathbf{d})}$ . This is true for the heat connection defined as above for any of the line bundles on abelian schemes that its construction applies to, but we will restrain the notation to only work with  $\mathcal{L}^B$  on  $\mathcal{E}^m/Y^*(\mathbf{d})$ .

Let  $g \in \mathcal{G}(\mathcal{L}^B)(T)$  where  $T \rightarrow Y^*(\mathbf{d})$  is flat. That is,  $g = (x, \varphi)$  where  $x \in \mathcal{E}^m(T)$  is such that  $\varphi : T_x^*(\mathcal{L}_T^B) \simeq \mathcal{L}^B$ . The action of  $g$  on  $\Theta_{\text{norm}} = R^s\pi_*\mathcal{L}_T^B \otimes \omega^{1/2}$  is by pullback to  $R^s\pi_*T_x^*\mathcal{L}_T^B$  followed by pullback along  $\varphi^{-1}$ . The connection  $\nabla$  eats a local tangent vector  $v \in T_{Y^*(\mathbf{d})/\mathbf{Z}'}(U)$  and acts on  $\Theta_{\text{norm}}$  via a heat operator

$$D_v \in \mathcal{D}\text{iff}^{\text{Heat}}(\mathcal{L}^B \otimes \omega^{m/2}).$$

We want to prove that  $\nabla_v$  and  $g$  commute, which we can do on the level of  $\mathcal{L}^B \otimes \omega^{m/2}$  instead of  $\Theta_{\text{norm}}$ . At the level of the projective connection, it was remarked in [55, §2.3.8] that this is true because of the uniqueness of the projective heat operator, as  $D_v$  and  $gD_vg^{-1}$  share the required properties to pin it down up to  $\mathcal{O}_{Y^*(\mathbf{d})}$ . It is likely possible to make a similar uniqueness argument to directly prove that  $D_v = gD_vg^{-1}$ , but we instead deduce it from the weaker fact (just mentioned) that  $gD_vg^{-1} - D_v \in H^0(Y^*(\mathbf{d}), \mathcal{O})$ . In other words,

$$g \mapsto (\nabla^{\text{Heat}})^g - \nabla^{\text{Heat}} = \eta_g \in H^0(U, \Omega_{Y^*(\mathbf{d})/\mathbf{Z}'}^1)$$

where  $\eta_g$  depends on  $g$  and our goal is to prove  $\eta_g = 0$ . This turns out to not require additional work:  $g \mapsto \eta_g$  has the property that

$$\eta_{gh} = \eta_g + \eta_h,$$

so by choosing  $\varphi$  such that  $g = (x, \varphi)$  satisfies  $g^{\mathbf{d}} = 1$  (where we can take the power to be  $\mathbf{d}$  because  $x$  is in the  $\mathbf{d}$ -torsion and  $\zeta \in \mathbf{Z}'$  so  $\varphi$  can be modified if necessary), we can ensure

$$0 = \eta_1 = \eta_{g^{\mathbf{d}}} = \mathbf{d}\eta_g,$$

hence  $\eta_g = 0$  since  $\mathbf{d}^{-1} \in \mathbf{Z}'$ . □

**Corollary III<sub>a</sub>.13.** *The flat connection  $\nabla_{\text{Hom}}$  on  $\underline{\text{Hom}}_{(H_B)_{Y^*(\mathbf{d})}}(\text{SVN}_{Y^*(\mathbf{d})}, (\Theta_{\text{norm}})_{Y^*(\mathbf{d})})$  descends to a flat connection on the descended line bundle  $\underline{\text{Hom}}_{\mathcal{G}(\mathcal{L}^B)_{Y^*}}(\mathbf{M}_{Y^*}, (\Theta_{\text{norm}})_{Y^*})$ .*

(vii): Constructing a horizontal section of  $\Theta_{\text{line}}$ .

Let us define

$$\Theta_{\text{line}} = \text{Hom}_{G(\mathcal{L}^B)}(\mathbf{M}, \Theta_{\text{norm}})$$

the line bundle on  $Y^*$  that we studied via descent in the previous section.

We proved that  $\Theta_{\text{line}}$  is descended from  $\text{Hom}_{H_B}(\text{SVN}_{Y^*(\mathbf{d})}, (\Theta_{\text{norm}})_{Y^*(\mathbf{d})})$  by restriction to  $H_B$ -fixed points of the descent datum for  $\text{SL}_2(\mathbf{Z}/\mathbf{d})$ . This makes it easy to describe via descent a section  $\underline{\theta}_{\text{line}}$  of  $\Theta_{\text{line}}$  in a standard punctured neighborhood of a cusp.

In particular: if  $\text{Spec} \mathbf{Z}'((q^{1/w})) \rightarrow Y_{\mathbf{Z}'}^*$  is a punctured neighborhood of a cusp of  $Y_{\mathbf{Z}'}^*$  of width  $w$  ( $w \in \{2, 4\}$ ), then a punctured neighborhood of a cusp of  $Y^*(\mathbf{d})$  lying over it always fits into the diagram

$$\begin{array}{ccc} \text{Spec} \mathbf{Z}'((q^{1/(w\mathbf{d})})) & \longrightarrow & Y^*(\mathbf{d}) \\ \downarrow \mathbf{Z}/\mathbf{d} & & \downarrow \text{SL}_2(\mathbf{Z}/\mathbf{d}) \\ \text{Spec} \mathbf{Z}'((q^{1/w})) & \xrightarrow{i} & Y_{\mathbf{Z}'}^* \end{array}$$

where the vertical map on the left is a  $\mathbf{Z}/\mathbf{d}$ -Galois covering with action  $q^{1/(w\mathbf{d})} \mapsto \zeta q^{1/(w\mathbf{d})}$ , where the  $\mathbf{Z}/\mathbf{d}$  is a unipotent radical in  $\text{SL}_2(\mathbf{Z}/\mathbf{d})$  preserving the particular cusp upstairs.

We only need one cusp of  $Y^*(\mathbf{d})$  lying over each cusp of  $Y^*$ , since our goal is just to construct a section downstairs by descent. We choose that cusp (over any choice of cusp of  $Y^*$  corresponding to a  $Y^*$ -level structure on  $E_{\text{Tate},w}^m$ ) to be the  $i : \text{Spec} \mathbf{Z}'((q^{1/(w\mathbf{d})})) \rightarrow Y^*(\mathbf{d})$  coming from the level structure on  $E_{\text{Tate},w\mathbf{d}}^m$  given by sending  $(\underline{x}, \underline{y})$  to  $q^{-\underline{x}}e(\underline{y})$  where  $e(\underline{y})$  is viewed algebraically as a  $\mathbf{d}$ -th root of unity.

To produce a flat section over the punctured neighborhood  $\text{Spec} \mathbf{Z}'((q^{1/w}))$  of  $i^* \Theta_{\text{line}}$ , it suffices to produce a flat section over  $\text{Spec} \mathbf{Z}'((q^{1/(w\mathbf{d})}))$  (representing a neighborhood of our choice of cusp lying over the chosen one of  $Y^*$ ) of the restriction of  $\text{Hom}_{(H_B)_{Y^*(\mathbf{d})}}(\text{SVN}_{Y^*(\mathbf{d})}, (\Theta_{\text{norm}})_{Y^*(\mathbf{d})})$ , and to check that it is fixed by the  $\mathbf{Z}/\mathbf{d}$ -action.

We have already computed a basis of sections for  $\Theta$  over all such cusps, where (following previous section)  $\mathcal{L}^B$  either restricts to  $\mathcal{L}_{0,0}^B$  (over all three cusps of  $Y^*$  lying over  $0 \in X_0(2)$ ) or  $\mathcal{L}_{1,0}^B$  (over the cusp lying over  $\infty$ ). In the case of  $\mathcal{L}_{0,0}^B$ , these sections are indexed by cosets  $\kappa + \mathbf{Z}^m$  inside  $B^{-1}\mathbf{Z}^m$  and are described by cocycles of the general form

$$(III_{\text{a.1}}) \quad \vartheta_{\kappa} : (v_0, \dots, v_s) \in (\mathbf{Z}^m)^{s+1} \mapsto \sum_{\underline{m} \in \mathbf{Z}^m \cap (\mathcal{C}(v) + \alpha)} \underline{z}^{B(\underline{m} + \kappa)} q^{\langle \underline{m} + \kappa, \underline{m} + \kappa \rangle / 2}$$

where  $\mathcal{C}$  is a region that depends only on  $v_0, \dots, v_s \in \mathbf{Z}^m$ .

Then the rule sending

$$\underline{\theta}_{\text{line}} : \text{indicator of } \kappa \in \text{SVN} \mapsto \theta_{\kappa} \otimes (dz/z)^{1/2}$$

defines a homomorphism from  $\text{SVN}$  to  $R^s \pi_* \mathcal{L}^B \otimes \omega^{1/2}$  over the punctured cusp neighborhood  $\text{Spec} \mathbf{Z}'((q^{1/(w\mathbf{d})}))$ .

The section  $\underline{\theta}_{\text{line}}$  is  $H_B$ -equivariant, because:

- elements  $(1, \underline{x}, 0)$  of  $1 \times \kappa_B^{(1)} \times 0$  act by sending  $\vartheta_\kappa$  to

$$\vartheta_\kappa(q^{-\underline{x}} \underline{z}) q^{Q(\underline{x})} \underline{z}^{-B\underline{x}},$$

which is

$$(v_0, \dots, v_s) \mapsto \sum_{\underline{n} \in \kappa + \mathbf{Z}^m} q^{Q(\underline{n}) - \langle \underline{x}, \underline{n} \rangle} \underline{z}^{B\underline{n}} q^{Q(\underline{x})} \underline{z}^{-B\underline{x}} = \sum_{\underline{n} \in \kappa + \mathbf{Z}^m} q^{Q(\underline{n} - \underline{x})} \underline{z}^{B(\underline{n} - \underline{x})},$$

i.e.  $\vartheta_{\kappa - \underline{x}} = \text{image of } (1, \underline{x}, 0) \cdot \mathbf{1}_\kappa$ .

- elements  $(1, 0, \underline{y})$  of  $1 \times 0 \times \kappa_B^{(2)}$  act by sending  $\vartheta_\kappa$  to

$$\vartheta_\kappa(e(\underline{y}) \underline{z}) = \sum_{\underline{n} \in \kappa + \mathbf{Z}^m} q^{Q(\underline{n})} \underline{z}^{B\underline{n}} \exp(2\pi i \langle \underline{n}, \underline{y} \rangle) = \exp(2\pi i \langle \kappa, \underline{y} \rangle) \vartheta_\kappa$$

which is the image of  $(1, 0, \underline{y}) \cdot \mathbf{1}_\kappa$ .

The section  $\underline{\theta}_{\text{line}}$  descends to a section of  $\Theta_{\text{line}}$  near the chosen cusp over 0 of  $Y^*$ , because (having defined it at the appropriate cusp over that one in  $Y^*(\mathbf{d})$  so that the isomorphism  $(\kappa_B \times \kappa_B)_{\mathbf{Z}'((q^{1/(wd)}))} \rightarrow \ker(B : E_{\text{Tate}}^m \rightarrow E_{\text{Tate}}^m)$  is the one being used above) making the coordinate change  $q^{1/(wd)} \mapsto \zeta_{\mathbf{d}} q^{1/(wd)}$  corresponds to the action of  $U^{-w} \in \text{SL}_2(\mathbf{Z}/\mathbf{d})$ , which acts on  $\mathbf{1}_\kappa$  by sending it to  $\zeta_{1/2}^{wd\langle \kappa, \kappa \rangle} \mathbf{1}_\kappa$ , while it sends  $\vartheta_\kappa$  to

$$\sum_{n \in \mathbf{Z}^m \cap \mathcal{C}} \zeta_{1/2}^{wd\langle n+\kappa, n+\kappa \rangle} q^{Q(n+\kappa)} \underline{z} = \zeta_{1/2}^{wd\langle \kappa, \kappa \rangle} \vartheta_\kappa.$$

Hence the section  $\underline{\theta}_{\text{line}}$  of the pullback of  $(\Theta_{\text{line}})_{Y^*(\mathbf{d})}$  to  $\mathbf{Z}'((q^{1/(wd)}))$  descends to  $Y^*$ . This is completely analogous to the analytic procedure of checking that the flat section  $\underline{\theta}_{\text{line}}$  over a suitable cusp of  $Y^*(\mathbf{d})$  is a certificate for the local monodromy of  $\nabla$  on  $(\Theta_{\text{line}})_{Y^*}$  being trivial for a loop around the cusp below it taking the coordinate  $q$  around a loop.

We can similarly carry this out for a suitable cusp over the cusp  $\infty$  of  $Y^*$ , where the formulas are a bit more complicated because the relevant half-theta characteristic is  $(1, 0)$  instead of  $(0, 0)$ .

In that situation, we let  $\eta \in B^{-1}\mathbf{Z}^m/\mathbf{Z}^m = \kappa_B$  be the unique element such that  $2\eta = B^{-1}\underline{b} \in B^{-1}\mathbf{Z}^m/\mathbf{Z}^m$ . Then let  $\vartheta_\kappa$  be the cocycle taking  $(v_0, \dots, v_s)$  to

$$\sum_{\underline{n} \in \mathbf{Z}^m \cap \mathcal{C}} q^{Q(\underline{n} + \underline{\kappa}) + \frac{\mathbf{d}}{2} \underline{b}^\top (\underline{n} + \underline{\kappa})} \underline{z}^{B(\underline{n} + \underline{\kappa}) + \frac{\mathbf{d}+1}{2} \underline{b}},$$

where  $\underline{b} = (B_{11}, \dots, B_{mm})$ .

It is worth remarking that this defines a well-defined section of  $\Theta_{\text{line}}$  over  $\mathbf{Z}'((q^{1/(4\mathbf{d}})))$ , since the worst possible denominators in the exponent of  $q$  are actually  $2\mathbf{d}$ .

We should verify that  $\underline{\theta}_{\text{line}} : \mathbf{1}_\kappa \mapsto \vartheta_\kappa$  is  $H_B$ -equivariant (in fact it was calculated to make it so):

- elements  $(1, \underline{x}, 0)$  of  $1 \times \kappa_B^{(1)} \times 0$  act by sending  $\vartheta_\kappa$  to

$$\vartheta_\kappa(q^{-\underline{x}} \underline{z}) q^{Q(\underline{x}) + \frac{1}{2} \underline{b}^\top \underline{x}} \underline{z}^{-B\underline{x}},$$

whose value on a simplex inducing a region  $\mathcal{C}$  is

$$\begin{aligned} q^{Q(x) + \frac{1}{2}b^\top x} \underline{z}^{-Bx} \sum_n q^{Q(n+\kappa) + \frac{d}{2}b^\top(n+\kappa)} (q^{-x} \underline{z})^{B(n+\kappa) + \frac{d+1}{2}b} \\ = \sum_n q^{Q(n+\kappa-x) + \frac{d}{2}b^\top(n+\kappa-x)} \underline{z}^{B(n+\kappa-x) + \frac{d+1}{2}b} \\ = \vartheta_{\kappa-x}. \end{aligned}$$

- elements  $(1, 0, \underline{y})$  of  $1 \times 0 \times \kappa_B^{(2)}$  act by sending  $\vartheta_\kappa$  to

$$\begin{aligned} \vartheta_\kappa(e(\underline{y}) \underline{z}) \exp(-(d+1)\pi i b^\top \underline{y}) &= \sum_{\underline{n} \in \mathbf{Z}^m \cap \mathcal{C}} q^{Q(n+\kappa) + \frac{d}{2}b^\top(n+\kappa)} (e(\underline{y}) \underline{z})^{B(n+\kappa) + \frac{d+1}{2}b} \exp(-(d+1)\pi i b^\top \underline{y}) \\ &= \sum_{\underline{n} \in \mathbf{Z}^m \cap \mathcal{C}} q^{Q(n+\kappa) + \frac{d}{2}b^\top(n+\kappa)} \underline{z}^{B(n+\kappa) + \frac{d+1}{2}b} \exp(2\pi i \langle n + \kappa, \underline{y} \rangle) \\ &= \vartheta_\kappa \zeta_d^{\mathbf{d} \langle \kappa, \underline{y} \rangle}. \end{aligned}$$

Thus  $\underline{\theta}_{\text{line}}$  is  $H_B$ -equivariant, as required.

To check that  $\underline{\theta}_{\text{line}}$  descends to  $\mathbf{Z}((q^{1/4}))$ , we need to check that the transformation taking  $q^{1/(4d)}$  to  $\zeta_d q^{1/(4d)}$  agrees with the Weil representation at  $U^{-4}$  after applying  $\underline{\theta}_{\text{line}} : \text{SVN} \rightarrow \Theta_{\text{norm}}$ .

That transformation sends  $\vartheta_\kappa$  to

$$\sum_{\underline{n} \in \mathbf{Z}^m \cap \mathcal{C}} \zeta_d^{4\mathbf{d}(Q(n+\kappa) + \frac{d}{2}b^\top(n+\kappa))} q^{Q(n+\kappa) + \frac{d}{2}b^\top(n+\kappa)} \underline{z}^{B(n+\kappa) + \frac{d+1}{2}b}.$$

where the exponent on  $\zeta_d$  simplifies to

$$\begin{aligned} 4\mathbf{d} \left( Q(n+\kappa) + \frac{d}{2}b^\top(n+\kappa) \right) &= 2\mathbf{d} \langle n+\kappa, n+\kappa \rangle + 2\mathbf{d}^2 b^\top(n+\kappa) \\ &\equiv 2\mathbf{d} \langle \kappa, \kappa \rangle \pmod{\mathbf{d}}. \end{aligned}$$

So  $U^{-4}$  sends  $\vartheta_\kappa$  to  $\zeta_{1/2}^{4\mathbf{d} \langle \kappa, \kappa \rangle} \vartheta_\kappa$ , which agrees with the Weil representation action of  $U^{-4}$  on  $\mathbf{1}_\kappa$  as a section of  $\text{SVN}$ ; hence  $\underline{\theta}_{\text{line}}$  descends to a flat section over  $\text{Spec} \mathbf{Z}((q^{1/4})) \rightarrow Y^*$  near the cusp at  $\infty$ .

*Remark.* The two descent calculations above are instructive and not overly burdensome to write down as we have, but it is relevant to remark that the full detail of that calculation was not actually necessary, thanks to the conclusion of [Lemma III<sub>0</sub>.3](#). This works as follows:  $i^* \Theta_{\text{line}}$  is identified (by looking at the image of  $\mathbf{1}_0$ ) with the  $\kappa_B^{(2)}$ -fixed vectors of the  $\mathbf{Z}'((q^{1/(wd)}))$ -module  $i^* \Theta_{\text{norm}}$ . This can (but need not) be viewed as an instance of Frobenius reciprocity. The special case  $\kappa = 0$  of the second bullet point<sup>21</sup> shows that  $\kappa_B^{(2)}$  fixes  $\vartheta_0$ , so  $\vartheta_0$  defines the section  $\underline{\theta}_{\text{line}}$  near a suitable cusp of  $Y^*(\mathbf{d})$  without having to define the other  $\vartheta_\kappa$ 's for  $\kappa \neq 0$ . The conclusion of [Lemma III<sub>0</sub>.3](#) says that  $U^{-w}$  (in fact  $U$ ) fixes  $\mathbf{1}_0$ , so the descent step is equivalent to showing that  $U^{-w}$  fixes  $\vartheta_0$ , which is again a special case of the calculation we

<sup>21</sup>That is:  $1/|\kappa_B|$  of half of the calculation that we did in the two bullet points in each case of defining  $\underline{\theta}_{\text{line}}$

already did that requires fewer symbols as there is no need to deal with any  $\vartheta_\kappa$  other than  $\vartheta_0$ . Especially in the case of the second cusp we considered, this makes things easier: we need only define

$$\vartheta_0(\underline{v}) = \sum_{\underline{n}} q^{Q(n) + \frac{d}{2}b^\top n} \underline{Z}^{Bn + \frac{d+1}{2}b}$$

from which everything else follows as just described. The extra  $\frac{d+1}{2}b$  in the exponent over  $\underline{Z}$  is there to make sure  $\vartheta_0$  is fixed by  $\kappa_B^{(2)}$ , and the extra  $\frac{d+1}{2}b^\top n$  in the exponent over  $q$  is there to ensure that  $\vartheta_0$  is a genuine cocycle.

We will use the perspective of this remark in [Section B](#) when we handle the case of even  $d$  in order to simplify our calculations.

### III<sub>b</sub>. THE $\theta$ -SECTION GLOBALIZES

Our goal is to prove

**Proposition III<sub>b</sub>.1.** *The line bundle  $\Theta_{\text{line}}$ , or its square if  $r + s$  is odd, together with the connection  $\nabla$  constructed earlier, is trivial as a bundle with flat connection on the generic fiber of  $Y^*$  over  $\mathbf{Q}'$ . Moreover, the four possible sections  $\underline{\theta}_{\text{line}}$  (or its square if  $r + s$  is odd) over the punctured cusp neighborhoods in  $Y^*$  extend to global sections of  $\Theta_{\text{line}}$  over all of  $Y_{\mathbf{Z}'}^*$ .*

**III<sub>b</sub>.1. The monodromy of  $(\Theta_{\text{line}}, \nabla)$  around the cusps is trivial on  $Y^*$ .** Let  $i : \mathbf{Z}((q^{1/w})) \rightarrow Y^*$  be one of the four cusp neighborhoods, where  $w \in \{2, 4\}$ . When we constructed the flat section  $\underline{\theta}_{\text{line}}$  of  $i^*\Theta_{\text{line}}$  by descent from a choice of cusp neighborhood  $\mathbf{Z}((q^{1/(wd)})) \rightarrow Y^*(d)$  lying over  $i$ , we performed exactly the same procedure that in the analytic context would verify that a local analytic solution to  $\nabla$  on  $Y^*$  near a point  $p$  near that cusp (that is, obtained from the explicit construction of sections in  $\text{Hom}_{H_B}(\text{SVN}, \Theta_{\text{norm}})$  over a small neighborhood in  $Y^*(d)$  which is isomorphic via  $p_d$  to a small neighborhood near  $p$ ) doesn't change when it is continued along a small loop around the cusp. In other words, we proved that the local monodromies of  $(\Theta_{\text{line}}, \nabla)$  around the four cusps are trivial, modulo the problem that  $\omega^{(r+s)/2}$  is not a well-defined line bundle unless  $r + s$  is even. We will not use this analytic formulation, instead only using the existence of the flat sections  $\underline{\theta}_{\text{line}}$  over the algebraic cusp neighborhoods, which we have already established.

**III<sub>b</sub>.2. The statement over  $\mathbf{Q}'$ .** We prove that  $\Theta_{\text{line}}$  is trivial on  $Y_{\mathbf{Q}'}^*$  and that  $\underline{\theta}_{\text{line}}$  extends there.

In the complex-analytic setting, one would argue that  $\Theta_{\text{line}}$  has trivial local monodromy (or trivial squared-monodromy if  $r + s$  is odd) at each cusp of  $Y^*$ . Since the fundamental group of  $\mathbf{P}^1 - \{4 \text{ points}\}$  is generated by loops around each cusp, this implies (after verifying that

$\nabla$  has regular singularities at the four cusps) that  $(\Theta_{\text{line}}, \nabla)$  corresponds via the Riemann–Hilbert correspondence to the trivial local system. In particular, any flat local section for  $\nabla$  (such as the analytic variants of the  $\underline{\theta}_{\text{line}}$  that we constructed) extends globally.

Being in the market for a stronger *algebraic* version of this anyway, we proceed purely algebraically. But our algebraic argument is completely analogous to the previous paragraph. We still use as crucial input the existence of the flat sections  $\underline{\theta}_{\text{line}}$  near the cusps: since they are by definition meromorphic sections of the structure sheaf of  $\mathbf{P}_{\mathbf{Q}'}$  near a cusp, there is no further analysis required for the “regular singularities” part. Still, we provide a complete argument, since the passage from  $\mathbf{Q}'$  to  $\mathbf{Z}'$  is somewhat delicate.

**Lemma III<sub>b</sub>.2.** *There is an isomorphism  $(\Theta_{\text{line}})_{Y_{\mathbf{Q}'}} \simeq \mathcal{O}_{Y_{\mathbf{Q}'}}$*

*Proof.* By [28, Ch. II, Proposition 6.5], there is a short exact sequence

$$\mathbf{Z}^4 \rightarrow \text{Pic}(\mathbf{P}_{\mathbf{Q}'}) \rightarrow \text{Pic}(Y^*) \rightarrow 0.$$

The curve  $\mathbf{P}_{\mathbf{Q}'}$  has Picard group  $\mathbf{Z}$ , and the map  $\mathbf{Z}^4 \rightarrow \mathbf{Z}$  here is surjective, because it takes  $\sum a_i[p_i]$  to the actual sheaf  $\mathcal{O}(\sum a_i[p_i]) \in \text{Pic}(\mathbf{P}_{\mathbf{Q}'}),$  from which we conclude that  $\text{Pic}(Y^*)$  is trivial.  $\square$

The line bundle  $(\Theta_{\text{line}})_{Y_{\mathbf{Q}'}} \simeq \mathcal{O}_{Y_{\mathbf{Q}'}}$  (resp.  $\Theta_{\text{line}}^2 \simeq \mathcal{O}$  if  $m$  is odd) is endowed with a flat connection  $\nabla$ . Letting  $e$  be a global section of  $(\Theta_{\text{line}})_{Y_{\mathbf{Q}'}}$  that provides the trivialization (resp. of  $(\Theta_{\text{line}}^2)_{Y_{\mathbf{Q}'}}$ ) and letting

$$\omega := -\nabla(e) \in H^0(Y_{\mathbf{Q}'}, \Omega_{Y_{\mathbf{Q}'}/\mathbf{Q}'}^1),$$

we know that  $\nabla$  is globally of the form

$$\nabla = d + \omega,$$

where  $\omega$  is a global 1-form on  $Y_{\mathbf{Q}'}^* = \mathbf{P}_{\mathbf{Q}'}^1 - \{4 \text{ points}\}.$

**Lemma III<sub>b</sub>.3.** *For all  $i = 1, 2, 3, 4$ , there is a local coordinate  $z_i$  near  $P_i$  such that  $\nabla$  has the form*

$$(III_b.1) \quad \nabla f = df + n_i \frac{dz_i}{z_i} f.$$

where each  $n_i$  is integral.

*Proof.* The formal completion of  $\mathbf{P}_{\mathbf{Q}'}$  at any of the four missing points is isomorphic to  $\mathbf{Q}'[[z]]$ , so  $\mathbf{P}^1 - \{4 \text{ points}\}$  admits a  $\mathbf{Q}'$ -scheme morphism

$$t_i : \text{Spec} \mathbf{Q}'((z)) \rightarrow \mathbf{P}^1 - \{4 \text{ points}\}.$$

Indeed, one can take this to be the Tate curve maps

$$\text{Spec} \mathbf{Q}'((q^{1/w})) \rightarrow Y^*, \quad w \in \{2, 4\}$$

corresponding to the four cusps.

Pulling back  $\nabla$  by  $t_i$  yields a differential form

$$t_i^* \omega \in H^0(\mathrm{Spec} \mathbf{Q}'((z)), \Omega_{\mathbf{Q}'((z))/\mathbf{Q}'}^1).$$

We constructed a section  $\underline{\theta}_{\mathrm{line}}$  of  $t_i^* \Theta_{\mathrm{line}} \simeq t_i^* \mathcal{O} \simeq \mathcal{O}_{\mathrm{Spec} \mathbf{Q}'((z))}$  (that is, an element  $f \in \mathbf{Q}'((z))$ ) that is flat for  $\nabla$ . In fact, it can be checked directly that

$$\underline{\theta}_{\mathrm{line}} \neq 0$$

(not literally obvious in degree- $s$  cohomology once  $s > 0$ ). So there is an

$$0 \neq f \in \mathbf{Q}'((z)) \text{ such that } df + (t_i^* \omega)f = 0,$$

in other words,

$$t_i^* \omega = -d \log f.$$

Letting  $n = v_z(f)$ , we have  $f = z^n u$ , where  $u \in \mathbf{Q}'[[z]]^\times$ . Then we can write

$$t_i^* \omega = -n dz - d \log u.$$

By changing local coordinates from  $z$  to  $uz$  we eliminate the extra  $d \log u$  term.

□

**Corollary III<sub>b</sub>.4.** *As line bundles with flat connection on  $Y_{\mathbf{Q}'}^*$ , we have*

$$((\Theta_{\mathrm{line}})_{Y_{\mathbf{Q}'}^*}, \nabla) \simeq (\mathcal{O}_{Y_{\mathbf{Q}'}^*}, d).$$

*Proof.* The lemma does NOT guarantee that  $\nabla = d + \omega$  has  $\omega$  of the form  $ndz/z$  for any particular coordinate  $z$ . It only guarantees this near the cusps. In other words, it guarantees that  $\omega$  extends to a meromorphic differential on the curve  $\mathbf{P}_{\mathbf{Q}'}^1$  with poles only at the four cusps, with integral residues  $n_1, \dots, n_4 \in \mathbf{Z}$ .

By changing coordinates we may assume that all four missing points are of the form  $a_1, a_2, a_3, a_4 \in \mathbf{A}^1(\mathbf{Q}') \subset \mathbf{P}_{\mathbf{Q}'}^1$ , that is, they are finite.

Then

$$\omega - \sum_{i=1}^4 n_i \frac{dt}{t - a_i} \text{ is regular except possibly a simple pole at } \infty$$

where  $t$  is the global coordinate on  $\mathbf{A}^1 \subset \mathbf{P}^1$ , since the terms  $dt/(t - a_i)$  all have a simple pole at  $\infty$ . Since this difference is regular on  $\mathbf{A}_{\mathbf{Q}'}^1$  and is therefore of the form  $P(t)dt$ , any pole would actually imply that it has a double pole at  $\infty$ , so we conclude that

$$\omega = \sum_{i=1}^4 n_i \frac{dt}{t - a_i}$$

as a consequence of the non-existence of regular differentials on  $\mathbf{P}_{\mathbf{Q}'}^1$ . This also implies that the sum of the residues of  $\omega$  is  $\sum_{i=1}^4 n_i = 0$ , since  $\omega$  must be regular at  $\infty$ . As a result, the rational function

$$R(t) := \prod_{i=1}^4 (t - a_i)^{-n_i} \in \mathbf{Q}'(t)$$

is regular at  $\infty$  and everywhere else except possibly the  $a_i$ , and satisfies (by the formula defining it)

$$\mathrm{dlog} R = -\omega.$$

In particular, viewed as a regular function on  $Y_{\mathbf{Q}'}^* \simeq \mathbf{P}^1 - \{a_1, a_2, a_3, a_4\}$ , we have concluded that  $R$  is a global section of the structure sheaf that is flat for  $\nabla$ , thus proving the desired triviality of the connection  $\nabla$ .  $\square$

At this stage, the flatness of  $\underline{\theta}_{\text{line}}$  for  $\nabla$  immediately implies that after base change to  $\mathbf{Q}'$ ,  $\underline{\theta}_{\text{line}}$  extends to a global section of  $(\Theta_{\text{line}})_{Y_{\mathbf{Q}'}^*}$ .

**III<sub>b</sub>.3. The statement over  $\mathbf{Z}'$ .** Now let us verify that  $\underline{\theta}$  in fact extends to a global section on  $Y_{\mathbf{Z}'}^*$ . The idea here is that the divisor of poles of  $\theta$  is disjoint from the generic fiber, so, if nonempty, it must be a single vertical fiber of  $Y$  over  $\mathrm{Spec} \mathbf{Z}'$ . But that cannot be the case either, because we know that it extends over the cusp.

To write this formally takes a little care, since we have not actually compactified the situation. We know that  $\Theta_{\text{line}}$  is, over  $\mathbf{Q}'$ , isomorphic to the trivial bundle with trivial connection, and  $\underline{\theta}_{\text{line}}$  extends to  $Y_{\mathbf{Q}'}^*$ . Therefore there exists  $M \in \mathbf{Z}'$  such that  $M \cdot \underline{\theta}_{\text{line}}$  extends to a section of  $\Theta_{\text{line}}$  on  $Y_{\mathbf{Z}'}^*$ . Fix a prime ideal  $\mathfrak{p}$  of  $\mathbf{Z}'$  and let  $D_{\mathfrak{p}}$  be the corresponding divisor on  $Y_{\mathbf{Z}'}^*$  (the fiber over  $[\mathfrak{p}]$  of  $Y^* \rightarrow \mathrm{Spec} \mathbf{Z}'$ ).

The order of vanishing along  $D_{\mathfrak{p}}$  defines a valuation on sections of  $\Theta_{\text{line}}$ . We will show that

$$M \cdot \underline{\theta}_{\text{line}} \text{ vanishes along } D_{\mathfrak{p}}, \text{ if } \mathfrak{p} | M.$$

Suppose otherwise. Consider the restriction of  $\Theta_{\text{line}}$  to the special fiber  $Y_k^*$  of  $Y^*$  over the residue field  $k = \mathbf{Z}'/\mathfrak{p}$ ; this is a line bundle on a smooth open curve; and  $M \cdot \underline{\theta}_{\text{line}}$  gives a section of this bundle that is (by assumption) not identically vanishing. However, by inspection of  $q$ -expansion, it *vanishes* when pulled back by the map  $\mathrm{Spec} k((q)) \rightarrow Y_k^*$  that gives a formal neighbourhood of one of the missing points.<sup>22</sup>

This state of affairs is impossible:

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<sup>22</sup>This vanishing requires a little justification, given our setup. The map from  $k((q))$  to  $\tilde{Y}$  factors through  $\mathbf{Z}((q))$ . By construction, our original section  $\underline{\theta}_{\text{line}}$  already extends from  $\mathbf{Q}((q))$  to  $\mathbf{Z}((q))$ , call that extension  $\underline{\theta}'_{\text{line}}$ ; now  $M \cdot \underline{\theta}'_{\text{line}}$  and  $M \cdot \underline{\theta}_{\text{line}}$  are sections of a line bundle on  $\mathbf{Z}((q))$  agreeing on  $\mathbf{Q}((q))$  and so they agree. Therefore,  $M \cdot \theta$  becomes zero when reduced to  $k((q))$ .

**Lemma III<sub>b</sub>.5.** *Let  $C$  be a proper smooth curve over a field  $k$ ,  $U \subset C$  an open subset. If a section  $s$  of a line bundle on  $U$  vanishes identically when restricted to the punctured formal neighbourhood of a point of  $C - U$ , then it vanishes identically.*

*Proof.* Shrinking  $U$  we suppose that the original line bundle is trivial and that there is a regular function  $z$  on  $U$  that defines a local coordinate at  $p$ . As such, we can identify  $s$  with a nonvanishing regular function on  $U$ . If we write  $n$  for the order of pole of  $s$  at  $p$ , then  $z^n \cdot s$  extends over the point  $p$  and is nonzero there. It therefore defines a nonvanishing class in the completed local ring  $\widehat{\mathcal{O}}_p$ . Since that completed local ring is an integral domain, in fact isomorphic to  $k[[z]]$  by means of “Taylor series”, the map

$$\widehat{\mathcal{O}}_p \rightarrow \widehat{\mathcal{O}}_p \left[ \frac{1}{z} \right]$$

is injective, and so the image of  $z^n s$  and so also  $s$  in the right-hand ring is nonzero. This amounts to say that  $s$  is not identically zero in the formal neighbourhood. □

It follows by repeated division by the prime below  $\mathfrak{p}$  (which is OK because we only care about getting rid of primes away from  $2d$ , where  $\mathbf{Q}'/\mathbf{Q}$  is unramified anyway) that one can assume that  $v_{\mathfrak{p}}(M) = 0$  for all  $\mathfrak{p}$  away from  $2d$ , and hence that  $\theta_{\text{line}}$  extends to  $Y_{\mathbf{Z}'}^*$ .

**III<sub>b</sub>.4. Various equivalent formulations.** In constructing a global section of  $\Theta_{\text{line}}$ , we have constructed a global homomorphism

$$\mathbf{M} \longrightarrow \Theta_{\text{norm}}.$$

from the model bundle  $\mathbf{M}$  to  $\Theta_{\text{norm}}$  over  $Y^*$ , and in particular over  $Y^*(\mathbf{d})$ . But over  $Y^*(\mathbf{d})$  the bundle  $\mathbf{M}$  is trivial, and so we have correspondingly extended the flat sections  $\vartheta_x \otimes (dz/z)^{m/2}$  to global sections of  $\Theta_{\text{norm}}$ . This is the foundation of how we will produce globally-defined modular forms on  $Y^*(\mathbf{d})$  with certain explicit  $q$ -expansions.

To conclude, let us spell out a more precise formulation in the case that  $r + s$  is odd.

A square root of  $\omega^{r+s}$  means that we have a line bundle  $\mathcal{Q}$  with an isomorphism

$$\rho : \mathcal{Q}^{\otimes 2} \simeq \omega^{r+s}.$$

We will additionally require that  $\sqrt{\rho^*(\frac{dq}{q})^{r+s}}$  has trivial monodromy at each cusp. This is the structure that permits one to speak of  $q$ -expansions for sections of  $\mathcal{Q}$  (although the  $q$ -expansion is itself defined only up to a sign).

In this situation, our flat connection gives rise to a flat connection on the genuine bundle  $R^q \pi_* \mathcal{L} \otimes \mathcal{Q}$ , and our previous construction produces a basis for flat sections near the cusp; the assertion is then that these flat sections extend to  $Y^*(\mathbf{d})$  for a suitable choice of  $(\mathcal{Q}, \rho)$ . To write down  $\mathcal{Q}$  we note that  $\Theta^{\otimes 2}$  has the general form of a square of a line bundle twisted

by  $\omega^{r+s}$  and has a global flat section, which induces an isomorphism from that square to  $\omega^{-r-s}$ ; we must just verify that  $\theta^2$  has even valuation at each cusp.

Finally, it may be useful to note that what we have proved can be boiled down to an interesting statement in the analytic theory of our non-projective version of Hitchin's connection:

**Theorem III<sub>b</sub>.6.** *The monodromy of  $(\Theta_{\text{norm}}, \nabla^{\text{Heat}})$  on  $Y^*$  can be identified with the composition*

$$\Gamma^* \rightarrow \text{SL}_2(\mathbf{Z}/\mathfrak{d}) \xrightarrow{\text{Weil representation}} \text{GL}_{\mathfrak{d}}(\mathbf{C}).$$

*In particular, it is finite.*

This is not a surprise (see the remarks in [50] about the Grothendieck–Katz  $p$ -curvature conjecture for Hitchin's connection in the abelian scheme case), but provides a pleasing relationship between the proofs of modularity of theta functions by Poisson summation and by the heat equation.

### III<sub>c</sub>. AN EXAMPLE: JACOBI'S FORMULA $\theta^2 = 2K/\pi$

We will sketch here a purely algebraic proof of Jacobi's identity

$$(III_{c.1}) \quad \theta_3(q)^2 = \frac{2K}{\pi}$$

relating the  $\theta$ -function

$$\theta_3(q) := \sum_{n \in \mathbf{Z}} q^{n^2/2}$$

to an elliptic integral. Here, we follow Jacobi's notation from [33, *fundamenta nova*], where everything depends on a variable  $k$  ranging over  $\mathbf{P}^1 - \{0, \pm 1, \infty\}$  and  $q = e^{-\pi K'/K}$  where

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

and

$$K'(k) = K(\sqrt{1-k^2}).$$

This very beautiful formula is in effect a classical incarnation of an isomorphism  $\Theta^2 \simeq \omega$  between the square  $\Theta^2$  of the theta-bundle, and the Hodge bundle  $\omega$  over a suitable modular curve, as alluded to in §1.2. It also provides a concrete instance of the defect in the previous algebraic theories of theta functions: they are not explicit enough to pin down the right hand side of (III<sub>c</sub>.1).

We will sketch this proof for two reasons. Firstly, it gives us a good excuse to set up a modular curve  $Y^*$  whose integral model over  $\mathbf{Z}[1/2]$  will provide a convenient home for the whole paper. It is not quite a standard choice, but is a classical one, in that it parameterizes

elliptic curves in Jacobi's family. Secondly, the ideas of the proof illustrate the broader ideas of the paper in a concrete case.

**III<sub>c</sub>.1. Monodromy of the flat connection.** Let  $\Theta$  denote the pushforward  $\pi_*\mathcal{L}$  of the universal line bundle  $\mathcal{L}$  on the universal elliptic curve  $\pi : \mathcal{E} \rightarrow Y_0(2)$  to  $Y_0(2)$ . It is a line bundle on  $Y_0(2)$ , and near the cusp 0 (resp.  $\infty$ ) discussed in [Section 3.2](#) it admits a local section  $\vartheta_2$  (resp.  $\vartheta_3$ ) described at the local coordinate  $q$  by  $\vartheta_3(-, q) \in H^0(E_q, \mathcal{L}_q)$  (resp.  $\vartheta_2(-, q) \in H^0(E_q, \mathcal{L}_q)$ ).

A key part of the paper, executed in a general context in [Section II<sub>a</sub>](#) and [Section II<sub>b</sub>](#), will be to write down a globally-defined flat connection  $\nabla^{\text{Heat}}$  on  $\Theta \otimes \omega^{1/2}$ ; it has the property that

The local sections  $\vartheta_i(q, z) \otimes \sqrt{\frac{dq}{q}}$  near the cusps are flat for  $\nabla^{\text{Heat}}$ .

In concrete terms, this flatness is equivalent to the classical fact that

$$\left( q \frac{\partial}{\partial q} - \frac{1}{2} \left( z \frac{\partial}{\partial z} \right)^2 \right) \vartheta_i = 0,$$

the differential operator here agreeing up to a constant scalar with the classical heat operator

$$\left( \frac{\partial}{\partial d\tau} - \frac{1}{4\pi i} \frac{\partial^2}{\partial w^2} \right)$$

in the coordinates  $q = \exp(2\pi i\tau)$ ,  $z = \exp(2\pi iz)$ . Indeed, functions like the  $\vartheta_i$  were written down by Fourier [\[16\]](#) to solve the heat equation on a circle.

To avoid any confusion about  $\omega^{1/2}$ , we will work instead with the squared bundle  $\Theta^2\omega$ . This flat connection defines, as does any flat connection, a monodromy homomorphism  $\Gamma_0(2) \rightarrow \mathbf{C}^\times$ .<sup>23</sup> This can be readily computed from the flatness of the Jacobi theta functions: the local monodromy of  $\vartheta_2^2$  as the local coordinate  $q$  near the cusp  $\infty$  encircles the origin equals a fourth root of unity, whereas the local monodromy of  $\vartheta_3^2$  when the local coordinate  $\sqrt{q}$  near the cusp  $0 = S(\infty)$  encircles the origin is trivial. Therefore, the monodromy homomorphism has the property that

$$(III_c.2) \quad T \mapsto i, U^{-2} = S^{-1}TS \mapsto 1$$

where, here and throughout, we use the notation for elements of  $SL_2$  over an arbitrary ring:

$$(III_c.3) \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

<sup>23</sup>If we don't square, and look instead at  $\Theta\omega^{-1/2}$ , we don't get a genuine connection, only a twisted one, and so the monodromy would only be defined up to sign.

This uniquely determines a homomorphism  $\Gamma_0(2)$  to  $\{\pm 1, \pm i\}$ , and it really is a homomorphism, unlike the root of unity that appears in the classical transformation law. Note that it sends the negative of the identity matrix to  $-1$ .

**III<sub>c</sub>.2. Jacobi's computation**  $\theta^2 = \frac{2K}{\pi}$ . Continue to consider Jacobi's family of curves  $E_k$ . The complete integrals of the first kind

$$K(k) = \int_0^1 \frac{dx}{y} \text{ and } iK'(k) = \int_1^{1/k} \frac{dx}{y}$$

can be interpreted as one-quarter (respectively one-half) of a period of the differential form  $\frac{dx}{y}$ : in the first case, we integrate from the origin to the 4-torsion point  $\tilde{P}$ ; and in the second case we integrate from the origin to the 2-torsion point  $Q$ .

Write  $\tau = \frac{2iK'}{4K} = \frac{iK'}{2K}$  for the ratio of periods, and  $q = e^{2\pi i\tau} = e^{-\pi K'/K}$ . Integrating the differential form  $\frac{1}{4K} \frac{dx}{y}$  identifies  $E_k$  with  $\mathbf{C}/\langle 1, \tau \rangle$  so that the origin  $(1, 0)$  of  $E_k$  is carried to zero, the point  $Q = (1/k, 0)$  is carried to  $\tau/2$ ,  $Q' := (-1/k, 0)$  is carried to  $(1 + \tau)/2$ , and the point  $\tilde{P}$  is carried to  $\frac{1}{4}$ . Exponentiating, we see that we may identify

$$\left( E_k, \frac{dx}{y}, \tilde{P}, Q \right) \simeq \left( \mathbf{C}^*/q^{\mathbf{Z}}, \frac{2K}{\pi i} \frac{dw}{w}, i, \sqrt{q} \right).$$

Under that identification, the following sections of  $\mathcal{O}(Q') \otimes \omega$  match up:

$$\left[ \frac{\vartheta_3(0, q)^2}{2K/\pi i} \right] \cdot \underline{1} \otimes \left( \frac{dx}{y} \right) \longleftrightarrow \vartheta_3(z, q)^2 \otimes \frac{dz}{z}$$

where  $\underline{1}$  is the section of  $\mathcal{O}(Q')$  on  $E_k$  which corresponds to the constant function with value 1. Indeed, the space of relevant sections is one-dimensional, and one can therefore check the equality of two sections merely by evaluating at the identity, using the specified rigidification.

Note that the left hand side is:

$$\left[ \frac{\vartheta_3(0, 2\tau)^2}{2K/\pi i} \right] \cdot \frac{\vartheta_3(0, \tau)^2}{\vartheta_3(0, 2\tau)^2} \cdot \underline{1} \otimes \left( \frac{dx}{y} \right) = \left[ \frac{\vartheta_3(0, \tau)^2}{2K/\pi i} \right] \cdot (1 + k) \cdot \underline{1} \otimes \left( \frac{dx}{y} \right)$$

Now, our general theory implies that  $\Theta^2 \otimes \omega$  admits a flat connection, which is described in purely algebraic terms. We will verify that  $\vartheta_3^2 \otimes \frac{dz}{z}$  is flat later in this paper; we will also prove:

**Proposition III<sub>c</sub>.1.** *Everywhere on  $Y^*$ ,*

$$\nabla_{\frac{\partial}{\partial k}} \left( (1 + k) \underline{1} \otimes \left( \frac{dx}{y} \right) \right) \Big|_e = 0.$$

For an algebraic example of the computational usefulness of our technique of normalizing Hitchin's projective connection, we provide an algebraic proof of [Proposition III<sub>c</sub>.1](#), thus completing our heat equation proof of Jacobi's identity  $\theta_3(q)^2 = 2K/\pi$ .

*Proof of flatness.* Recall that the connection  $\nabla$  is characterized by the fact that for tangent vectors  $Z = \frac{\partial}{\partial k}$  on the base,

$$\nabla_Z(s^2 \otimes \alpha) = (\text{proj}_s \circ \nabla_{Z,GM})(\alpha)$$

for all symmetric sections  $s$  of  $\mathcal{L}$  defined near  $e$ .

We can use this characterization directly, because the section 1 of  $\mathcal{L} = \mathcal{O}(Q)$  is symmetric. In particular, it suffices to show that at all  $k$ ,

$$(\text{the line in } H_{dR}^1(E_k/\mathbf{C}) \text{ spanned by } \nabla_{Z,GM}((1+k)dx/y)) = \varphi_s(\omega^\vee) = \varphi_s(T_e E_k).$$

To this end, we just compute both sides and verify that we get the same thing up to scalar.

The left hand side is a standard computation in this subject: first of all,

$$\nabla_{Z,GM}((1+k)dx/y) = \left[ \frac{dx}{y} \right] + (1+k)\nabla_{\frac{\partial}{\partial k},GM}(dx/y)$$

so it suffices to compute  $\nabla_{\frac{\partial}{\partial k},GM}(dx/y)$ . We will include the details of this, since the one most commonly found in the literature is for the Legendre family, not the Jacobi family. First of all,  $dx$  is an exact differential of the second kind, so it vanishes and is therefore flat for Gauss–Manin in  $H_{dR}^1(E_k/\mathbf{C})$ . Therefore,

$$\nabla_{\frac{\partial}{\partial k},GM} \left( \frac{dx}{y} \right) = \frac{\partial}{\partial k} \left( \frac{1}{y} \right) dx.$$

Differentiating both sides of the equation for  $E_k$  yields

$$\frac{\partial y}{\partial k} = \frac{kx^2(x^2 - 1)}{y}$$

from which we deduce that

$$\nabla_{\frac{\partial}{\partial k},GM} \left( \frac{dx}{y} \right) = \frac{\partial}{\partial k} \left( \frac{1}{y} \right) dx = \left( \frac{kx^2(1 - x^2)}{y^3} \right) dx$$

as a differential of the second kind. Finally, we need to rewrite this in the basis  $[dx/y], [x^2 dx/y]$ .

Our dsk is

$$\frac{kx^2}{1 - k^2 x^2} \cdot \frac{dx}{y},$$

which has poles of order 2 at  $(x, y) = (\pm 1/k, 0)$ . On the other hand,  $dx/y$  is regular, and  $x^2 dx/y$  has its only pole at  $x = \infty$ . So to rewrite our dsk in our chosen basis for  $H_{dR}^1$  requires finding a rational function  $f$  on  $E_k$  such that  $f$  has simple poles at  $(\pm 1/k, 0)$  and is allowed

a simple pole at  $\infty$  but nowhere else. An obvious choice (since by Griffiths transversality we know  $f$  should have a simple pole at  $\infty$ ) is

$$f = \frac{x(x^2 - 1)}{y},$$

which gives

$$\begin{aligned} df &= d\left(\frac{x(x^2 - 1)}{y}\right) \\ &= \frac{d(x(x^2 - 1))}{y} + x(x^2 - 1)d\left(\frac{1}{y}\right) \\ &= \left((3x^2 - 1)\frac{dx}{y} - \frac{x(x^2 - 1)}{y^2}dy\right) \\ &= \left((3x^2 - 1)\frac{dx}{y} - \frac{x(x^2 - 1)}{y^2} \cdot \frac{x(2k^2x^2 - k^2 - 1)}{y}dx\right) \\ &= \left(\frac{(3x^2 - 1)(1 - k^2x^2)}{1 - k^2x^2} + \frac{x^2}{1 - k^2x^2} \cdot (2k^2x^2 - k^2 - 1)\right) \frac{dx}{y} \\ &= \left(\frac{-1}{1 - k^2x^2} + \frac{x^2}{1 - k^2x^2}(3 + k^2 - 3k^2x^2) + \frac{x^2}{1 - k^2x^2} \cdot (2k^2x^2 - k^2 - 1)\right) \frac{dx}{y} \\ &= \left(\frac{-1 + 2x^2 - k^2x^4}{1 - k^2x^2}\right) \frac{dx}{y} \\ &= \left(\frac{(1 - k^2)x^2}{1 - k^2x^2} + x^2 - 1\right) \frac{dx}{y}. \end{aligned}$$

from which we conclude that in  $H_{dR}^1(E_k/\mathbf{C})$ ,

$$\nabla_{\frac{\partial}{\partial k}, GM} \left( \frac{dx}{y} \right) = \frac{k}{1 - k^2} \left( \left[ \frac{dx}{y} \right] - \left[ x^2 \frac{dx}{y} \right] \right).$$

In other words, it only remains to prove that the image of  $\varphi_s$  is the line spanned by

$$\left[ \frac{dx}{y} \right] - k \left[ x^2 \frac{dx}{y} \right].$$

Note that this step does NOT depend on the variation of  $k$ : we can compute it for the fixed elliptic curve, as it is a subspace of  $H_{dR}^1(E_k/\mathbf{C})$  given by the image of the right section  $\varphi_s$  of the Hodge filtration

$$0 \rightarrow \Omega_{E_k}^1 \Big|_e \rightarrow H_{dR}^1(E_k/\mathbf{C}) \rightarrow T_{E_k^\vee} \Big|_e \rightarrow 0.$$

Here  $s$  denotes the global section  $1 \in H^0(E_k, \mathcal{O}(P))$ , where  $P = (-1/k, 0)$ , which is symmetric and has  $s|_e = 1$  for the symmetric rigidified structure on  $\mathcal{O}(P)$ . Recall the definition of  $\varphi_s$ : a neighborhood of the identity in  $E_k^\vee$  is parametrized by a neighborhood of the identity in  $E_k$  via the principal polarization  $\phi_{\mathcal{L}} : E_k \rightarrow E_k^\vee$  given on points by

$$x \mapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

where  $\mathcal{L} = \mathcal{O}(P)$ . Given the point

$$\phi_{\mathcal{L}}(x) = T_x^* \mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{O}([P - x] - [P]) \in E_k^{\vee},$$

one can produce the section near  $e$

$$\frac{T_x^*(s)}{s} = 1.$$

In fact, 1 viewed here as a section near  $e$  of  $\mathcal{O}([P - x] - [P])$  is NOT regular: it has a simple zero at  $P - x$  and a simple pole at  $P$  and is invertible everywhere else. Since  $x$  is close to the identity, 1 is still regular near the identity. The line bundle  $T_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \in \text{Pic}^0(E_k)$  admits a flat connection, and since the flat connections are a torsor under  $H^0(E_k, \Omega^1) = \mathbf{C} \cdot dx/y$ , there is a unique flat connection  $\nabla_s$  on  $T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$  such that  $\nabla_s(1) = 0$ . The map  $\phi_{\mathcal{L}}(x) \mapsto (\phi_{\mathcal{L}}, \nabla_s)$  defines a section of the universal vector extension near  $e$ , and the section

$$\varphi_s : T_{E_k}|_e \xrightarrow{d\phi_{\mathcal{L}}} T_{E_k^{\vee}}|_e \rightarrow H_{dR}^1(E_k/\mathbf{C})$$

of the Hodge exact sequence is defined as the derivative of that map.

Let us carry this out in rigorous notation so that we can be certain of the answer. A vector in  $T_{E_k}|_e = \text{Lie}(E_k)$  is the same as a point

$$D \in E_k(\mathbf{C}[\epsilon])$$

such that the image of  $D$  in  $E_k(\mathbf{C})$  is  $e = (1, 0)$ . In particular,

$$T_{E_k}|_{(x,y)} = \mathbf{C} \left( y \frac{\partial}{\partial x} - ((1 + k^2)x - 2k^2x^2) \frac{\partial}{\partial y} \right)$$

so for instance  $T_{E_k}|_e$  is spanned by  $(1, \epsilon) \in E_k(\mathbf{C}[\epsilon])$ , and  $T_{E_k}|_P$  is spanned by  $(-1/k, \epsilon) \in E_k(\mathbf{C}[\epsilon])$ . In fact, the vector field given at  $(x, y)$  by

$$y \frac{\partial}{\partial x} - ((1 + k^2)x - 2k^2x^2) \frac{\partial}{\partial y}$$

is invariant under the group action of  $E_k$ , because it always pairs to 1 with the invariant differential  $dx/y$ . We will prove that  $\varphi_s(1, \epsilon)$  is a scalar multiple of  $[dx/y] - k[x^2 dx/y]$ , which will conclude the proof of [Proposition III<sub>c</sub>.1](#).

Since we wrote down a formula for the invariant vector field on  $E_k$ , we know that  $(1, (k^2 - 1)\epsilon) \in T_e E_k$  gets pushed to

$$d(z \mapsto P - z)(1, (k^2 - 1)\epsilon) = \left( -1/k, \frac{1 - k^2}{k} \epsilon \right) \in T_P E_k.$$

This is important because  $z = (1, (k^2 - 1)\epsilon) \in E_k(\mathbf{C}[\epsilon])$  gets mapped to the point

$$\mathcal{N} := \mathcal{O}([P - z] - [P]) = T_z^* \mathcal{L} \otimes \mathcal{L}^{-1} \in E_k^{\vee}(\mathbf{C}[\epsilon]) = \text{Pic}^0(E_k \times_{\mathbf{C}} \mathbf{C}[\epsilon])$$

and we just showed that  $P - z = (-1/k, \frac{1 - k^2}{k} \epsilon)$ . We get a rational section 1 of  $\mathcal{O}([P - z] - [P])$  induced by translating and dividing the regular section 1 of  $\mathcal{O}([P])$ . The rational section 1 has a simple pole at  $P$  and a simple zero at  $P - z$  and is invertible elsewhere. And of course

$\mathcal{O}([P - z] - [P])$  as a point in  $E_k^\vee(\mathbf{C}[\epsilon])$  is really in  $\text{Lie}(E_k^\vee)$  (it is the image of  $D$  under  $d\phi_{\mathcal{Z}}$ ). All we have to do is calculate what it maps to under  $d\varphi_s$ .

To do that, we need to write down the connection  $\nabla_s$  on  $\mathcal{N}$  with the property that  $\nabla_s(1)|_e = 0$ , which is a well-defined thing because 1 is regular at  $e$ . All of this is taking place in the Lie algebra of the universal vector extension, viewed inside of its  $\mathbf{C}[\epsilon]$ -points. Let us first of all write down the 1-cocycle defining  $\mathcal{N}$  as a  $\mathbf{C}[\epsilon]$ -point of  $E_k^\vee$ . Let  $U_0$  be the affine  $(E_k - P) \times \mathbf{C}[\epsilon]$ . Over  $U_0$ , 1 is a regular nonvanishing section of  $\mathcal{N}$ , so it is a basis for it. Now let  $U_1$  be the base change to  $\mathbf{C}[\epsilon]$  of a small affine around  $P$ , which in particular excludes all the other points where  $y = 0$ . In  $U_1$ ,  $P$  is given by the equation  $y = 0$  and  $P - z$  is given by the equation  $y = \frac{1-k^2}{k}\epsilon$ . So

$$\frac{y}{y - \frac{1-k^2}{k}\epsilon} = 1 + \frac{1-k^2}{ky}\epsilon$$

is a basis for  $\mathcal{N}$  over  $U_1$ . In particular the cocycle defining  $\mathcal{N}$  is given by

$$1 + \frac{1-k^2}{ky}\epsilon \in \Gamma(U_0 \cap U_1, \mathcal{O}^\times).$$

The data of the invertible function  $\frac{1-k^2}{ky}$  on  $U_0 \cap U_1$  is of course equivalent to this, and it defines  $\mathcal{N}$  as a deformation of the identity element of  $\text{Pic}^0(E_k)$ , by viewing it as a Čech cocycle for  $H^1(E_k, \mathcal{O}) = \text{Lie}(E_k^\vee)$ .

Now for the connection: we know that when we restrict to  $\epsilon = 0$ , 1 restricts to the section 1 of the structure sheaf of  $E_k$ , and the unique connection that kills it at the identity is nothing but the trivial connection  $d$ . Therefore,  $\nabla_s(1)$  (regular wherever 1 is) must restrict to 0 when  $\epsilon = 0$ , i.e.

$$\nabla_s(1) = \epsilon\omega_s \otimes 1,$$

where  $\omega_s$  is a rational 1-form on  $E_k$ .

In fact, the data of  $\omega_s$  determines  $\nabla_s$  completely, as for all rational functions  $f$  on  $E_k \times \mathbf{C}[\epsilon]$

,

$$\nabla_s(f \cdot 1) = (df + \epsilon\omega_s) \otimes 1.$$

It is also easy to “guess” the value of  $\omega_s$  up to addition of  $\mathbf{C}[\epsilon]$ -multiples of the invariant differential  $dx/y$ , because of the requirement that  $\nabla_s$  is regular on all of  $E_k \times \mathbf{C}[\epsilon]$ . Recall that over  $U_1$ , there is a regular nonvanishing section  $1 + \frac{1-k^2}{ky}\epsilon$ . The requirement that  $\nabla_s$  be regular works out to requiring that

$$\nabla_s \left( \left( 1 + \frac{1-k^2}{ky}\epsilon \right) \cdot 1 \right) \text{ is regular on } U_1,$$

and that  $\nabla_s(1)$  is regular on  $U_0$ . The second condition is just saying that  $\omega_s$  has to be regular away from  $P$ , since  $\nabla_s(1) = \epsilon\omega_s \otimes 1$ . On the other hand, the first condition says that

$$\nabla_s \left( 1 + \epsilon \frac{(1-k^2)/k}{y} \right) = \left( d \left( 1 + \epsilon \frac{(1-k^2)/k}{y} \right) + \epsilon\omega_s \right) \otimes 1 = \epsilon \left( -\frac{1-k^2}{k} \frac{dy}{y^2} + \omega_s \right) \otimes 1.$$

So we need  $\omega_s$  to have a double pole at  $P = (-1/k, 0)$  with principal part  $(1 - k^2)/kdy/y^2$ . This automatically implies that

$$\omega_s = \frac{1 - k^2}{2k} \left( A(k) + \frac{1}{x + 1/k} \right) \frac{dx}{y}$$

for some  $A(k) \in \mathbf{C}$ . Finally, the requirement that  $\nabla_s(1)|_e = 0$  tells us that  $A(k) = \frac{-1}{1+1/k}$ , that is, we have calculated the differential of the second kind

$$\omega_s = \frac{1 - k^2}{2k} \left( -\frac{1}{1 + 1/k} + \frac{1}{x + 1/k} \right) \frac{dx}{y}.$$

together with the cocycle for  $\mathcal{N}$ , of course, this is already an element of  $H_{dR}^1(E_k/\mathbf{C})$ . But we want to compare it to the line we computed earlier. So let us compute this disk (up to scalars) in the basis  $[dx/y], [x^2dx/y]$ .

To do so requires us to find a rational function  $f$  on  $E_k$  which is regular except for simple poles at  $\infty$  and  $P$ . The obvious choice is

$$d \left( \frac{y}{x + 1/k} \right) = \left( k^2 x^2 - 1 + \frac{1 - k^2}{k(x + 1/k)} \right) \frac{dx}{y}$$

from which one concludes that the class of  $\omega_s$  in  $H_{dR}^1(E_k/\mathbf{C})$  is in fact

$$k([dx/y] - k[x^2dx/y])$$

as desired. □

Applying  $\nabla_{\frac{\partial}{\partial k}}$  to both sides and then restricting to  $e$ , we find that

$$\frac{\partial}{\partial k} \left[ \frac{\vartheta_3(0, \tau)^2}{2K/\pi i} \right] = 0$$

for all  $k \in Y^*$ , i.e. that the ratio between  $\vartheta_3(0, \tau)^2$  and  $2K/(\pi i)$  must be constant:

$$\vartheta_3(0, \tau)^2 = \left( \sum q^{n^2/2} \right)^2 = c \frac{2K}{\pi i}.$$

One can readily compute  $c = i$  by degeneration (as  $q \rightarrow 0, K \rightarrow \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$ ).

This concludes our sketch of the proof of Jacobi's formula.

## IV. Promoting to the $\Gamma$ -equivariant situation

IV<sub>0</sub>. CONSTRUCTION OF COCYCLES FOR  $\Gamma \ltimes \mathbf{Z}^m$ 

So far, we have constructed explicit cocycles in  $H^s(\mathbf{Z}^m, \mathcal{O}^{\text{an}})$  for the actions from (I.2) for  $\underline{\epsilon} = (0, 0), (1, 0)$ , and we identified those explicit cocycles with sections of  $\Theta_{\text{norm}}$  over various cusp neighborhoods  $\text{Spec} \mathbf{Z}'((q^{1/w_d})) \rightarrow Y^*(\mathbf{d})$  that extend to all of  $Y^*(\mathbf{d})$  thanks to the theory we developed in steps (II) and (III).

The  $s$ -cocycles  $\vartheta_\kappa$  are constructed by first *fixing* a rational rank- $r$  positive-definite subspace  $P \subset \mathbf{R}^{r+s}$ ; then the value of it on  $(v_0, \dots, v_s)$  is given by summing the usual terms  $q^{Q(\underline{n})} \underline{z}^{B\underline{n}}$  (or variants thereof for nontrivial thetas with characteristics), not over all  $\underline{n} \in \mathbf{Z}^m$ , but rather over a region determined by  $P$  and  $(v_0, \dots, v_s)$ : namely a small irrational shift of the convex hull of the prism over  $v_0, \dots, v_s$  that is swept out in the direction  $P$ .

This construction therefore only accounts for the ability to choose a single positive-definite direction  $P$ . But we already know from Section 1.1 and Section 1.3 that the modular theta functions we wish to construct as arithmetic-geometric objects by restricting to  $\underline{z} = 1$  will involve ranging over a family of choices of  $P$  as well as of translates of  $P$ .

In the next section, we will explain the geometric significance of such a construction. In this section, we explain the full detail of how to upgrade our Jacobi-style cocycles to include information about  $P$  varying at the same time as the translation of  $P$ . The type of the object we will construct is a cocycle for

$$H^s(\Gamma \ltimes \mathbf{Z}^m, \mathcal{O}^{\text{an}}),$$

where the action of  $\mathbf{Z}^m$  on  $\mathcal{O}^{\text{an}}$  is that of (I.2) for  $\underline{\epsilon} = 0$ , and the action of  $\Gamma$  on  $\mathcal{O}^{\text{an}}$  is by

$$(\gamma \cdot f)(\underline{z}) := f\left(\left(\prod_{j=1}^m z_j^{\gamma_{ij}}\right)_{i=1}^m\right).$$

There is no obstacle towards repeating this exercise for other  $\underline{\epsilon}$  and for a complete basis of cocycles (as we did for the  $\mathbf{Z}^m$ -cocycles earlier) — one just has to make sure to shrink  $\Gamma$  so that it fixes the choice of characteristic. But for now we do not include this, since we only promised to construct a single modular form, which will be the simplest possible one.

Our goal here is to make a generating function (analogous to Jacobi's  $\vartheta_3(z, \tau)$  and the other elliptic theta functions) for the  $\underline{\lambda} \in \mathbf{Z}^m$  that keeps track of the values of  $Q$  by summing terms of the form  $q^{n/2} \underline{z}^{B\underline{\lambda}}$ . But the set of  $\lambda$  with  $Q(\lambda) = n$  is typically infinite, so the count must be “regularized” by summing over only a family of positive-definite directions. Which family to sum over is encoded in the input to a cocycle.

Though this is not the exact formal mechanism by which we will make the construction, it can be convenient to think in terms of how the desired cocycle in  $H^s(\Gamma \ltimes \mathbf{Z}^m, \mathcal{O}^{\text{an}})$  pairs with each homology class  $h \in H_s(\Gamma \ltimes \mathbf{Z}^m, \mathbf{k})$  for every ring  $\mathbf{k}$  in which  $2\mathbf{d}$  is invertible.

The class  $h$  will be used to define a region  $\mathcal{R}(h) \subset \mathbf{R}^m$  — or, more precisely, a map from the region  $\mathcal{R}(h)$  to  $\mathbf{R}^m$  with compact fibers — which we call a *sweepout domain*. In the following section, we will interpret the sweepout domains as a family of translates of positive-definite subspaces of  $\mathbf{R}^m$  by interpreting  $h$  as a dimension- $s$  simplicial chain in the symmetric space for  $\mathrm{SO}(r, s) \ltimes \mathbf{R}^m$ , which parametrizes such translates. Our cocycle  $\tilde{\vartheta}$  for  $H^s(\Gamma \ltimes \mathbf{Z}^m, \mathcal{O}^{\mathrm{an}})$  will be constructed so that after pairing it with  $h$ , it becomes a power series obtained by summing  $q^{Q(\lambda)} \underline{z}^{B\lambda}$  over  $\mathcal{R}(h)$  with multiplicity (since the map  $\mathcal{R}(h) \rightarrow \mathbf{R}^m$  might hit  $\lambda$  more than once) and signs that depend on orientation.

There is nothing canonical about  $\mathcal{R}(h)$ , but everything will be “unique up to homotopy,” in a sense to be prescribed.

The reason that this deserves a section, rather than a couple of paragraphs, is that one has to be very careful that the regions  $\mathcal{R}(h)$  do not have lattice points on their boundaries! Otherwise the construction would not yield a cocycle — recall [Figure 4](#). We accomplish this using the tools of transversality theory in differential topology, the bespoke version of which we need here (in the case of manifolds with corners and with boundary conditions) we can thankfully cite from the work of [\[17, 18\]](#). In fact, we recommend that the reader glance at the very short [\[18\]](#) for a different writeup of many of the ideas of this section.

**IV<sub>0</sub>.1. The symmetric space.** Recall that  $m := r + s$  is the rank of the quadratic form  $Q$ . Write

$$G = \mathrm{SO}(r, s) \ltimes \mathbf{R}^{r+s}, \quad \tilde{\Gamma} = \Gamma \ltimes \mathbf{Z}^m, \quad K = \mathrm{S}(\mathrm{O}(r) \times \mathrm{O}(s)),$$

so that  $G$  is a Lie group,  $\tilde{\Gamma}$  a lattice in  $G$ , and  $K$  a maximal compact subgroup of  $G$ ; and also consider the symmetric space for  $\mathrm{SO}(r, s)$ ,

$$D := \mathrm{SO}(r, s)/K = \{\text{rank-}r \text{ positive-definite subspaces of } \mathbf{R}^m\}.$$

For  $p \in D$  let  $V_p$  be the associated subspace of  $\mathbf{R}^m$ . We want a similar picture for affine subspaces, so consider the symmetric space for  $G$

$$E = D \times \mathbf{R}^m = \{\text{affine positive definite subspaces in } \mathbf{R}^{r+s}, \text{ with a distinguished basepoint}\}$$

so that  $E$  is identified with  $G/K$  by means of the rule  $(gK, g^{-1}x) \mapsto (g, x)K$ .

Now, for  $e = (p, x) \in E$ , we denote by  $\mathrm{Aff}_e$  the corresponding *affine* subspace, where we translate the positive definite subspace by the point in  $\mathbf{R}^m$ , i.e. for  $e = (p \in D, x \in \mathbf{R}^m)$ ,

$$\mathrm{Aff}_e := (x + V_p) \subset \mathbf{R}^{r+s}.$$

*Note that the quadratic form goes to infinity in all directions along this affine subspace; this will guarantee that all the sums we construct will lie in  $\mathcal{O}^{\mathrm{an}}$ .*

The  $\mathrm{Aff}_e$  fit together into a rank- $r$  affine bundle over  $E$ , which comes equipped with a tautological inclusion as a closed submanifold of the trivial rank- $m$  vector bundle  $E \times \mathbf{R}^m$ , and therefore a map  $\mathrm{Aff}_E \rightarrow \mathbf{R}^m$ .

**IV<sub>0</sub>.2. Bad loci and transversality.** For each  $\alpha \in \mathbf{Z}^{r+s}$  we may consider the locus  $Z_\alpha \subset E$  of those  $e$  for which the affine subspace  $\text{Aff}_e$  contains  $\alpha$ . By definition, for each fixed  $\alpha$ ,  $Z_\alpha$  is a rank  $r$  affine sub-bundle of  $E$ , and in particular it is a closed submanifold of  $E$ . Define  $Z$  as the union  $\bigcup_{\alpha \in \mathbf{Z}^{r+s}} Z_\alpha$  of all the  $Z_\alpha$ , this is a “bad locus” that we want to avoid. Note that  $Z$  is quite an unpleasant set: for example, it is dense. We only need that it is a countable union of closed submanifolds of the manifold  $E$ .

A homology class  $h$  for  $H^s(\Gamma \ltimes \mathbf{Z}^m, \mathbf{k})$  can be represented by a  $\mathbf{k}$ -linear combination of *smooth* maps  $\sigma : \Delta^s \rightarrow E$ . The definition of smooth here can be taken as follows: Fix once and for all a closed embedding of  $E$  in some  $\mathbf{R}^N$ ; the choice will not matter. A continuous map  $\sigma : \Delta^q \rightarrow E$  is smooth if after postcomposing with the inclusion of  $E$  into  $\mathbf{R}^N$  it extends<sup>24</sup> to a smooth map from a neighborhood of  $\Delta^q$  in  $\mathbf{R}^q$ .

To construct the regions  $\mathcal{R}(h)$  that are necessary for the construction of our cocycles for  $\Gamma \ltimes \mathbf{Z}^m$ , following [Figure 4](#), we need to be able to deform a smooth singular chain  $\sigma : \Delta^q \rightarrow E$  to one that is *transverse* to  $Z$ . One has to take some care in the definition of transversality, because we are in the setting of *manifolds with corners* thanks to the presence of simplices. We follow the definition of [17, Definition 2.12], of which we only require a special case:

**Definition IV<sub>0</sub>.1.** Fix a closed submanifold  $N \subset E$ . A smooth map  $\sigma : \Delta^q \rightarrow E$  is *transverse* to  $N$  if and only if  $\sigma$  restricted to the interior of every face (of any dimension) of  $\Delta^q$  is transverse to  $N$ .

This is exactly the definition that we need, since we need the boundary of  $\sigma : \Delta^s \rightarrow E$  to avoid all the  $Z_\alpha$  (which is equivalent to the interiors of the boundary faces being transverse to  $Z_\alpha$ , as the interiors of the boundary faces are of dimension  $\leq s - 1$ , and  $Z_\alpha$  is of codimension  $s$  in  $E$ ).

As for the transversality condition for  $\sigma$  on the interior of  $\Delta^s$  itself, we also need to arrange so the family of translations of positive-definite subspaces of  $\mathbf{R}^m$  parametrized by  $\sigma$  only contains any given  $\underline{\alpha}$  a finite number of times. Indeed, it guarantees by [17, Theorem 2.28] that

(IV<sub>0</sub>.1) If  $\sigma : \Delta^q \rightarrow E$  is transverse to a codimension- $c$  closed submanifold  $N \subset E$ , then  $\sigma^{-1}(N)$  is a codimension- $c$  submanifold with corners of  $\Delta^q$ .

which in fact subsumes the remark of the previous paragraph about the boundary avoiding the bad loci, and should be considered the ultimate source of the importance of the transversality condition.

**IV<sub>0</sub>.2.1. An example of deforming to achieve transversality.** Here is an example that may help to get the sense of the meaning of transversality, and connects the construction from [Section I](#) to the framework of this section.

<sup>24</sup>This smooth function is not uniquely defined, but nonetheless all derivatives of  $\varphi$  on  $\Delta^q$  do not depend on the extension.

Suppose that  $1, \alpha_1, \dots, \alpha_m \in \mathbf{R}$  are linearly independent over the rational numbers, and  $\varphi$  is a linear simplex  $\Delta^q \rightarrow \mathbf{R}^m$  with rational vertices. Then  $\varphi + \underline{\alpha}$  is transverse to any coordinate subspace  $\mathbf{R}^r \subset \mathbf{R}^m$  (which plays the role of the rational subspace  $P$  of [Section I](#)).

We reduce by projection to  $\mathbf{R}^m/\mathbf{R}^r$  to the case when  $r = 0$  – so we are asking to prove that  $\varphi + \underline{\alpha}$  is transverse to the subspace  $\{0\}$ . Take then any  $q$ -dimensional face of this linear simplex. Then either:

- this face spans  $\mathbf{R}^m$  as an affine subspace, in which case  $q \geq m$  and it is clearly transverse to  $\{0\}$  or
- it spans a proper rational affine subspace; we can make a rational affine map  $\mathbf{R}^s \rightarrow \mathbf{R}$  that kills this affine space, and it cannot send  $(\alpha_1, \dots, \alpha_s)$  to zero. So the face misses  $0$  entirely.

*IV<sub>0</sub>.2.2. Filling in while maintaining transversality.* The crucial statement that we need is the following generalization of [Section IV<sub>0</sub>.2.1](#).

**Lemma IV<sub>0</sub>.2.** *Let  $(N_i)_i$  be a countable collection of closed submanifolds of  $E$ . Let  $\sigma : \Delta^q \rightarrow E$  be a smooth map, such that  $\sigma|_{\partial\Delta^q}$  is transverse to all  $N_i$ . Then  $\sigma$  may be homotoped (relative to  $\partial\Delta^q$ ) to a smooth  $\sigma' : \Delta^q \rightarrow E$  with the property that  $\sigma'$  is transverse to all  $N_i$ , and  $\sigma'|_{\partial\Delta^q} = \sigma|_{\partial\Delta^q}$ .*

*Proof.* We refer to [\[18, Lemma 5\]](#) for the full detail, and provide a sketch here. Note that the proof is *not* explicit in the same way that [Section IV<sub>0</sub>.2.1](#) is. The statement that smooth maps of manifolds can be homotoped to ones that are transverse to a given closed submanifold is a cornerstone of the foundations of differential topology [\[27\]](#), being one of the most important consequences of Sard’s theorem. This version of that principle requires several additional technical improvements:

- (i) It has to be relative to the boundary. Such results for manifolds with boundary go by the name “strongly relative transversality theorem” and can be proved using partitions of unity to ensure that the homotopy doesn’t affect the boundary.
- (ii) It has to work for manifolds with corners instead of just manifolds with boundary. This may not have been present in the literature prior to [\[18\]](#), but certainly requires some care in handling the strata of the domain.
- (iii) It has to maintain transversality with a countably infinite collection of closed submanifolds. This is handled by applying the Baire category theorem: since the Sard’s theorem machine behind the proofs of transversality theorem (see e.g. [\[27\]](#)) ultimately shows that the space of smooth maps transverse to any given  $N_i$  is not just nonempty but *open and dense* for some appropriate topology on the set of smooth maps to  $E$  that makes it a Baire space. So intersecting countably many of these conditions doesn’t change the conclusion.

□

**IV<sub>0.3</sub>. Group cohomology as equivariant transverse singular cohomology.** We now construct the desired cocycle, which requires us to first explain (i.e., recap from [18]) how one can compute the usual group cohomology cocycles for  $\tilde{\Gamma} = \Gamma \ltimes \mathbf{Z}^m$  by using the complex of smooth singular chains on  $E$  that are transverse to all the  $Z_\alpha$ .

We start with the standard fact ( $E$  is so-named because it is  $E\tilde{\Gamma}$  — see [56, Ch. 8]) relating group cohomology of  $\tilde{\Gamma}$  with singular cohomology of  $E$ . The relevant chain complexes of  $\tilde{\Gamma}$ -modules are: the group cohomology complex

$$(\text{Chains}_\Gamma)_\bullet : \mathbf{Z} \xleftarrow{\delta_0} \mathbf{Z}[\tilde{\Gamma}] \xleftarrow{\delta_1} \mathbf{Z}[\tilde{\Gamma}^2] \xleftarrow{\delta_2} \mathbf{Z}[\tilde{\Gamma}^3] \xleftarrow{\delta_3} \dots \xleftarrow{\delta_n} \mathbf{Z}[\tilde{\Gamma}^{n+1}] \xleftarrow{\delta_{n+1}} \dots$$

and the singular chain complex

$$(\text{Chains}_E)_\bullet : \mathbf{Z} \xleftarrow{\partial_0} C_0(E, \mathbf{Z}) \xleftarrow{\partial_1} C_1(E, \mathbf{Z}) \xleftarrow{\partial_2} C_2(E, \mathbf{Z}) \xleftarrow{\partial_3} \dots \xleftarrow{\partial_n} C_n(E, \mathbf{Z}) \xleftarrow{\partial_{n+1}} \dots$$

Since we are assuming that  $\tilde{\Gamma}$  acts freely on  $E$ , these are both free resolutions of  $\mathbf{Z}$  by  $\tilde{\Gamma}$ -modules (the latter because  $E$  is a vector bundle over the  $\text{SO}(r, s)$ -symmetric space  $D$ , so it is contractible and connected) and therefore have a  $\tilde{\Gamma}$ -equivariant quasiisomorphism  $\Psi$  between them inducing the identity on  $\mathbf{Z}$  and unique up to homotopy.

The further contribution of **Lemma IV<sub>0.2</sub>** is that we can choose  $\Psi$  to be  $Z_\alpha$ -transversal for all  $\alpha$ :

**Proposition IV<sub>0.3</sub>.** *There exists a  $\tilde{\Gamma}$ -equivariant map of chain complexes  $\Psi : \text{Chains}_\Gamma \rightarrow \text{Chains}_E$  whose image in degree  $q$  consists entirely of  $q$ -singular chains whose components  $\sigma : \Delta^q \rightarrow E$  are transverse to  $Z_\alpha$  for all  $\alpha \in \mathbf{Z}^m$ . Any two such maps  $\Psi, \Psi'$  are homotopic by means of a map  $\Phi$  with the same transversality property. In particular,  $\text{Chains}_\Gamma$  and the subcomplex of  $\text{Chains}_E$  consisting of smooth chains transverse to all  $Z_\alpha$ , denoted  $\text{Chains}_E^\natural$  admit a  $\tilde{\Gamma}$ -equivariant quasiisomorphism.*

*Proof.* Both parts amount to iterative application of **Lemma IV<sub>0.2</sub>**: one constructs  $\Psi$  or  $\Phi$  by induction on the degree.  $\square$

One way to explicitly define a cocycle for  $H^s(\tilde{\Gamma}, \mathcal{O}^{\text{an}})$  is (after having implicitly fixed  $\Psi$ ) therefore to provide a cocycle in

$$\text{Hom}_{\tilde{\Gamma}}((\text{Chains}_E^\natural)_s, \mathcal{O}^{\text{an}}),$$

which is what we will do following the strategy already prescribed, replacing the rôle of the homology class  $h$  with a chain  $\sum a_i \sigma_i \in (\text{Chains}_E^\natural)_s$ .

**IV<sub>0.4</sub>. Constructing the cocycle.** For  $\sigma : \Delta^s \rightarrow E$  transverse to all  $Z_\alpha$ , the sweepout domain  $\mathcal{R}(\sigma)$  that we intend to sum over is *defined* to be

$$\mathcal{R}(\sigma) := \sigma^* \text{Aff}_E,$$

that is,  $\mathcal{R}(\sigma)$  the affine bundle over  $\Delta^s$  which is naturally a subbundle of  $\Delta^s \times \mathbf{R}^m$  whose fiber at  $p \in \Delta^s$  is precisely the fiber

$$(\text{Aff}_E)_{\sigma(p)} = \text{the affine subspace of } \mathbf{R}^m \text{ corresponding to } \sigma(p).$$

Its relative dimension over  $\Delta^s$  is  $r$  and its total space has dimension  $r + s$ .

So  $\mathcal{R}(\sigma)$  carries a family of translates of positive-definite subspaces of  $\mathbf{R}^m$ , which is essentially the “region” we would like to sum  $q^{Q(\underline{\lambda})} \underline{z}^{B\underline{\lambda}}$  over. More specifically, a term  $q^{Q(\underline{\lambda})} \underline{z}^{B\underline{\lambda}}$  should be counted (with sign according to orientation) *every time* it appears in one of those translates, i.e., every time  $\underline{\lambda} \in \mathbf{Z}^m$  appears in the image of the smooth map  $\text{sweep} : \mathcal{R}(\sigma) \rightarrow \mathbf{R}^m$  fitting into the diagram

$$\begin{array}{ccccccc} \mathcal{R}(\sigma) & \xrightarrow{\psi} & \text{Aff}_E & \longrightarrow & E \times \mathbf{R}^m & \longrightarrow & \mathbf{R}^m \\ \downarrow & & \downarrow & & \downarrow & & \\ \Delta^s & \longrightarrow & E & \xlongequal{\quad} & E & & \end{array}$$

As promised, the cocycle we construct will be obtained by sending  $\sigma$  to such a sum and extending linearly. We carry out the construction now, where it only remains to be careful about the meaning of the signed multiplicities — the summation procedure over a family of affine subspaces might hit the same point more than once or sweep over it in opposite directions that cancel out. All of this is well-behaved and can easily be defined in terms of intersection theory, thanks to the transversality condition for  $\sigma$  that was arranged. In particular:

**Lemma IV<sub>0</sub>.4.** *For all  $\underline{\lambda} \in \mathbf{Z}^m \subset \mathbf{R}^m$ , the smooth map*

$$\text{sweep} : \mathcal{R}(\sigma) \rightarrow \mathbf{R}^m$$

*is transverse to the closed submanifold  $\{\underline{\lambda}\} \subset \mathbf{R}^m$ .*

*Proof.* We claim that this is a formal consequence of the fact that  $\sigma : \Delta^s \rightarrow E$  is transverse to the closed submanifold  $Z_\alpha \subset E$ . Since the manifold with corners  $\mathcal{R}(\sigma)$  has total dimension  $m$  and  $\{\underline{\lambda}\}$  has codimension  $m$  in  $\mathbf{R}^m$ , there are two parts to this claim:

- (i) If  $x \in \partial \mathcal{R}(\sigma) = \sigma|_{\partial \Delta^s}^* \text{Aff}_E$  lies over  $p \in \partial \Delta^s$ , then the transversality of  $\sigma$  to  $Z_\lambda$  ensures that  $\sigma(p)$  is not in  $Z_\lambda$ , i.e.  $\mathcal{R}(\sigma)|_p$  does not contain  $\lambda$ , hence it is impossible for  $\lambda$  to be in  $\text{sweep}(\mathcal{R}(\sigma)|_p)$ , which is where  $\text{sweep}(x)$  is.
- (ii) Suppose  $x$  is in the interior of  $\mathcal{R}(\sigma)$ , so it lies over a point  $p$  in the interior of  $\Delta^s$ , and that  $\text{sweep}(x) = \lambda$ . We want to prove that

$$d\text{sweep} : T_x \mathcal{R}(\sigma) \rightarrow T_\lambda \mathbf{R}^m$$

is surjective, for which it suffices (by the fact that the natural map  $\text{Aff}_E \rightarrow \mathbf{R}^m$  is a submersion) to prove that

$$d\psi(T_x \mathcal{R}(\sigma)) + T_{\psi(x)} \tilde{Z}_\lambda = T_{\psi(x)} \text{Aff}_E,$$

where  $\tilde{Z}_\lambda$  is the preimage of  $\lambda$  under the map  $\text{Aff}_E \rightarrow \mathbf{R}^m$ . This follows formally from the transversality of  $\sigma$  to  $Z_\lambda$ . In particular,  $T_x \mathcal{R}(\sigma) \subset T_p \Delta^s \oplus T_{\psi(x)} \text{Aff}_E$  is exactly the collection of pairs  $(v_1, v_2)$  such that  $d\sigma(v_1) = d\pi(v_2)$ , and  $d\psi$  is identified with the projection to  $v_2$ ; so

$$d\psi(T_x \mathcal{R}(\sigma)) = \{w \in T_{\psi(x)} \text{Aff}_E : d\pi(w) \in d\sigma(T_p \Delta^s)\}.$$

By transversality of  $\sigma$  to  $Z_\lambda$ , every  $w \in T_{\psi(x)} \text{Aff}_E$  has the property that there exist  $v \in T_p \Delta^s$  and  $z \in T_{\sigma(p)} Z_\lambda$  such that  $d\pi(w) - z = d\sigma(v)$ . The fact that  $\pi$  restricts to a diffeomorphism  $\tilde{Z}_\lambda \rightarrow Z_\lambda$  means there is a unique  $\tilde{z} \in T_{\psi(x)} \tilde{Z}_\lambda$  such that

$$d\pi(w - \tilde{z}) = d\sigma(v),$$

which means that  $w - \tilde{z} \in d\psi(T_x \mathcal{R}(\sigma))$  as desired. □

The signs that intervene when we count points in  $\text{sweep}^{-1}(\lambda)$  come from orientations. In particular, we put the standard orientation on  $\mathbf{R}^m$ , we orient  $D$  as in [38, p. 11] by propagating an orientation of

$P_0$  = a fixed choice of positive-definite dimension- $r$  subspace of  $\mathbf{R}^m$

to all other such  $P$  continuously, we orient  $E = D \times \mathbf{R}^m$  by taking the product, and finally we orient  $\text{Aff}_E$  by orienting the fibers  $x + P$  using the orientation of  $P$  we already chose. The affine bundle  $\mathcal{R}(\sigma)$  is oriented by orienting its fibers over  $\Delta^s$  in the same way as the corresponding fibers over  $E$ .

**Definition IV<sub>0</sub>.5.** Let  $\underline{\lambda} \in \mathbf{Z}^m \subset \mathbf{R}^m$ . Then  $\text{sweep}^{-1}(\underline{\lambda}) \subset \mathcal{R}(\sigma)$  is an oriented 0-dimensional closed submanifold of  $\mathcal{R}(\sigma)$ , so it is a finite disjoint union of points in  $\mathcal{R}(\sigma)$ , weighted with  $\pm 1$ . The finiteness comes from the fact that it is in bijection with  $\sigma^{-1}(Z_\lambda)$ , and the fact that it is a submanifold of dimension 0 comes from the transversality we just proved and (IV<sub>0</sub>.1). The orientation on it is induced by the orientations we defined (see for example [17, §3.1.1]). Define the *sweepout-multiplicity of  $\underline{\lambda}$*  to be the sum of all the orientations of the points of  $\text{sweep}^{-1}(\underline{\lambda})$ .

Then the formal definition of what we have been describing slightly less formally for our  $\tilde{\Gamma}$ -cocycle is:

**Definition IV<sub>0</sub>.6.** We define our cocycle

$$\vartheta \in \text{Hom}(C_s^{\text{rh}}(E, \mathbf{Z}), \mathcal{O}^{\text{an}})$$

by extending linearly from

$$\vartheta(\sigma) = \sum_{\underline{n} \in \mathbf{Z}^m} (\text{sweep}_\sigma\text{-multiplicity of } \underline{n}) q^{Q(\underline{n})} \underline{z}^{B\underline{n}}.$$

The only point to be verified in order to define  $\vartheta$  as an element of  $\text{Hom}(C_s^\natural(E, \mathbf{Z}), \mathcal{O}^{\text{an}})$  is that the convergence criterion that defines  $\mathcal{O}^{\text{an}}$  is indeed satisfied by  $\vartheta(\sigma)$ . But the set of  $n$  with nonzero  $\sigma$ -multiplicity is entirely contained in a set of the form  $\{k_0 + k_1 t : k_0 \in K_0, k_1 \in K_1, t \in \mathbf{R}_{\geq 1}\}$  with  $K_i$  compact and  $Q$  bounded away from zero on  $K_1$ ; on such a set  $Q$  is bounded from below by an increasing linear function of  $\|n\|^2$ , for any Euclidean norm  $\|n\|$ . But any  $\sigma : \Delta^s \rightarrow E$  has compact image, also after projection to  $D$ .

It remains to show that  $\vartheta$  is a cocycle. The fact that it is  $\tilde{\Gamma}$ -equivariant follows for  $\Gamma$  using the definition of the  $\Gamma$ -action on  $\mathcal{O}^{\text{an}}$  and the fact that  $\Gamma$  preserves  $\mathbf{Z}^m$  and  $Q$ , and for  $\mathbf{Z}^m$  using exactly the same argument as in [Lemma I.5](#). So we just need to prove:

**Lemma IV<sub>0</sub>.7.** *If  $\sigma \in C_s^\natural(E, \mathbf{Z})$  is the boundary of a  $Z$ -transverse simplex  $\sigma' : \Delta^{s+1} \rightarrow E$  then the sweep $_\sigma$ -multiplicity of  $\underline{n}$  is zero for all  $\underline{n} \in \mathbf{Z}^m$ .*

*Proof.* We follow the argument of [\[18, p. 2\]](#) (see also the much more general [\[17, Theorem 7.8\]](#)). For the fiber  $F$  of any fixed  $\underline{n} \in \mathbf{Z}^m$  under

$$\text{sweep}_{\sigma'} : \mathcal{R}(\sigma') \rightarrow \mathbf{R}^m$$

is, by transversality, a one-manifold with boundary. Since each “entry point” of  $F$  matches with an “exit point,” the total multiplicity is zero.

To give a little more detail, and spell out transversality issues: By classification of 1-manifolds  $F$  is a union of images of  $S^1$  by smooth embeddings  $g_i$ , and images of  $[0, 1]$  by smooth embeddings  $f_i$ . The transversality that we have imposed entails that  $g_i(S^1)$  is disjoint from the boundary of  $\mathcal{R}(\sigma')$  and  $f_i([0, 1])$  meets it only at  $f_i(0)$  and  $f_i(1)$ , where the derivatives of  $f_i$  at 0 and 1 are transverse to the faces containing it. Now write  $\sigma = \partial\sigma'$  as a sum of various singular  $s$ -simplices  $\sigma_0 + \cdots + \sigma_{s+1}$  corresponding to the boundary of  $\Delta^{s+1}$ . The preimage of  $\underline{n}$  under

$$(IV_0.2) \quad \coprod_{j=0}^{s+1} \mathcal{R}(\sigma_j) \rightarrow \mathbf{R}^m$$

is given by the collection of all the points  $f_i(0)$  and  $f_i(1)$ , although these may contribute to different  $j$  in [\(IV<sub>0</sub>.2\)](#) say  $j_0, j_1$ . One then checks that the contribution of  $f_i(0)$  to the sweep $_{\sigma_{j_0}}$ -multiplicity of  $n$  and the contribution of  $f_i(1)$  to the sweep $_{\sigma_{j_1}}$ -multiplicity have opposite signs and so cancel.

□

**Corollary IV<sub>0</sub>.8.**  *$\vartheta$  defines a cocycle for  $H^s(\tilde{\Gamma}, \mathcal{O}^{\text{an}})$ .*

We remark that what we proved also implies that “multiplicity of  $n$ ” defines a functional on the  $s$ th homology group of the chain complex of  $Z$ -transversal simplices, and therefore we have a functional

$$H_s(\tilde{\Gamma}, \mathbf{k}) \longrightarrow \mathbf{k}.$$

for all  $\underline{n} \in \mathbf{Z}^m$ . These are the functionals given by intersection pairing with the special cycle  $Z_{\underline{n}}$  in  $E$ . These are closely related to the special cycles on  $D$  used by Kudla–Millson as the coefficients of their modular forms.

#### IV. ADDING $\Gamma$ -EQUIVARIANCE TO THE GEOMETRIC STORY

We are equipped with a global section  $\vartheta$  of  $\Theta_{\text{norm}} = R^s \pi_* \mathcal{L}^B \otimes \omega^{m/2}$  with a specific Fourier expansion near a cusp of  $Y^*(\mathbf{d})$ . If  $s = 0$ , then we can use the map

$$\text{evaluation at } 1 : \pi_* \mathcal{L}^B \rightarrow \mathcal{O}_{Y^*(\mathbf{d})}$$

(which uses the rigidification of  $\mathcal{L}^B$  induced by the rigidification of  $\mathcal{L}$ ) to evaluate  $\vartheta$  and produce a global section of  $\omega^{m/2}$  on  $Y^*(\mathbf{d})$  — that is, a modular form of weight  $m/2$  — whose  $q$ -expansion near 0 can be calculated to be exactly  $\sum q^{Q(\underline{n})}$ , thus proving that this  $q$ -expansion is modular and generalizing the procedure of proving that  $\theta_3(q)$  is modular by exhibiting it as  $\vartheta_3$  evaluated at  $z = 1$ . This doesn't use  $\Gamma$  at all, which makes sense because  $\Gamma$  can only be the trivial group in this case.

On the other hand, if  $s > 0$ , then there is no nonzero evaluation map along 1 for  $\Theta = R^s \pi_* \mathcal{L}^B$ . To remedy this, as suggested in the introduction, we exploit the fact that  $\mathcal{L}^B$  has a natural  $\Gamma$ -equivariant structure covering the  $\Gamma$ -action on  $\mathcal{E}^m$ , with the property that  $\Gamma$  acts trivially on the rigidification of  $\mathcal{L}^B$ . As such, we can replace  $R^s \pi_* \mathcal{L}^B$  with the  $\Gamma$ -equivariant derived pushforward  $\Theta^\Gamma := R^s \pi_*^\Gamma \mathcal{L}^B$ , and, as we will explain, the construction of [Section IV<sub>0</sub>](#) gives a section of near a cusp of  $\Theta^\Gamma$ . The benefit of this is that unlike  $\Theta$ , the sheaf  $\Theta^\Gamma$  naturally comes with an evaluation morphism to  $H^s(\Gamma, \mathbf{Z}') \otimes \mathcal{O}_{Y^*(\mathbf{d})}$ . By upgrading one of the global sections we constructed for  $\Theta \otimes \omega^{m/2}$  to a global section of  $\Theta^\Gamma \otimes \omega^{m/2}$ , which agrees near a cusp of our choice with the cocycle of [Section IV<sub>0</sub>](#), and evaluating it at 1, we will therefore obtain a modular form valued in  $H^s(\Gamma, \mathbf{Z}')$ , which is exactly what we wanted.

In this section we will therefore handle the technicalities by doing the following:

- (i) Define the  $\Gamma$ -equivariant derived pushforward  $R^s \pi_*^\Gamma \mathcal{L}^B$  and the evaluation morphism to  $H^s(\Gamma, \mathbf{Z}') \otimes_{\mathbf{Z}'} \mathcal{O}_{Y^*(\mathbf{d})}$ .
- (ii) Explain the geometric significance of [Section IV<sub>0</sub>](#), that is, how the cocycle we defined there actually defines a section of  $R^s \pi_*^\Gamma \mathcal{L}^B$  near a cusp. This will be pulled from [\[34\]](#).
- (iii) Explain how to bootstrap the non-equivariant results of step (III) to show that the section in (ii) extends globally.

##### (i): Incorporating $\Gamma$ -equivariant structure

Let us first state exactly the structure that is necessary: not only does  $\mathcal{L}^B$  have a  $\Gamma$ -equivariant structure  $\gamma^* \simeq \mathcal{L}^B$ , but also a rigidification  $e^* \mathcal{L}^B \simeq \mathcal{O}_{Y^*(\mathbf{d})}$  that is *compatible with the equivariant structure* in the sense that for all  $\gamma \in \Gamma$ , the rigidification of  $e^* \mathcal{L}^B$  obtained by using  $\gamma \circ e = e$  and identifying  $e^* \mathcal{L}^B$  with  $e^* \gamma^* \mathcal{L}^B \simeq e^* \mathcal{L}^B$  using the  $\Gamma$ -equivariance agrees with the rigidification  $e^* \mathcal{L}^B \simeq \mathcal{O}_S$  that it comes with automatically.

The sheaf  $\mathcal{L}^B$  is a  $\Gamma$ -equivariant  $\mathcal{O}_{\mathcal{E}^m}$ -module, so  $\pi_* \mathcal{L}^B$  is a  $\Gamma$ -equivariant  $\mathcal{O}_{Y^*(d)}$ -module, for the trivial  $\Gamma$ -action on  $S$ . We define the  $\Gamma$ -equivariant theta-bundle  $\Theta^\Gamma$  by using  $\Gamma$ -equivariant pushforward:  $\Theta^\Gamma = R^s \pi_*^\Gamma \mathcal{L}$ , which is the  $s$ -th derived functor of the functor  $\text{Mod}_{\mathcal{O}_X}^\Gamma \rightarrow \text{Mod}_{\mathcal{O}_{Y^*(d)}}^\Gamma \rightarrow \text{Mod}_{\mathcal{O}_{Y^*(d)}}$  given by  $(-)^\Gamma \circ \pi_*$ . It is a sheaf on  $Y^*(d)$ , and it has a key advantage over  $\Theta$ , namely we can evaluate along the identity section and get something interesting:

$$(IV.1) \quad \text{evaluate at } 1 : \Theta^\Gamma \rightarrow H^s(\Gamma, \mathbf{Z}') \otimes_{\mathbf{Z}'} \mathcal{O}_{Y^*(d)}.$$

We now explain exactly how to define the map (IV.1). First, evaluation at 1 provides a  $\Gamma$ -equivariant map of  $\Gamma$ -equivariant  $\mathcal{O}_X$ -modules

$$\mathcal{L}^B \rightarrow e_* e^* \mathcal{L}^B$$

where  $e_* e^* \mathcal{L}^B = e_* \mathcal{O}_{Y^*(d)}$  has the trivial  $\Gamma$ -equivariant structure thanks to the compatibility between the  $\Gamma$ -action on  $\mathcal{L}^B$  and the rigidification on  $\mathcal{L}^B$ . Therefore, it provides a map

$$R^s \pi_*^\Gamma \mathcal{L}^B \rightarrow R^s \pi_*^\Gamma (e_* e^* \mathcal{L}^B) = R^s \pi_*^\Gamma (e_* \mathcal{O}_{Y^*(d)}).$$

To define (IV.1), we just carefully examine  $R^s \pi_*^\Gamma (e_* \mathcal{O}_{Y^*(d)})$  to see its (very plausible) relationship to  $H^s(\Gamma, \mathbf{Z}') \otimes_{\mathbf{Z}'} \mathcal{O}_{Y^*(d)}$ . In particular, the Grothendieck spectral sequence (which applies because of [23, Proposition 5.1.3]) is of the form

$$E_2 = H^i(\Gamma, R^j \pi_* e_* \mathcal{O}_{Y^*(d)}) \implies R^{i+j} \pi_*^\Gamma e_* \mathcal{O}_{Y^*(d)}$$

which provides an isomorphism  $H^s(\Gamma, \mathcal{O}_{Y^*(d)}) \simeq R^s \pi_*^\Gamma e_* \mathcal{O}_{Y^*(d)}$  due to the vanishing of  $R^j \pi_* e_* \mathcal{O}_{Y^*(d)}$  when  $j > 0$ . That is,  $R^s \pi_*^\Gamma (e_* \mathcal{O}_{Y^*(d)})$  is the  $s$ -th right derived functor of the functor  $(-)^\Gamma$  from the category  $\text{Mod}_\Gamma(S)$  of  $\Gamma$ -equivariant sheaves of abelian groups on  $Y^*(d)$  to  $\text{Mod}(Y^*(d))$ , applied to  $\mathcal{O}_{Y^*(d)}$  with the trivial  $\Gamma$ -action.

Another way to say this is that  $\text{Mod}_\Gamma$  is the category of sheaves of  $\mathbf{Z}'[\Gamma]$ -modules on  $Y^*(d)$ , where  $\mathbf{Z}'[\Gamma]$  is viewed as a constant sheaf of rings on  $Y^*(d)$ , and  $H^s(\Gamma, -) : \text{Mod}_\Gamma(Y^*(d)) \rightarrow \text{Mod}(Y^*(d))$  is the right derived functor of  $\text{Hom}_{\mathbf{Z}'[\Gamma]}(\mathbf{Z}', -)$ , that is, by  $H^s(\Gamma, \mathcal{O}_{Y^*(d)})$ , we mean the sheaf  $\text{Ext}^s(\mathbf{Z}', \mathcal{O}_{Y^*(d)})$  of abelian groups on  $Y^*(d)$  (see [23, §4.4, Remarque 1]).

By [23, Corollaire 1, to Proposition 4.2.4], that Ext group can be calculated by means of taking a resolution

$$\mathbf{Z}' \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

of  $\mathbf{Z}'$  by finitely generated free  $\mathbf{Z}'[\Gamma]$ -modules, applying  $\text{Hom}_{\mathbf{Z}'[\Gamma]}(-, \mathcal{O}_{Y^*(d)})$ , and then taking cohomology. Such a resolution actually exists, by [7, §11.1(c)].

The complex  $\text{Hom}_{\mathbf{Z}'[\Gamma]}(P_\bullet, \mathcal{O}_{Y^*(d)})$  of sheaves of abelian groups on  $Y^*(d)$  that computes  $H^\bullet(\Gamma, \mathcal{O}_{Y^*(d)})$  is isomorphic to the complex

$$\text{Hom}_{\mathbf{Z}'[\Gamma]}(P_\bullet, R) \otimes \mathcal{O}_{Y^*(d)}$$

via the multiplication map, since the  $P_\bullet$  are finite free over  $\mathbf{Z}'[\Gamma]$ . But this complex computes  $H^\bullet(\Gamma, \mathbf{Z}') \otimes \mathcal{O}_{Y^*(\mathbf{d})}$ , thanks to the fact that  $Y^*(\mathbf{d}) \rightarrow \text{Spec} \mathbf{Z}'$  is flat. This completes the full detail of the justification of why the evaluation at 1 map has the target claimed by (IV.1).

(ii) Sections of  $\Theta^\Gamma$  near a cusp

Let  $i : \text{Spec} \mathbf{Z}'((q^{1/(2\mathbf{d})})) \rightarrow Y^*(\mathbf{d})$  be a cusp over the cusp  $0_i^*$  of  $Y^*$  (see Table 1). Then “sections of  $\Theta^\Gamma$  near the cusp” means

$$H^0(\text{Spec} \mathbf{Z}'((q^{1/(2\mathbf{d})})), i^* \Theta^\Gamma) = H_\Gamma^s(E_{\text{late}, 2\mathbf{d}}^g, \mathcal{L}^B).$$

The following key proposition is proved by the first-named author in a separate paper [34].

**Proposition IV.1.** *The sections of  $\Theta^\Gamma$  near the chosen cusp are identified with  $H^s(\tilde{\Gamma}, \mathcal{O}^{\text{an}})$ , in such a way that the evaluation map (IV.1) corresponds under the isomorphism of (I.8) to the natural evaluation map  $\mathcal{O}^{\text{an}} \rightarrow \mathbf{Z}'$  (given by substituting  $\underline{z} = 1$ ), composed with restriction to  $\Gamma$ . In symbols, the diagram*

$$\begin{array}{ccc} H^0(\text{Spec} \mathbf{Z}'((q^{1/(2\mathbf{d})})), i^* \Theta^\Gamma) & \longrightarrow & H^s(\Gamma, \mathbf{Z}'((q^{1/(2\mathbf{d})}))) \\ \downarrow & & \downarrow \\ H^s(\Gamma \ltimes \mathbf{Z}^m, \mathcal{O}^{\text{an}}) & \longrightarrow & H^s(\Gamma, \mathbf{Z}'((q^{1/(2\mathbf{d})}))) \end{array}$$

commutes, where the top horizontal map is the pullback of the restriction map we just defined, and the bottom one is the one given by restricting to  $\Gamma$  and applying the evaluation at  $\underline{z} = 1$  map to  $\mathcal{O}^{\text{an}}$ .

This provides the geometric significance of the cocycle  $\tilde{\vartheta}$  we constructed in Section IV<sub>0</sub>: it provides an explicit section of  $\Theta^\Gamma$  near a cusp of  $Y^*(\delta)$ .

(iii): extending  $\tilde{\vartheta}$  globally

One plausible way to prove that our section  $\tilde{\vartheta}$  of  $\Theta^\Gamma$  extends globally after tensoring with  $(dz/z)^{m/2}$  would be to redo the entire content of step (II)  $\Gamma$ -equivariantly. However, we accomplish this less laboriously, by exploiting the direct relationship between  $\Theta^\Gamma$  and  $\Theta$ , plus the fact that we already extended certain sections of  $\Theta_{\text{norm}}$  from cusp neighborhoods to global sections.

If we forget about the evaluation structure,  $\Theta^\Gamma$  is very easy to understand.

**Lemma IV.2.** *The morphism  $\Theta^\Gamma \rightarrow \Theta$  identifies  $\Theta^\Gamma$  with the  $\Gamma$ -invariant subbundle of  $\Theta$ .*

*Proof.* The Grothendieck spectral sequence reads

$$H^p(\Gamma, R^q \pi_* \mathcal{L}^B) \implies R^{p+q} \pi_*^\Gamma \mathcal{L}^B.$$

However,  $R^q\pi_*\mathcal{L}$  vanishes for all  $q \neq s$ , and therefore the spectral sequence necessarily degenerates to an isomorphism

$$H^0(\Gamma, R^s\pi_*\mathcal{L}^B) \simeq H^p\pi_*^\Gamma \mathcal{L}^B,$$

by means of which the image of  $\Gamma$ -equivariant sections corresponds precisely to the  $\Gamma$ -invariant sections of  $\Theta$ .  $\square$

The material of (ii) further extends to describe this natural map. Following [34], we will use:

**Proposition IV.3.** *The natural map  $\Theta^\Gamma \rightarrow \Theta$  becomes, after applying the Fourier expansion isomorphism of [Proposition IV.1](#), the natural restriction-to- $\mathbf{Z}^m$  map*

$$H^s(\Gamma \ltimes \mathbf{Z}^m, \mathcal{O}^{\text{an}}) \rightarrow H^s(\mathbf{Z}^m, \mathcal{O}^{\text{an}})$$

where  $H^s(\mathbf{Z}^m, \mathcal{O}^{\text{an}})$  is identified with the sections of  $i^*\Theta$  via [Equation \(I.9\)](#).

**IV.1. The lift  $\tilde{\vartheta}_0$  of  $\vartheta_0$ .** We already constructed a section  $\vartheta_0$  of  $i^*\Theta_{\text{norm}}$ , whose Fourier expansion in  $H^s(\mathbf{Z}^m, \mathcal{O}^{\text{an}})$  was

$$\vartheta_0(v_0, \dots, v_s) = \sum_{\underline{n} \in \alpha + \mathcal{C}(v)} q^{Q(\underline{n})} \underline{z}^{\underline{n}}.$$

We know that  $\vartheta_0 \otimes (dz/z)^{m/2}$  extends to a global section of  $\Theta_{\text{norm}}$ .

We will prove that  $\vartheta_0$  is fixed by  $\Gamma$  and therefore corresponds to a section of  $i^*\Theta^\Gamma$  via [Lemma IV.2](#). We will also prove that that section is precisely  $\tilde{\vartheta}$  from [Section IV<sub>0</sub>](#), via [Proposition IV.1](#), therefore proving that  $\tilde{\vartheta}$  extends.

Recall that  $\tilde{\vartheta}_0$  was defined so that for each  $\sigma : \Delta^s \rightarrow E$  as above,

$$\tilde{\vartheta}_0(\sigma) = \sum_{n \in \mathbf{Z}^m} (\sigma\text{-multiplicity of } n) \cdot q^{Q(n)} \underline{z}^n.$$

We now check the first part of our claim, which is that the inclusion  $\Theta^\Gamma \rightarrow \Theta$  from [Lemma IV.2](#) sends  $\tilde{\vartheta}_0$  to  $\vartheta_0$ .

**Proposition IV.4.** *After taking cohomology, the composition*

$$(IV.2) \quad C_s(\tilde{\Gamma}) \rightarrow C_s^\natural(E) \xrightarrow{\tilde{\vartheta}_0} \mathcal{O}^{\text{an}}$$

is carried to  $\vartheta_0$  under restriction from  $\tilde{\Gamma}$  to  $\mathbf{Z}^m$  (which corresponds to  $\Theta^\Gamma \rightarrow \Theta$  near the cusp, by [Proposition IV.3](#))

*Proof.* Fix a *rational* positive definite subspace  $P$  corresponding to a basepoint  $p$  in  $D$  so that we can consider  $\mathbf{R}^m \subset E$  as the fiber over  $p$ , which therefore contains  $P$  as a subspace. The  $\tilde{\Gamma}$ -cocycle is given by the composition

$$C_s(\Gamma \ltimes \mathbf{Z}^m) \xrightarrow{\Psi} C_s^\natural(E) \xrightarrow{\tilde{\vartheta}_0} \mathcal{O}^{\text{an}}.$$

On the other hand, the definition of the cocycle  $\vartheta_0$  for  $\mathbf{Z}^m$  is of the form

$$C_s(\mathbf{Z}^m) \xrightarrow{\Psi_0} C_s^{\natural}(\mathbf{R}^m) \rightarrow \mathcal{O}^{\text{an}}$$

where  $\Psi_0$  is the quasiisomorphism which restricts to  $C_s(\mathbf{Z}^m)$  sending  $(v_0, \dots, v_s)$  to  $\alpha + \mathcal{C}(v)$  (recall from [Section I](#)), and the decoration  $\natural$  means adding the requirement that the simplices  $\sigma : \Delta^s \rightarrow \mathbf{R}^m$  are transverse to all  $\mathbf{Z}^m$ -translates of  $P$ . Recall from [Section I](#) (or better yet, from [Section IV<sub>0</sub>.2.1](#)) that the map  $\Psi_0$  does indeed map to  $C_s^{\natural}(\mathbf{R}^m)$ .

It follows from the definitions that the inclusion  $C_s^{\natural}(\mathbf{R}^m) \rightarrow C_s(E)$  induced by the inclusion of  $\mathbf{R}^m$  as the fiber over  $p$  actually factors through  $C_s^{\natural}(E)$  and (by looking at the definition of  $\text{sweep}_{\Psi_0(v_0, \dots, v_s)}$ ) that the cocycle  $\vartheta_0$  is actually given by the composition

$$C_s(\mathbf{Z}^m) \rightarrow C_s^{\natural}(\mathbf{R}^m) \rightarrow C_s^{\natural}(E) \xrightarrow{\tilde{\vartheta}_0} \mathcal{O}^{\text{an}}.$$

The desired result follows from this and the fact that the composition  $C_s(\mathbf{Z}^m) \rightarrow C_s^{\natural}(E)$  has to be homotopic to the restriction of  $\Psi$ , following exactly the same argument as [Proposition IV<sub>0</sub>.3](#).  $\square$

It remains to conclude that what we just proved allows us to conclude that we have actually produced the Fourier expansion of a global section of  $\Theta^{\Gamma} \otimes \omega^{m/2}$

**Lemma IV.5.**  $\tilde{\vartheta}_0 \otimes (dz/z)^{m/2}$ , as defined above, extends to a global section of  $\Theta^{\Gamma} \otimes \omega^{m/2}$ .

*Proof.* We first verify that  $\vartheta_0$  is  $\Gamma$ -invariant. Since  $\Theta$  is a vector bundle, to check that a given section is  $\Gamma$ -invariant it's enough to check it on the generic fiber, or even just on complex points, or even just on complex points in a formal neighbourhood of a single cusp. The  $\Gamma$ -invariance of  $\vartheta_0$  then follows from [Proposition IV.4](#). Therefore,  $\vartheta_0$  promotes to *some* section of  $\Theta^{\Gamma}$ ; we must verify that this section is given by  $\tilde{\vartheta}_0$  in a neighbourhood of the cusp. But this follows by the uniqueness of the  $\Gamma$ -equivariant lift ([Lemma IV.2](#)), applied near the cusp.  $\square$

All these arguments are very formal, and they provide us with a global section  $\tilde{\vartheta}_0$  of  $R^s \pi_*^{\Gamma} \mathcal{L}^B \otimes \omega^{m/2}$  with *explicit* Fourier expansion near a cusp over 0 in  $Y^*(\mathbf{d})$ . We know that after applying the restriction-to-1 map, we get a section of  $H^s(\Gamma, \mathbf{Z}') \otimes \omega^{m/2}$ . That is, a modular form of weight  $m/2$ , level  $Y^*(\mathbf{d})$ , valued in  $H^s(\Gamma, \mathbf{Z}')$ :

**Definition IV.6.** Consider the  $\mathbf{Z}[1/(2\mathbf{d})]$ -module  $H^s(\Gamma, \mathbf{Z}[1/(2\mathbf{d})])$ . By base-change to  $\text{Spec } \mathbf{Z}'$  and then to  $Y^*(\Delta)$ ,  $H^s(\Gamma, \mathbf{Z}[1/(2\mathbf{d})])$  becomes a coherent sheaf on the  $\mathbf{Z}'$ -scheme  $Y^*(\Delta)$ . We therefore define a modular form of weight  $k/2$  and level  $\Gamma^*(\Delta)$  with coefficients in  $H^s(\Gamma, \mathbf{Z}[1/(2\mathbf{d})])$  to be a global section of  $H^s(\Gamma, \mathbf{Z}[1/(2\mathbf{d})]) \otimes \omega^{k/2}$  over  $Y^*(\Delta)$ .

The  $q$ -expansion of such a modular form at any cusp is therefore an element of the formal power series module

$$H^s(\Gamma, \mathbf{Z}[1/(2\mathbf{d})]) \otimes \mathbf{Z}[1/(2\mathbf{d})]((q^{1/(4\mathbf{d})})),$$

from which its  $n$ -th  $q$ -expansion coefficient is of course a class in  $H^s(\Gamma, \mathbf{Z}[1/(2d)])$  in an obvious way.

This is exactly the type of object we hoped to produce, but there is something missing:

(IV.3) We still need to calculate its  $q$ -expansion at a cusp.

We will address this in step (V), the final section.

## V. Restricting to the identity section: algebraic construction of Kudla–Millson series

### V. THE FOURIER COEFFICIENTS OF THE EVALUATION OF $\tilde{\vartheta}_0$ AT THE IDENTITY

We will complete the task set out in (IV.3) by proving that the  $q$ -expansion of the evaluation at 1 of  $\tilde{\vartheta}_0$  is the same as the one defined by Kudla and Millson [38, §2] whose coefficients are cohomology classes defined by special cycles.

We first have to introduce the cycles used by Kudla–Millson to define their  $q$ -expansions.

Consider the vector bundle  $V$  over  $D$  whose fiber over a point  $p \in D$  corresponding to a positive definite subspace  $P \subset \mathbf{R}^m$  is simply  $P$ . We will call this the *tautological positive vector bundle* or *positive bundle* for short; and the similar vector bundle with fiber  $P^\perp$  the *tautological negative vector bundle*  $V^\perp$ . The vector bundle  $V$  is naturally a rank- $r$  subbundle of  $E \rightarrow D$ , and  $V^\perp$  is naturally a rank- $s$  subbundle of  $E \rightarrow D$ . They descend to vector bundles  $V_\Gamma$  and  $V_\Gamma^\perp$  on the locally symmetric space  $\Gamma \backslash D = D_\Gamma$ . Since  $V_\Gamma^\perp$  comes with an orientation as in [38, §2], the Euler class  $e$  of  $V_\Gamma^\perp$  is well-defined and lies in  $H^s(D_\Gamma, \mathbf{Z}) = H^s(\Gamma, \mathbf{Z})$ . This is the constant term of the Kudla–Millson theta series.

In the absence of any compactness assumption on  $D_\Gamma$  that would allow us to view  $e$  as Poincaré dual to anything, we may think of  $e$  as the Borel–Moore homology class for  $V_\Gamma^\perp$  obtained by pushing forward the codimension- $s$  submanifold  $D_\Gamma \subset V^\perp$  included by the zero section. In other words, using a  $\Gamma$ -equivariant

$$C_*(\Gamma) \rightarrow C_*^{\text{rd}}(V^\perp)$$

as in Proposition IV.0.3, the class  $e$  is defined by the  $\Gamma$ -invariant function  $C_s^{\text{rd}}(V^\perp) \rightarrow \mathbf{Z}$  taking  $\sigma : \Delta^s \rightarrow V^\perp$  to the signed intersection number  $\#(\sigma \times_{V^\perp} D)$  as in [17].

Now we introduce the other Kudla–Millson coefficients for  $n \in \frac{1}{2}\mathbf{Z}$ ,  $n > 0$ . A point in the total space of  $V$  is an element of  $\mathbf{R}^m$  and a positive definite subspace containing it. Inside of  $D$ , we may consider the locus  $Y^\lambda$  where  $\lambda \in P$ . And for a half-integer  $n$  we put  $\mathcal{Y}^n$  for the

formal sum over lattice points where  $Q$  takes the value  $n$ :

$$\mathcal{Y}^n := \sum_{Q(\underline{\lambda})=n} Y^{\underline{\lambda}}.$$

This  $\mathcal{Y}^n$  is a  $\Gamma$ -stable formal sum of smooth submanifolds of the total space of the positive bundle; its components are each of dimension equal to  $\dim(D) - s$ , because to contain a given vector is a codimension  $s$  condition on a positive definite subspace. In fact, there are only finitely many  $\Gamma$ -orbits of  $\underline{\lambda}$  as above (by [8, 9.11]). If we choose representatives  $\underline{\lambda}_1, \dots, \underline{\lambda}_k$ , then the quotient  $\Gamma \backslash \mathcal{Y}^n$  is identified with a union  $\Gamma_i \backslash Y^{\underline{\lambda}_i}$ , which is itself a locally symmetric space. Since  $\Gamma$  has no torsion, the map

$$\Gamma \backslash \mathcal{Y}^n \rightarrow \Gamma \backslash D = D_\Gamma$$

is a proper immersion of manifolds (modulo the issues resolved in [38, §2]). We define the class  $C_n \in H^s(\Gamma, \mathbf{Z})$  to be the integral Borel–Moore homology class for  $D_\Gamma$  obtained by pushing forward the fundamental class of  $\Gamma \backslash \mathcal{Y}^n$ . In our preferred language, using a  $\Gamma$ -equivariant

$$C_*(\Gamma) \rightarrow C_*^{\cap \text{ all } Y^{\underline{\lambda}}}(D)$$

as in [Proposition IV<sub>0</sub>.3](#), the class  $C_n$  is given by sending a simplex  $\sigma : \Delta^s \rightarrow D$  that is transverse to all  $Y^{\underline{\lambda}}$  to the sum of the signed intersection numbers with all  $Y^{\underline{\lambda}}$  such that  $Q(\underline{\lambda}) = n$ .

Our interpretation of Kudla–Millson theory in terms of intersections with transverse chains already appears at least implicitly in the literature on indefinite theta functions — for instance in [19–22, 62]. In any case, the Kudla–Millson theta series  $\theta_0$  is the series in  $H^s(\Gamma, \mathbf{Z}) \otimes \mathbf{Z}((q^{1/(2d)}))$  whose  $n$ -th coefficient is the class  $C_n$ . To prove that we have produced a modular form with  $q$ -expansion agreeing with that of Kudla–Millson, it suffices to verify that the  $q^n$ -coefficient of  $\tilde{\vartheta}_0|_{z=1}$  is the class  $C_n$ . We do this now, thereby completing the proof of our algebraic Kudla–Millson theory.

Suppose  $n > 0$ . The  $n$ -th coefficient of  $\tilde{\vartheta}_0|_{z=1}$  is given by the composition

$$\begin{array}{c} C_*(\Gamma) \\ \downarrow \\ C_*(\Gamma \ltimes \mathbf{Z}^m) \longrightarrow C_*^{\cap}(E) \longrightarrow \mathbf{Z} \end{array}$$

where the last arrow is taking the sum of the signed counts of  $\text{sweep}_\sigma^{-1}(\lambda)$  over  $\lambda$  such that  $Q(\lambda) = n$ . We define  $C_*^\heartsuit(D)$  to be the chain complex of singular chains that are transverse to  $Y^\lambda$  for all  $\lambda \neq 0$ . We have to omit  $\lambda = 0$  because all subspaces pass through the origin. All such singular chains are transverse to all the  $Z_\lambda$  in  $E$ , so we can extend to a commutative diagram up to homotopy

$$\begin{array}{ccc}
C_*(\Gamma) & \longrightarrow & C_*^\heartsuit(D) \\
\downarrow & & \downarrow \\
C_*(\Gamma \ltimes \mathbf{Z}^m) & \longrightarrow & C_*^\heartsuit(E) \longrightarrow \mathbf{Z}
\end{array}$$

The  $n$ -th coefficient of  $\tilde{\vartheta}_0|_{\underline{z}=1}$  is therefore given (starting with  $\sigma \in C_*^\heartsuit(D)$ ) by sending  $\sigma : \Delta^s \rightarrow D$  to

$$\sum_{Q(\lambda)=n} \# \text{sweep}_\sigma^{-1}(\lambda).$$

Recall that  $\mathcal{R}(\sigma) = \sigma^* \text{Aff}_E$  is just  $\sigma^* V$  in this case, since  $\sigma$  only maps to  $D \subset E$  along the zero section. So this cocycle is really just counting (over all  $\lambda$  with  $Q(\lambda) = n$ ) the oriented number of times that  $\lambda$  appears in the fiber of  $V$  over the point  $\sigma(p)$ , i.e. the intersection number between  $\sigma$  and  $Z_\lambda$ . So it indeed coincides with the class  $C_n$ .

Finally, suppose that  $n = 0$ . We wish to rewrite (up to homotopy) the cocycle  $\tilde{\vartheta}_0|_{\underline{z}=1}$  as a  $\Gamma$ -equivariant homomorphism from  $C_*^\heartsuit(V^\perp)$ , using the fact that it fits into the commutative up to homotopy diagram

$$\begin{array}{ccc}
C_*(\Gamma) & \longrightarrow & C_*^\heartsuit(V^\perp) \\
\downarrow & & \downarrow \\
C_*(\Gamma \ltimes \mathbf{Z}^m) & \longrightarrow & C_*^\heartsuit(E) \longrightarrow \mathbf{Z}
\end{array}$$

The diagram we used for  $n \neq 0$  wouldn't work here, because of what we already mentioned: being transverse to  $Y_0 = D$  is vacuous and therefore doesn't imply anything about the image being in  $C_*^\heartsuit(E)$ , hence the diagram itself doesn't even exist in this case.

Using this diagram, one sees that the composite cocycle

$$C_*^\heartsuit(V^\perp) \rightarrow C_*^\heartsuit(E) \xrightarrow{\tilde{\vartheta}_0|_{\underline{z}=1}} \mathbf{Z}$$

is explicitly given by taking  $\sigma : \Delta^s \rightarrow V^\perp \subset E$  to the oriented count of  $\text{sweep}_\sigma^{-1}(0)$ . But  $\text{Aff}_E$  having a point in the fiber over  $(\sigma(p), x)$ ,  $x \in P^\perp$  implies that  $0 \in x + P$ , so  $x \in P \cap P^\perp = 0$ . In other words, it just counts the oriented intersection of  $\sigma$  with the zero section of  $V^\perp$ , as desired.

Thus we have proved:

**Theorem V.1.** *The formal series*

$$\theta_0 = \tilde{\vartheta}_0|_{\underline{z}=1} = \sum_{n \in \frac{1}{2}\mathbf{Z}} C_n q^n \in H^s(\Gamma, \mathbf{Z}[1/(2d)]) \otimes \mathbf{Z}((q^{1/(2d)}))$$

*is the  $q$ -expansion of a modular form of weight  $m/2$  and level  $4d$ , valued in  $H^s(\Gamma, \mathbf{Z}[\frac{1}{2d}])$ .*

## A. NOTIONS OF THETA GROUP SCHEMES

Since much of step (III) is about choosing the right level structure (as opposed to working only over  $Y^*$ , where everything is defined at first), we are forced to recall the standard definitions in general — in reality we will only apply them to  $\mathcal{L}^B$  pulled back to various covers of  $Y^*$  (with the caveat that there will also be a change of base ring involved). We pull all of this from [52, §2B], itself a summary of arguments in [43] and [45, §6] which were mostly originally written mostly for relatively ample line bundles but actually apply in general with no substantive changes.

**Definition A.1.** Let  $\pi : X \rightarrow S$  be an abelian scheme, and let  $\mathcal{F}$  be a line bundle on  $X$ . The functor taking an  $S$ -scheme  $T$  to the group of  $x \in X(T)$  such that  $T_x^*(\mathcal{F}_T) \simeq \mathcal{F}_T \otimes M$  where  $M$  is pulled back from  $T$  is representable by an  $S$ -group scheme, since it is the kernel of the natural map

$$\phi_{\mathcal{F}} : X \rightarrow X^\vee.$$

We denote this kernel by  $K(\mathcal{F})$ .

**Definition A.2.** Let  $\pi : X \rightarrow S$  be an abelian scheme, and let  $\mathcal{F}$  be a line bundle on  $X$ . The *theta-group scheme*  $\mathcal{G}(\mathcal{F})$  over  $S$  is defined to represent the functor taking an  $S$ -scheme  $T$  to the group of pairs  $(x, \varphi)$  such that  $\varphi$  is an isomorphism  $\varphi : \mathcal{F} \simeq T_x^* \mathcal{F}$ . The fact that this functor is representable is noted by [52, p. 1728], and follows from exactly the same argument as [45, §6, Proposition 1], which is based on the following lemma.

**Lemma A.3.** *Let  $\pi : X \rightarrow S$  be an abelian scheme and  $\mathcal{F}$  an arbitrary line bundle on  $X$ . The obvious sequence*

$$1 \rightarrow \mathbf{G}_m \rightarrow \mathcal{G}(\mathcal{F}) \rightarrow K(\mathcal{F}) \rightarrow 0$$

*is an exact sequence of Zariski sheaves on  $S$ .*

(making  $\mathcal{G}(\mathcal{F})$  a  $\mathbf{G}_m$ -torsor over  $K(\mathcal{F})$  which is therefore representable).

The sequence

$$1 \rightarrow \mathbf{G}_m \rightarrow \mathcal{G}(\mathcal{F}) \rightarrow K(\mathcal{F}) \rightarrow 0$$

is therefore an exact sequence of  $S$ -group schemes regardless of what  $\mathcal{F}$  is.

This implies that the commutator map  $\mathcal{G}(\mathcal{F}) \times \mathcal{G}(\mathcal{F}) \rightarrow \mathcal{G}(\mathcal{F})$  derived from the group scheme structure of  $\mathcal{G}(\mathcal{F})$  factors through the natural inclusion of  $\mathbf{G}_m$  into  $\mathcal{G}(\mathcal{F})$ .

**Lemma A.4.** *If  $\mathcal{F}$  is nondegenerate, then  $K(\mathcal{F})$  is a finite flat  $S$ -group scheme. The commutator pairing*

$$\mathcal{G}(\mathcal{F}) \times \mathcal{G}(\mathcal{F}) \rightarrow \mathbf{G}_m$$

*factors through the quotient to  $K(\mathcal{F}) \times K(\mathcal{F})$  and is a nondegenerate alternating pairing of  $S$ -group schemes.*

*Proof.* See [52, Lemma 2.2, Lemma 2.8] and references therein. □

We now begin the (also standard) discussion of global trivialization of theta-groups, so that we may directly relate some  $\mathcal{G}(\mathcal{L}^B)$  to a model finite Heisenberg group  $H_B$  after suitable base change. The functorial description of  $\mathcal{G}(\mathcal{F})$  and  $K(\mathcal{F})$  also shows that they are compatible with base change:

$$\mathcal{G}(\mathcal{F}_{S'}) = \mathcal{G}(\mathcal{F}) \times_S S', \quad K(\mathcal{F}_{S'}) = K(\mathcal{F}) \times_S S'$$

for all  $S$ -schemes  $S'$ . This is relevant because we will trivialize the theta group after change of base ring and after passing to a cover of the base scheme.

**Definition A.5.** Let  $\pi : X \rightarrow S$  be an abelian scheme. Let  $\mathcal{F}$  be a nondegenerate line bundle on  $X$  relative to  $\pi$  such that  $K(\mathcal{F})$  is étale over  $S$ . Let  $\underline{\delta}$  be a tuple of positive integers such that  $d_i | d_{i+1}$ . A *theta structure* for  $\mathcal{F}$  of type  $\underline{\delta}$  is an  $S$ -group scheme isomorphism

$$\mathcal{G}(\mathcal{F}) \simeq \mathcal{G}(\underline{\delta}),$$

where  $\mathcal{G}(\underline{\delta})$  is the *standard finite Heisenberg group scheme* on  $S$  given as a scheme by

$$\mathcal{G}(\underline{\delta}) = \mathbf{G}_m \times \prod_{i=1}^m (\mathbf{Z}/d_i) \times \prod_{i=1}^m \mu_{d_i}$$

with multiplication law

$$(\lambda_1, x_1, \ell_1) \cdot (\lambda_2, x_2, \ell_2) = (\lambda_1 \lambda_2 \ell_2^{x_1}, x_1 + x_2, \ell_1 + \ell_2).$$

Defining  $K(\underline{\delta}) = \prod_{i=1}^m (\mathbf{Z}/d_i) \times \prod_{i=1}^m \mu_{d_i}$ , the standard finite Heisenberg group scheme fits into the short exact sequence of  $S$ -group schemes

$$1 \rightarrow \mathbf{G}_m \rightarrow \mathcal{G}(\underline{\delta}) \rightarrow K(\underline{\delta}) \rightarrow 0$$

whose induced commutator pairing  $K(\underline{\delta}) \times K(\underline{\delta}) \rightarrow \mathbf{G}_m$  is the standard alternating pairing  $e((x_1, \ell_1), (x_2, \ell_2)) = \ell_2^{x_1} \ell_1^{-x_2}$ .

*Remark.* If  $\mathcal{O}_S$  contains a  $d_m$ -th root of unity, then the group schemes  $\mu_{d_i}$  all become constant, and the theta structure simply becomes an isomorphism with the finite Heisenberg group built from  $\mathbf{G}_m \times (\prod \mathbf{Z}/d_i)^2$ , which is exactly what the group  $H_B$  is a special case of: take the  $d_i$  to be the elementary divisors of  $B$ . That specific tuple of integers depending only on  $B$  is the only tuple  $\underline{\delta}$  that will be relevant to us.

**Definition A.6.** Let  $S$  be a scheme such that  $2d$  is invertible in  $\mathcal{O}_S$ . Define the functor  $\underline{\text{Aut}}_{\mathbf{G}_m}(\mathcal{G}(\underline{\delta}))$  from  $S$ -schemes to sets given by taking  $T \rightarrow S$  to the set of automorphisms  $\varphi$  of the  $T$ -group scheme  $\mathcal{G}(\underline{\delta}) \times_{\mathbf{Z}} T$  with the property that  $\varphi$  restricts to the identity on  $\mathbf{G}_{m,S}$ .

**Lemma A.7.** Suppose that  $S$  is a  $\mathbf{Z}'$ -scheme. There is an exact sequence of étale sheaves of groups

$$1 \rightarrow K(\underline{\delta}) \rightarrow \underline{\text{Aut}}_{\mathbf{G}_m}(\mathcal{G}(\underline{\delta})) \rightarrow \underline{Sp}(K(\underline{\delta})) \rightarrow 0,$$

where:

- The inclusion of the constant group scheme  $K(\underline{\delta})_T = \prod_i (\mathbf{Z}/d_i)^2$  takes  $(a, b)$  to the automorphism of  $\mathcal{G}(\underline{\delta})_T$  given by  $(\lambda, x, y) \mapsto (\lambda, x, y) \cdot \zeta^{xb-ay}$ .

- The map  $\underline{Aut}_{\mathbf{G}_m}(\mathcal{G}(\underline{\delta})) \rightarrow \underline{Sp}(K(\underline{\delta}))$  just takes an automorphism  $\tilde{\varphi}$  to the symplectic automorphism that fits in the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathcal{G}(\underline{\delta}) & \longrightarrow & K(\underline{\delta}) \longrightarrow 0 \\
 & & \downarrow 1 & & \downarrow \tilde{\varphi} & & \downarrow \varphi \\
 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathcal{G}(\underline{\delta}) & \longrightarrow & K(\underline{\delta}) \longrightarrow 0
 \end{array}$$

*Proof.* This lemma is common in the literature on this subject: see for instance [39, Lemma 3.9], which cites [49, Lemma 6.3.7], which is ultimately based on [44, Corollary to Theorem 1]. But we remark that the entire content of the lemma is already contained in the longer but more direct argument of [57, §5], since  $\mathbf{d}$  is odd. If  $\mathbf{d}$  is even, then one must make sure to include a primitive  $2\mathbf{d}$ -th root of unity in  $\mathbf{Z}'$  so that one has a square root of  $\zeta$ .  $\square$

**Corollary A.8.**  $\underline{Aut}_{\mathbf{G}_m}(\mathcal{G}(\underline{\delta}))$  is represented by a finite étale  $S$ -group scheme.

*Proof.* The functor  $\underline{Sp}(K(\underline{\delta}))$  is represented by a constant  $S$ -group scheme with group  $Sp(K(\underline{\delta}))$ . The functor we are interested in is a  $K(\underline{\delta})$ -torsor over it, which is finite étale since  $\delta$  is invertible, so it is representable by a finite étale  $S$ -group scheme.  $\square$

In particular, if one fixes a symplectic automorphism of  $K(\underline{\delta})$ , there is a whole  $K(\underline{\delta})$  worth of ambiguity in how to lift it to an automorphism of  $\mathcal{G}(\underline{\delta})$  fixing  $\mathbf{G}_m$ . [11, 2.5.3] provides the exact rigidity property that is typically used.

**Lemma A.9.** Let  $S$  be any  $\mathbf{Z}'$ -scheme. Then the functor  $\underline{Aut}^{Sym}(\mathcal{G}(\underline{\delta}))$  taking an  $S$ -scheme  $T$  to the group of symmetric automorphisms of  $\mathcal{G}(\underline{\delta})_T$  fixing  $\mathbf{G}_{m,T}$  is representable and fits in an exact sequence

$$0 \rightarrow K(\underline{\delta})[2] \rightarrow \underline{Aut}^{Sym}(\mathcal{G}(\underline{\delta})) \rightarrow \underline{Sp}(K(\underline{\delta})) \rightarrow 1$$

*Proof.* This appears in the unpublished preprint [11, Lemma 2.5.3] and is also standard. The only point that is not completely obvious is the surjectivity, which we do following the argument of [44, p. 318]. The point is that the operation  $\tilde{\varphi} \mapsto D_{-1} \circ \tilde{\varphi} \circ D_{-1}$  maps  $\underline{Aut}(\mathcal{G}(\underline{\delta}))$  to itself, fitting into the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K(\underline{\delta}) & \longrightarrow & \underline{Aut}(\mathcal{G}(\underline{\delta})) & \longrightarrow & \underline{Sp}(K(\underline{\delta})) \longrightarrow 1 \\
 & & \downarrow -1 & & \downarrow \iota & & \downarrow 1 \\
 0 & \longrightarrow & K(\underline{\delta}) & \longrightarrow & \underline{Aut}(\mathcal{G}(\underline{\delta})) & \longrightarrow & \underline{Sp}(K(\underline{\delta})) \longrightarrow 1
 \end{array}$$

Taking the fixed points of the exact sequence immediately yields (for all  $T \rightarrow S$ )

$$0 \rightarrow K(\underline{\delta})[2] \rightarrow \underline{Aut}^{Sym}(\mathcal{G}(\underline{\delta})) \rightarrow \underline{Sp}(K(\underline{\delta})),$$

and we still need to prove surjectivity. Given a section  $\varphi$  of  $\underline{Sp}(K(\underline{\delta}))$ , we already know we can lift it to an automorphism  $\tilde{\varphi}$  of  $\mathcal{G}(\underline{\delta})$  by the previous lemma. We want to prove that it can be modified by a section  $k$  of  $K(\underline{\delta})$  to make it symmetric. Letting

$$u = \tilde{\varphi}^{-1}\iota(\tilde{\varphi})$$

(which is a section of  $K(\underline{\delta})$ ) we have

$$\iota(k \cdot \tilde{\varphi}) = \iota(k)\iota(\tilde{\varphi}) = k^{-1}u\tilde{\varphi} = k^{-2}u \cdot (k \cdot \tilde{\varphi})$$

so all that is required is to produce a section  $k$  such that  $k^2 = u$ . If  $[2]$  is an automorphism of  $K(\underline{\delta})$ , as is the case when  $d_m$  is odd, then this is immediate. The case where  $d_m$  is even will be handled in the appendix. A direct proof more along the lines of [57, §5] may be found in [11, Lemma 2.5.3].  $\square$

In particular, if  $d_m$  is odd, then the restriction map from  $\underline{Aut}^{Sym}$  to  $\underline{Sp}(K(\underline{\delta}))$  is an isomorphism of finite flat  $S$ -group schemes.

## B. THE EVEN DISCRIMINANT CASE

Especially in step (III), we have leaned on the assumption that  $\mathbf{d}$  is odd to streamline our proofs. In this appendix, we explain how to modify our arguments to show that our main theorem, [Theorem V.1](#), is still true as stated even when  $\mathbf{d}$  is even.

**B.1. Weil representation for even  $\mathbf{d}$ .** The first issue is that  $\mathrm{SL}_2(\mathbf{Z}/\mathbf{d})$  does not act on the Heisenberg group; it must be modified: Consider the subgroup  $\Xi$  of  $\mathrm{SL}_2(\mathbf{Z}/2\mathbf{d})$  that preserves the quadratic form  $(x_1, x_2) \mapsto x_1x_2$ , explicitly:

$$(B.1) \quad \Xi = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}/2\mathbf{d}) : ac = bd \equiv 0 \pmod{2} \right\}$$

Equivalently,  $\Xi$  is the Galois group of  $Y(2\mathbf{d}) \rightarrow Y_0(2)$ .

**Lemma B.1.** *The action of  $\Xi$  on  $\kappa_B \times \kappa_B$  lifts to a symmetric action on  $H_B$ , that is, there is a section*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \kappa_B[2] \times \kappa_B[2] & \longrightarrow & \mathrm{Aut}_{\mathbf{G}_m}^{Sym}(H_B) & \longrightarrow & \mathrm{Sp}(\kappa_B \times \kappa_B) \longrightarrow 1 \\ & & & & \nwarrow & & \uparrow \\ & & & & & & \Xi \end{array}$$

*Proof.* Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Xi$ . Since  $ac$  is even, we can make sense of  $\frac{ac}{2}$  modulo  $\mathbf{d}$ . Then the rule

$$(B.2) \quad \lambda_\gamma : x \in \kappa_B \mapsto \exp \left( 2\pi i \left( \frac{ac}{2} \text{ in } \mathbf{Z}/\mathbf{d} \right) \langle x, x \rangle \right)$$

is engineered to enjoy the property

$$\frac{\lambda_\gamma(x_1 + x_2)}{\lambda_\gamma(x_1)\lambda_\gamma(x_2)} = \exp(2\pi i \langle ax_1, cx_2 \rangle)$$

and therefore gives a lift of the homomorphism  $\gamma : \kappa_B^{(1)} \rightarrow \kappa_B^{(1)} \times \kappa_B^{(2)}$  (the one given by  $x \mapsto (ax, cx)$ , i.e. the restriction of the natural action of  $\gamma$ ) to a homomorphism  $\kappa_B \rightarrow H_B$ . The same formula applies to lifting the restriction  $\gamma : \kappa_B^{(2)} \rightarrow \kappa_B^{(1)} \times \kappa_B^{(2)}$  to have target  $H_B$ , except with  $(a, c)$  replaced with  $(b, d)$ .

So from  $\gamma \in \Xi$  we obtain two new copies of  $\kappa_B$  inside  $H_B$  still satisfying the commutation relation III<sub>0</sub>.1 and therefore this extends to an isomorphism  $H_B \xrightarrow{\gamma} H_B$ .  $\square$

Because  $\mathbf{d}$  is now even, the group  $\kappa_B[2]$  is nontrivial, and the action from Lemma B.1 may not be unique. However, the *proof* of Lemma B.1 does specify a well-defined such action, which we take to be the definition of “the” action of  $\Xi$  or any subgroup thereof on  $H_B$  from now on.

$\Xi$  has a subgroup which corresponds to the reduction modulo  $\mathbf{d}$  of the level structure defining  $Y^*$ , namely

$$\mathrm{SL}_2(\mathbf{Z}/2\mathbf{d})^* = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}/2\mathbf{d}) : a \equiv d \equiv 1, b \equiv 2c \equiv 0 \text{ modulo } 4 \right\}$$

That is to say;  $\mathrm{SL}_2(\mathbf{Z}/2\mathbf{d})^*$  is the Galois group of  $Y(2\mathbf{d}) \rightarrow Y^*$ .

Lemma III<sub>a</sub>.3 fails for  $\mathbf{d}$  even; we need to increase the level to trivialize the group scheme. *However it continues to be valid for if we replace  $\mathbf{d}$  by  $\Delta = 2\mathbf{d}$ :*

**Lemma B.2.** *There exists a symmetric theta structure over  $Y(2\mathbf{d})$ :*

$$H_B := \mathbb{G}_m \times \kappa_B \times \kappa_B \xrightarrow{\sim} \mathcal{G}(\mathcal{L})$$

compatible with the actions of  $\mathrm{SL}_2(\mathbf{Z}/\Delta)^*$ .

*Proof.* A symmetric theta structure exists étale locally, by the proof of [44, §2, Remark 2] (which in fact we could have invoked in the proof of Proposition III<sub>a</sub>.3 — it is the same procedure of taking the necessary square roots étale locally). This uses the fact that  $e_{\mathcal{L}}^*|_{K(\mathcal{L})[2]}$  is trivial, itself a consequence of the calculation of the pairing  $e_{\mathcal{L}}^*$  (the obvious algebraic analog of Lemma I.3).

But unlike the situation of step (III) in the main body of this paper, it is no longer true that symmetric theta structures are unique, due to the nontrivial kernel in the exact sequence of Lemma A.9 — so more care is required to do the gluing. We accomplish this by technique of normalization, where the extra level structure at 2 is essential.

Fixing generators  $g_1, \dots, g_t$  for the cyclic components of  $K(\mathcal{L})$ , the odd order ones have canonical global square roots in  $K(\mathcal{L})$  already without the level structure. The even-order

ones are responsible for the nontriviality of  $\kappa_B[2]$ , and thanks to the level  $2\mathbf{d}$  being used they admit a non-unique *global choice* of  $g_i/2$ , that is, sections  $g_i/2$  of  $\mathcal{E}^m$  over  $Y(2\mathbf{d})$  which are not in  $K(\mathcal{L})$  but square to  $g_i \in K(\mathcal{L})$ . Symmetry implies<sup>25</sup> that if  $\tilde{g}_i$  is the preimage of  $g_i$  in  $\kappa_B \times \kappa_B$ , then the image of  $(1, \tilde{g}_i)$  in  $\mathcal{G}(\mathcal{L})$  under the étale local choice of symmetric theta structure induces  $\pm\varphi : \mathcal{L}|_{-g_i/2} \rightarrow \mathcal{L}|_{g_i/2}$ , where  $\varphi$  is the global symmetric structure on  $\mathcal{L}$ . We normalize the étale local choice of symmetric theta structure by possibly modifying it by a sign so that the induced map  $\mathcal{L}|_{-g_i/2} \rightarrow \mathcal{L}|_{g_i/2}$  agrees exactly with the symmetric structure. Since  $g_i/2$  has been defined globally, this allows for the gluing together of the étale local symmetric theta structures to a global symmetric theta structure.

The  $\mathrm{SL}_2(\mathbf{Z}/2\mathbf{d})^*$ -equivariance statement follows from viewing the proof of [Lemma III<sub>a</sub>.5](#) in light of the above. The point is that a symmetric theta-structure is determined by the element of  $\mathrm{Hom}(\kappa_B \times \kappa_B, \mu_2)$  coming from looking at the signs  $\epsilon$  obtained from each generator of cyclic parts of  $\kappa_B \times \kappa_B$  as above. Since those signs are globally defined, they are unchanged by pullback under elements of  $\mathrm{SL}_2(\mathbf{Z}/2d\mathbf{Z})$ .  $\square$

**Lemma B.3.** *The conclusion of Lemma [III<sub>0</sub>.3](#) remains valid for general  $\mathbf{d}$ , replacing however the role of  $\mathrm{SL}_2(\mathbf{Z}/\mathbf{d})$  by  $\mathrm{SL}_2(\mathbf{Z}/2\mathbf{d})^*$ .*

The statement of the lemma is adapted to the “worst case;” our proof will show something a little more precise: If we write  $\mathbf{d} = 2^r \times \text{odd}$ , the proof actually will show that Lemma [III<sub>0</sub>.3](#) remains valid if both  $r$  and  $B$  are even. If  $r$  is odd and  $B$  is even we can replace  $\mathrm{SL}_2(\mathbf{Z}/\mathbf{d})$  with the image of  $\Gamma_0(2)$  in  $\mathrm{SL}_2(\mathbf{Z}/2d\mathbf{Z})$ ; if  $r$  is even and  $B$  is arbitrary, we can replace it with  $\Xi$ .

*Proof.* It is sufficient to prove the claim when  $\mathbf{d}$  is a power of 2. To make the reduction, we decompose  $\kappa_B$  into prime-to-2 and 2-parts, giving rise to a corresponding tensor decomposition of SVN, and describe the actions of the corresponding factors of  $\mathrm{SL}_2(\mathbf{Z}/\mathbf{d})$  on each part separately.

So assume  $\mathbf{d}$  is a power of 2. Just as in [Lemma III<sub>0</sub>.3](#), we get a projective action of  $\Xi$  on SVN. In that action, the matrix  $U^{-2}$  acts by<sup>26</sup> a constant multiple of  $\zeta^{\langle x, x \rangle}$ . Similarly for  $T^2$ ;

<sup>25</sup>The symmetry of the theta structure implies that the image  $a$  of  $(1, \tilde{g}_i)$  induces the symmetric structure  $\iota : \mathcal{L}|_{-g_i/2} \rightarrow \mathcal{L}|_{g_i/2}$  up to sign, because symmetry means that, viewed as isomorphisms  $\mathcal{L}|_{-g_i/2} \rightarrow \mathcal{L}|_{g_i/2}$ ,  $\iota \circ a \circ \iota^{-1} = a^{-1}$ . So if  $a = \epsilon\iota$ , then this implies  $\epsilon^2 = 1$ .

<sup>26</sup>This is exactly the same calculation that is implicit in the proof of [Lemma III<sub>0</sub>.3](#) and dates back to the beginnings of the subject [\[57\]](#): when we conjugate a translation by  $a$  by this operator we get

$$\Phi(x) \mapsto \Phi(x + a) \zeta_{\mathbf{d}}^{-\mathbf{d}\langle x, 2a \rangle - \mathbf{d}\langle a, a \rangle},$$

on the other hand the map  $\kappa_B^{(1)} \rightarrow H_B$  coming from  $U^{-2}$  via [Lemma B.1](#) sends  $a$  to  $(\exp(-2\pi i \langle a, a \rangle), a, -2a)$ . which acts on the Schrödinger model in exactly the same way.

whereas  $S$  acts by a multiple of Fourier transform, and  $\Xi$  is generated by these three<sup>27</sup>, or even the pairs  $T^2, S$  or  $U^2, S$ .

Take a subgroup  $M \subset \kappa_B$  that is maximal isotropic for the bilinear form, i.e.  $\langle m, m' \rangle = 0$  for all  $m, m' \in M$  and  $M$  is maximal subject to this. There is an induced perfect bilinear form  $\bar{B}$  on  $M^\perp/M$ , valued in  $\mathbf{Q}/\mathbf{Z}$ . Moreover, this new form satisfies  $\bar{B}(\mu, \mu) \neq 0$  for all  $\mu$ , otherwise we could enlarge  $M$ . It follows from this readily that either  $M^\perp/M$  is trivial or of order 2; these two possibilities are differentiated by the parity of  $r$  when we write  $d = 2^r \times \text{odd}$ .

We now consider the submodule that is pointwise fixed by  $M$  viewed in both  $\kappa_B^{(1)}$  and  $\kappa_B^{(2)}$ :

$$\mathbf{SVN}^M \subset \mathbf{SVN}.$$

The subset  $M \subset \kappa_B^{(2)}$  fixes only the vectors supported on  $M^\perp$ , and  $M \subset \kappa_B^{(1)}$  acts by translation, so:

- If  $M^\perp = M$ , then  $\mathbf{SVN}^M$  is of rank one over  $\mathbf{Z}'$ , generated by  $\sum_{x \in M} \mathbf{1}_x$ .
- If  $[M^\perp : M] = 2$ , then  $\mathbf{SVN}^M$  is free of rank 2 over  $\mathbf{Z}'$  with basis given by

$$(B.3) \quad \sum_{x \in M} \mathbf{1}_x, \quad \sum_{x \in M^\perp - M} \mathbf{1}_x$$

The first case is simpler: an inspection of the explicit forms of  $U^{-2}, S$  show that they preserve the line  $\sum_{x \in M} \mathbf{1}_x$ . Then we can lift the projective  $\Xi$ -action to a genuine action on  $\mathbf{SVN}$  by normalizing it to act trivially on the rank one submodule  $\mathbf{SVN}^M$ . In this action,  $U^{-2}$  acts by multiplication by the function  $\zeta_d^{\mathbf{d}\langle x, x \rangle}$ ,  $S$  acts by the normalized Fourier transform

$$\Phi \mapsto \frac{1}{|M|} \sum_{y \in \kappa} \Phi(y) \zeta_d^{\mathbf{d}\langle x, y \rangle}$$

and  $T^2$  acts by the Fourier conjugate of  $U^{-2}$ ; this implies the claim of the lemma – namely, the fixed space for  $\kappa_B^{(1)}$  is also fixed by the pointwise stabilizer of  $\kappa_B^{(1)}$  inside  $\Xi$  (namely, the subgroup generated by  $U^2$ ) and similarly for  $\kappa_B^{(2)}$ .

In the second case, when  $M^\perp/M$  has index 2 the situation is a little more complicated. In the basis (B.3), the elements  $T^2, U^{-2}, S$  act projectively as:

$$U^{-2} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S \mapsto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, T^2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In other words,  $\mathbf{SVN}^M$  is the Schrödinger model of the projective Stone–von Neumann representation we have already defined, coming from the 2-dimensional  $\mathbf{F}_2$ -vector space

<sup>27</sup>That  $\Xi$  is generated by  $T^2, S$  follows from the fact that  $\Gamma(2)$  is generated by those same matrices and the surjectivity of  $\mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathrm{SL}_2(\mathbf{Z}/2\mathbf{d})$ , as  $\Xi$  is equivalently the subgroup of matrices that are congruent to  $I$  or  $S$  mod 2

$M^\perp/M$  with the alternating form  $\exp(2\pi i\langle -, - \rangle)$ . In particular, it factors through maps

$$\Xi \rightarrow \Xi_4 \rightarrow \text{projective symmetries of an octagon},$$

where  $\Xi_4$  means the group  $\Xi$  in the case  $d = 2$  (so a subgroup of  $\text{SL}_2(\mathbf{Z}/4)$ ), and the last group being a dihedral group of order 8. The latter map does not lift to a genuine representation, but it does so lift if we restrict to the subgroup  $\langle U^2 \rangle$  of order 2; we can thereby produce a lift of the projective representation on  $\text{SL}_2(\mathbf{Z}/2d)^*$  by requiring that it agree with this specified lift, since the image of  $\text{SL}_2(\mathbf{Z}/2d)^*$  in  $\Xi_4$  is by definition  $\{I, U^{-2}\}$ . That is to say, for each  $\gamma \in \text{SL}_2(\mathbf{Z}/2d)^*$ , we declare  $\gamma$  to act on  $\text{SVN}$  in such a way that its action on  $\text{SVN}^M$  is trivial (if  $\gamma$  modulo 4 is the identity matrix) or given by the action of  $U^{-2}$  above (if  $\gamma$  modulo 4 is  $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ ).

In this choice of lift,  $U^{-2}$  acts by multiplication by  $\zeta_d^{d\langle x, x \rangle}$ , so  $T^4$  acts by the Fourier conjugate of  $\zeta^{2\langle x, x \rangle}$ , so again the conclusion of the lemma holds.  $\square$

Just as we consider the proof of [Lemma B.1](#) to be the *definition* of a certain action of  $\text{SL}_2(\mathbf{Z}/2d)^*$  on  $H_B$ , we consider the proof of [Lemma III<sub>0</sub>.3](#) to be the *definition* of a lift of the resulting projective Weil representation of  $\text{SL}_2(\mathbf{Z}/2d)^*$  to a genuine representation of  $\text{SL}_2(\mathbf{Z}/2d)^*$  on  $\text{SVN}$ .

Since  $\text{SL}_2(\mathbf{Z}/2d)^*$  is identified with the Galois group of  $Y(2d) \rightarrow Y^*$ , just as in the  $d$  odd version of step (III), we have the following data:

- The group scheme  $\mathcal{G}(\mathcal{L})$  pulled back from  $Y^*$  to  $Y(2d)$ , which admits an action of  $\text{SL}_2(\mathbf{Z}/2d)^*$  by construction.
- The group scheme  $H_B$  viewed over  $Y(2d)$ , which admits the action of  $\text{SL}_2(\mathbf{Z}/2d)^*$  constructed in [Lemma B.1](#).
- The  $\text{SL}_2(\mathbf{Z}/2d)^*$ -equivariant theta structure from [Lemma B.2](#) together with the action of  $\text{SL}_2(\mathbf{Z}/2d)^*$  on  $\text{SVN}$  from [Lemma B.3](#) allows us to descend the line bundle  $\text{Hom}_{H_B}(\text{SVN}, \Theta_{\text{norm}})$  on  $Y(2d)$  to  $\Theta_{\text{line}}$  on  $Y^*$ .

The construction of our theta series near a cusp is achieved by descent of specific sections to  $\Theta_{\text{line}}$ , which can be done in exactly the same way as in step (III) for  $d$  odd now that we have established these preliminaries. This time, we use [Lemma B.3](#) as indicated in the remark at the end of [Section III<sub>a</sub>](#) to make the calculation cleaner.

For every cusp of  $Y^*$  lying over the cusp 0 of  $Y_0(2)$ , we may construct a section by descent from exactly the same cocycle described in [Equation \(III<sub>a</sub>.1\)](#), since the formula for a symmetric theta structure in this case is unchanged from the odd case. As a warm up, we implement the remark from the end of [Section III<sub>a</sub>](#) in that case to show how much it simplifies the situation.

We simply define the cocycle  $\vartheta_0$ , which takes a tuple  $\underline{v}$  to the infinite sum

$$\sum_{n \in \mathbf{Z}^m \cap \mathcal{C}} q^{Q(n)} \underline{z}^{Bn}.$$

We verify that this is fixed by  $\kappa_B^{(2)}$ : plugging in  $e(y)z$  for  $z$  does nothing, for  $y \in \kappa_B$ . Therefore, the data of  $\vartheta_0$  already defines a section over  $\mathrm{Spec} \mathbf{Z}'((q^{1/(w\mathrm{d}})))$  of  $\mathrm{Hom}_{H_B}(\mathrm{SVN}, \Theta_{\mathrm{norm}})$ . To descend it under  $U^{-w}$  (the change of coordinates  $q^{1/(w\mathrm{d})} \mapsto \zeta_{\mathrm{d}} q^{1/(w\mathrm{d})}$ ), because of [Lemma B.3](#), we need only verify that  $\vartheta_0$  is fixed by that change of coordinates, which is manifestly true.

Now for a suitable cusp of  $Y(2\mathrm{d})$  over  $\infty$ , we define  $\vartheta_0$  to take a tuple  $\underline{v}$  with corresponding region  $\mathcal{C}$  to

$$\sum_n q^{Q(n) - \frac{1}{2}\langle c, n \rangle} \underline{z}^{Bn + \frac{1}{2}(b - Bc)}$$

where  $c \in \mathbf{Z}^m$  is such that  $Bc \equiv b \pmod{2}$ . Here the extra  $\frac{1}{2}(b - Bc)$  in the exponent of  $\underline{z}$  is there to make  $\vartheta_0$  fixed by  $\kappa_B^{(2)}$  according to [\[34, \(8.3\)\]](#), and the extra  $\frac{1}{2}\langle c, n \rangle$  instead of  $-\frac{1}{2}b^\top n$  in the exponent of  $q$  is there to ensure that  $\vartheta_0$  is still a cocycle after making the first modification. The change of variables  $q^{1/(2\mathrm{d})} \mapsto \zeta_{\mathrm{d}} q^{1/(2\mathrm{d})}$  does not affect  $\vartheta_0$ , so again one obtains a genuine section of  $\vartheta$  near the cusp  $\infty$  of  $Y^*$ .

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