Counting Integral Points on Thin Sets of Type II:

SINGULARITIES
Sieves & STRATIFICATION

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Geometric and Analytic Number Theory

Thin Sets: Let $S \subset \mathbb{P}^n(\mathbb{Q})$

- S is a thin set of type I if there exists a subvariety $V \subset \mathbb{P}^n(\mathbb{Q})$ s.t. $S \subset V$
- S is a thin set of type II if there exists a variety Z of dimension n and a dominant morphism
$$\pi : Z \to \mathbb{P}^n, \quad \deg(\pi) \geq 2$$
such that $S \subset \pi(Z)$

A (projective) thin set is a finite union of thin sets of type I + II

The affine setting: Let $S \subset \mathbb{A}^n$

- S is a thin set of type I if there exists a subvariety $V \subset \mathbb{A}^n$ s.t. $S \subset V$
- S is a thin set of type II if there exists a variety Z of dimension $n-1$ and dominant morphism $\pi : Z \to \mathbb{A}^n$, $\deg(\pi) \geq 2$ s.t. $S \subset \pi(Z)$.

An (affine) thin set is a finite union of thin sets of type I + II.

The affine setting:

Let $F(Y, X_1, \ldots, X_n) \in \mathbb{Z}[Y, X_1, \ldots, X_n]$ be absolutely irreducible and $\deg_Y F \geq 2$

$\{\vec{x} \in \mathbb{Z}^n : \exists y \in \mathbb{Z},\ F(y, \vec{x}) = 0\}$ is an affine thin set of type II

Example: let $G_0 \subsetneq S_d$

$$\left\{ \vec{a} \in \mathbb{Z}^d : \begin{array}{l} x^d + a_1 x^{d-1} + \dots + a_{d-1} x + a_d \\ \text{has Galois group } G \subset G_0 \end{array} \right\}$$

is an affine thin set of type II

if $G_0 = A_d \Rightarrow$

$$\left\{ \vec{a} \in \mathbb{Z}^d : \exists y \in \mathbb{Z},\ y^2 - \operatorname{disc}(a_1, \dots, a_d) = 0 \right\}$$

Serre's Question:

Let $S \subset \mathbb{P}^{n-1}(\mathbb{Q})$ be a (projective) thin set (of type II)

$$N(B) = \#\{x \in S(\mathbb{Q}) : h(x) \leq B\}$$

Is it true that $N(B) \ll_S B^{n-1}(\log B)^c$?

Dimension Growth: $S \subset \mathbb{P}^{n-1}(\mathbb{Q})$ projective subvariety of degree ≥ 2,

$$N(B) \ll_{S,\varepsilon} B^{n-2+\varepsilon}$$

(Selected) History:

- $N(B) \ll_{s,\varepsilon} B^{n-1/2+\varepsilon}$ (Serre '97 + Cohen* '81)
- $n \geq 10$, $F(y,\vec{x}) = y^d - f(\vec{x})$ for f nonsingular (Heath-Brown, Pierce '12)
- $N(B) \ll_{s,\varepsilon} B^{n-1+\frac{1}{n+1}+\varepsilon}$
  - $\rightarrow$ $F(y,\vec{x}) = g(y) - f(\vec{x})$ (Bonolis '21)
  - $\rightarrow$ $F(y,\vec{x}) = y^{dm} + f_{d-1}(\vec{x})\, y^{(d-1)m} + \dots + f_0(\vec{x})$ (Bonolis–Pierce '24)

  } $F(y,\vec{x})$ nonsingular

  f_i homogeneous of degree $m \cdot e \cdot i$
- $\deg(\pi) \geq 4$ (Buggenhout, Cluckers, Santens, Vermeulen '25)

THE AFFINE SETTING:

Let $F(Y, \vec{X})$ be absolutely irreducible and $\deg_Y F \geq 2$

$$N(F,B) = \#\{|\vec{x}| \leq B : \vec{x} \in \mathbb{Z}^n, \exists y \in \mathbb{Z}, F(y,\vec{x}) = 0\}$$

Is it true that $N(F,B) \ll_F B^{n-1} (\log B)^c$?

No! $F(y,\vec{x}) = y^2 - (x_1 + \dots + x_n)$

$N(F,B) \asymp B^{n-1/2}$

$\rightsquigarrow$ when is it true?

(Selected) History:

- $N(F,B) \ll_F B^{n-1/2}(\log B)^c$ (Serre '97 and Cohen* '81)

- $n \geq 10$, $F(y,\vec{x}) = y^d - f(\vec{x})$, f nonsingular
  (Heath-Brown, Pierce '12)

- $N(B) \ll_{S,\varepsilon} B^{n-1+\frac{1}{n+1}+\varepsilon}$

  $\rightarrow F(y,\vec{x}) = g(y) - f(\vec{x})$ (Bonolis '21)

  $\rightarrow F(y,\vec{x}) = y^{dm} + f_{d-1}(\vec{x})\, y^{(d-1)m} + \dots + f_0(\vec{x})$ } $F(y,\vec{x})$ nonsingular

  f_i homogeneous of degree $m \cdot e \cdot i$ (Bonolis-Pierce '24)

weighted projective space

- $\deg(\pi) \geq 4$ (Buggenhout, Cluckers, Santens, Vermeulen '25)

  $F(y,\vec{x}) = y^d + y^{d-1} f_1(\vec{x}) + \dots + f_0(\vec{x})$ + [lower degree component]

  f_i homogeneous of degree $e \cdot i$

TALK OUTLINE

THE CYCLIC CASE!

H($\vec{x}$) is allowed to be singular and has no homogeneity condition

Thm (Bonolis–Pierce–W. '25)
Let $d \geq 2$, $n \geq 2$.
$H(\vec{x}) \in \mathbb{Z}[\vec{x}]$ be absolutely irreducible in n variables
Assume there does not exist $L \in GL_n(\mathbb{Q})$ such that $H(L(\vec{x}))$ is a polynomial in $<n$ variables

Let $F(Y, \vec{x}) = Y^d - H(\vec{x})$. Then $\forall \varepsilon > 0$,

$$N(F, B) \ll_{n,d,\varepsilon} \log(\|F\| + 2)^{e(n)} B^{n-1+\frac{1}{n+1}+\varepsilon}$$

POLYNOMIAL SIEVE I

① Observe that for any prime p,

$$\mathbb{1}_{\exists y \in \mathbb{Z} : F(y,\vec{x})=0} \leq \mathbb{1}_{\exists y \in \mathbb{F}_p : F(y,\vec{x})=0 \in \mathbb{F}_p}$$

Moreover, if $\zeta_d \in \mathbb{F}_p$ and $y^d - H(\vec{x}) = 0 \in \mathbb{F}_p$

then $(\zeta_d y)^d - H(\vec{x}) = 0 \in \mathbb{F}_p$

Thus, if $p \equiv 1 \bmod d$,

$$\mathbb{1}_{\exists y \in \mathbb{Z} : F(y,\vec{x})=0} \leq \#\{y \in \mathbb{F}_p : F(y,\vec{x})=0\} - 1 = \sum_{\substack{\chi \neq 1 \\ \operatorname{ord}(\chi)=d}} \chi(H(\vec{x}))$$

② $$N(F,B) = \sum_{|\vec{x}| \le B} \mathbb{1}_{\exists y \in \mathbb{Z}:\, F(y,\vec{x})=0}$$

$\mathcal{P} =$ $\left\{ \begin{array}{l} \text{primes } p \equiv 1 \bmod d \\ \text{and } p \in [P, 2P] \end{array} \right\}$

$$\le \sum_{|\vec{x}| \le B} \left(\frac{1}{|\mathcal{P}|^2} \left(\sum_{p \in \mathcal{P}} \sum_{\substack{\chi \ne 1 \\ \mathrm{ord}(\chi) = d}} \chi(H(\vec{x})) \right)^2 \right)$$

Heuristic: if $\exists y \in \mathbb{Z},\ F(y,\vec{x}) = 0$ then

$$\frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \#\{y \in \mathbb{F}_p : F(y,\vec{x}) = 0\} - 1 \gg 1$$

if $\nexists\, y \in \mathbb{Z},\ F(y,\vec{x}) = 0$ then under GRH

$$\frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \#\{y \in \mathbb{F}_p : F(y,\vec{x}) = 0\} - 1 \ll_\varepsilon P^{-1/2 + \varepsilon}$$

(3)

$$N(F,B) \le \frac{B^n}{|P|} + \frac{1}{|P|^2} \sum_{p \neq q \in P} \sum_{\substack{\chi_p \neq 1 \\ \chi_q \neq 1 \\ \text{order}=d}} \sum_{|\vec{x}| \le B} \chi_p(H(\vec{x}))\, \chi_q(H(\vec{x}))$$

POISSON SUMMATION

$$\sum_{|\vec{x}| \le B} \chi_p(H(\vec{x}))\, \chi_q(H(\vec{x})) \ll \frac{B^n}{(pq)^n} \sum_{\substack{\vec{r} \in \mathbb{Z}^n \\ |\vec{r}| \ll pq/B}} |S(p, \bar{q}\vec{r})|\, |S(q, \bar{p}\vec{r})|$$

where

$$S(p, \vec{r}) = \sum_{\vec{x} \in \mathbb{F}_p^n} \chi(H(\vec{x}))\, e\left(\frac{\vec{r} \cdot \vec{x}}{p}\right)$$

THE EXPONENTIAL SUM

$$\boxed{S(p,\vec{\gamma}) = \sum_{\vec{x}\in\mathbb{F}_p^n} \chi(H(\vec{x}))\, e\left(\frac{\vec{\gamma}\cdot\vec{x}}{p}\right)}$$

How big should $S(p,\vec{\gamma})$ be?

Trivial bound: $|S(p,\vec{\gamma})| \leq p^n$

Squareroot cancellation: $|S(p,\vec{\gamma})| \ll p^{n/2}$ choose $P = B^{n/n+1}$

(heuristic)

$$\Rightarrow N(F,B) \ll \frac{B^n}{|P|} + \frac{1}{|P|^2} \sum_{p\neq q} \sum_{\substack{\chi_p\neq 1\\ \chi_q\neq 1}} (pq)^{n/2} \ll_\varepsilon \frac{B^n}{|P|} + P^{n+\varepsilon} \ll_\varepsilon B^{n-1+\frac{1}{n+1}+\varepsilon}$$

STRATIFICATION PART I

<u>Thm</u> (Fouvry-Katz '01)

There exist constants $C, N > 0$ and homogeneous varieties

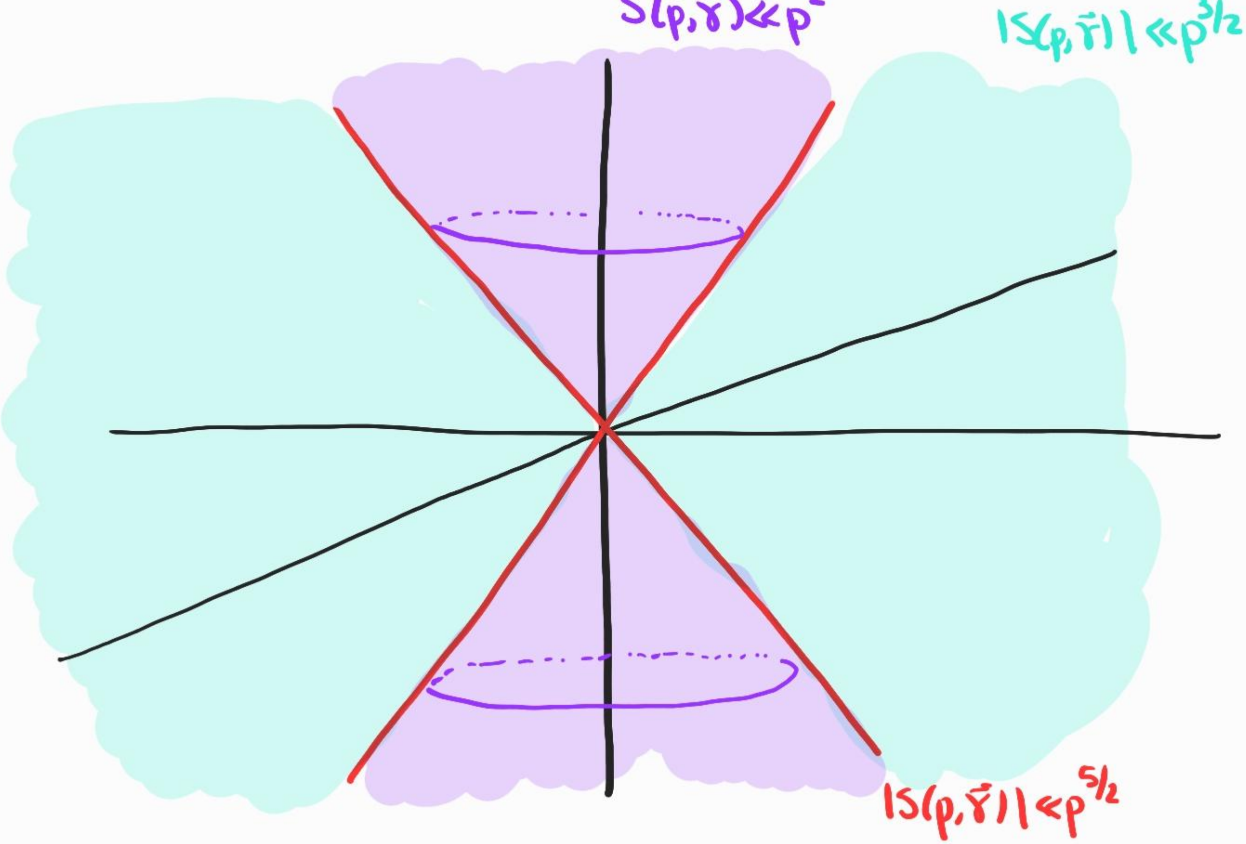
$$\mathbb{A}^n = X_0 \supset X_1 \supset \dots \supset X_n$$ ← "the stratification"

such that $\mathrm{codim}(X_j) \geq j$

and $\forall p \nmid N$ prime and $\vec{\gamma} \in (\mathbb{A}^n - X_j)(\mathbb{F}_p)$

$$|S(p, \vec{\gamma})| \leq C\, p^{\frac{n}{2} + \frac{j-1}{2}}$$

VISUALIZATION



STRATIFICATION PART I

$$S(p,\vec{\gamma}) = \sum_{\vec{x}\in\mathbb{F}_p^n} \chi(H(\vec{x}))\, e\left(\frac{\vec{\gamma}\cdot\vec{x}}{p}\right)$$

$$\rightsquigarrow$$

stratification $\mathcal{X} = \{X_j\}$ codim$(X_j) \geq j$

constants $C, N > 0$

independent of $\|H(\vec{x})\|_\infty$ and multiplicative character χ

CYCLIC CASE

$$N(F,B) \ll \frac{B^n}{|P|} + \frac{1}{|P|^2} \sum_{p \neq q} \sum_{\substack{\chi_p \neq 1 \\ \chi_q \neq 1}} \frac{B^n}{(pq)^n} \sum_{|\vec{\gamma}| \ll pq/B} |S(p,\vec{\gamma})|\,|S(q,\vec{\gamma})|$$

split up sum by the strata

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} p^{\frac{n}{2}+\frac{j}{2}} q^{\frac{n}{2}+\frac{k}{2}} \#\left\{ |\vec{\gamma}| \ll pq/B : \begin{array}{l} \vec{\gamma} \in X_j(\mathbb{F}_p) \\ \vec{\gamma} \in X_k(\mathbb{F}_q) \end{array} \right\}$$

$$N(F,B) \ll \frac{B^n}{|P|} + \frac{1}{|P|^2} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} p^{n+\frac{j+k}{2}} \sum_{|\vec{\gamma}| \ll P^2/B} \#\left\{ p \neq q \in P : \begin{array}{l} \vec{\gamma} \in X_j(\mathbb{F}_p) \\ \vec{\gamma} \in X_k(\mathbb{F}_q) \end{array} \right\}$$

$$\frac{1}{|P|^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} p^{n+\frac{j+k}{2}} \sum_{|\vec{\gamma}| \ll p^2/B} \#\left\{p \neq q \in P : \begin{array}{l}\vec{\gamma} \in X_j(\mathbb{F}_p) \\ \vec{\gamma} \in X_k(\mathbb{F}_q)\end{array}\right\}$$

① $\vec{\gamma} \in \mathbb{Z}^n$ is globally j-bad if $\vec{\gamma} \in X_j(\mathbb{Z}) \leftarrow$ $\forall p \in P$, $\vec{\gamma} \in X_j(\mathbb{F}_p)$

② $\vec{\gamma} \in \mathbb{Z}^n$ is locally j-bad if $\vec{\gamma} \notin X_j(\mathbb{Z})$ but $\underbrace{\vec{\gamma} \in X_j(\mathbb{F}_p) \text{ for some } p}$

since $\vec{\gamma} \notin X_j(\mathbb{Z})$, only finitely many primes p satisfy that $\vec{\gamma} \in X_j(\mathbb{F}_p)$

$\Rightarrow$ extra savings of $|P|^{-1}$ ✓

globally j-bad points:

could apply resolution of dimension growth

$$\#\left\{ |\vec{\gamma}| \ll P^2/B : \vec{\gamma} \in X_j(\mathbb{Z}) \right\} \ll_{n, \deg X_j, \varepsilon} \left(\frac{P^2}{B}\right)^{n-j-1+\frac{1}{\deg X_j}+\varepsilon}$$

<u>Fact</u>: If $V \subset \mathbb{P}^n$ is irreducible nondegenerate of degree e and dimension m, then
$$e > N - m$$

If X_j is not a plane, this is sufficient savings.

<u>But</u> X_1 can be a plane!

When X_j is a plane:

<u>Lemma</u> (second moment bound)

Let V be a plane defined over $\mathbb{Z}$.

Then $\forall p \notin \mathcal{E}$, (with $|\mathcal{E}| \ll_{n,\deg H} \log\|H\|\|V\| / \log\log\|H\|\|V\|$)

$$\sum_{\vec{\gamma} \in V(\mathbb{F}_p)} |S(p, \vec{\gamma})|^2 \ll_{n,\deg H} p^{2n-1}$$

$\Rightarrow$ the case when X_j is a plane and $\vec{\gamma}$ is globally j-bad is bounded by $B^{n+\varepsilon}/p$

Sketch of the second moment band:

If $V = \{ \gamma_1 = 0 \}$

$$\sum_{\vec{\gamma} \in V(\mathbb{F}_p)} |S(p, \vec{\gamma})|^2 = \sum_{\vec{\gamma} \in V(\mathbb{F}_p)} \sum_{\vec{x}, \vec{y} \in \mathbb{F}_p} \chi(H(\vec{x})) \overline{\chi(H(\vec{y}))}\, e\left(\frac{\vec{\gamma} \cdot (\vec{x} - \vec{y})}{p} \right)$$

$$= \sum_{\vec{x}, \vec{y} \in \mathbb{F}_p^n} \chi(H(\vec{x})) \overline{\chi(H(\vec{y}))} \sum_{\vec{\gamma} \in V(\mathbb{F}_p)} e\left(\frac{\vec{\gamma} \cdot (\vec{x} - \vec{y})}{p} \right)$$

$$= p^{n-1} \sum_{\vec{z} \in \mathbb{F}_p^{n-1}} \sum_{x, y \in \mathbb{F}_p} \chi(H(x, \vec{z})) \overline{\chi(H(y, \vec{z}))}$$

$$= p^{n-1} \sum_{\vec{z} \in \mathbb{F}_p^{n-1}} \left| \sum_{x \in \mathbb{F}_p} \chi(H(x, \vec{z})) \right|^2$$

$$p^{n-1} \sum_{\vec{z} \in \mathbb{F}_p^{n-1}} \left| \sum_{x \in \mathbb{F}_p} \chi(H(x,\vec{z})) \right|^2$$

Since χ is cyclic character of order d

$$\sum_{x \in \mathbb{F}_p} \chi(H(x,\vec{z})) = \begin{cases} p & \text{if } H(x,\vec{z}) \text{ is a } d^{th} \text{ power} \\ O_{d,\deg H}(p^{1/2}) & \text{otherwise} \end{cases}$$

$$\sum_{\vec{\gamma} \in V(\mathbb{F}_p)} |S(p,\vec{\gamma})|^2 \ll p^{n-1} \cdot (p^{n-2} \cdot p^2 + p^{n-1} \cdot p) \ll p^{2n-1}$$

When $V = \{\gamma_1 = 0\}$ AND $\deg_{x_1} H \geq 1$

generally need a linear transform $L \in GL_n(\mathbb{Q})$

CYCLIC CASE

$$F(Y,\vec{X}) = Y^d - H(\vec{X})$$

- absolutely irred
- $H(L(\vec{x}))$ has n variables $\forall L \in GL_n$

① Polynomial sieve + Poisson $\Rightarrow$

$$\frac{1}{|P|^2} \sum_{p \neq q \in P} \frac{B^n}{(pq)^n} \sum_{|\vec{\gamma}| \ll p^2/B} |S(p,\vec{\gamma})|\,|S(q,\vec{\gamma})|$$

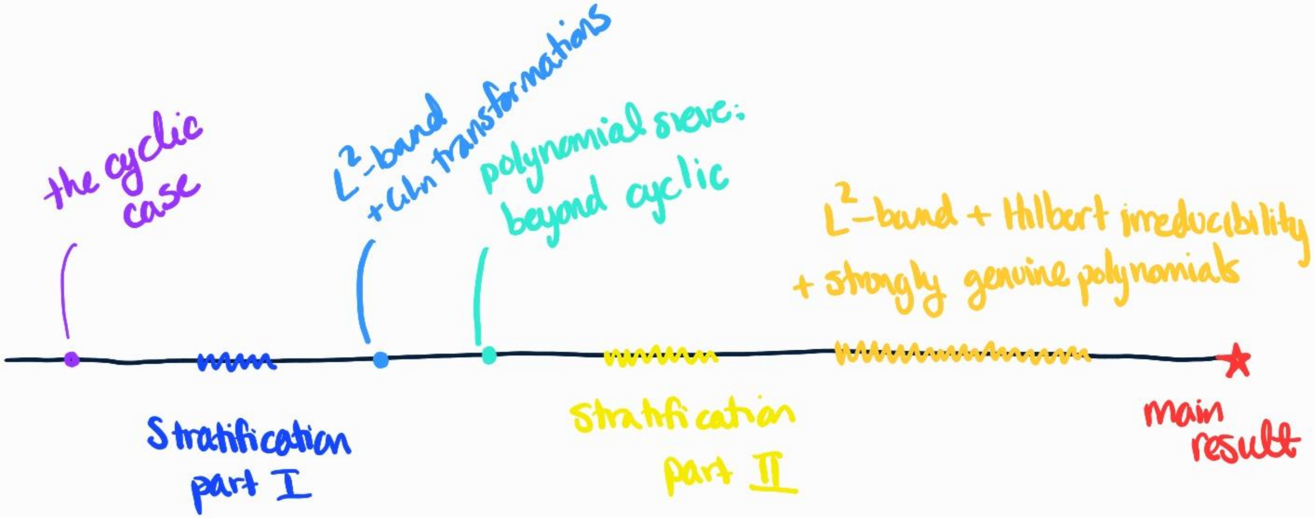
② Stratification of the exponential sum $\quad \mathcal{X} = \{X_j\}$ with $\operatorname{codim} X_j \geq j$

$$|S(p,\vec{\gamma})| \ll p^{\frac{n}{2} + \frac{(j-1)}{2}} \quad \text{for } \vec{\gamma} \in (\mathbb{A}^n - X_j)(\mathbb{F}_p)$$

③ Sum over $\vec{\gamma}$:

- $\vec{\gamma}$ is locally j-bad : savings from only finitely many p will satisfy that $\vec{\gamma} \in X_j(\mathbb{F}_p)$
- $\vec{\gamma}$ is globally j-bad and $\deg X_j > 1$: Savings from determinant method / dimension growth
- $\vec{\gamma}$ is globally j-bad and X_j is a plane : SECOND MOMENT BOUND
  $\rightarrow$ need that $H(L(\vec{x}))$ does not lose variables

TALK OUTLINE



Let $F(Y, \vec{X})$ be an absolutely irreducible polynomial

PROOF OUTLINE

① Polynomial siéve + Poisson band:

② Stratification of the exponential sum:

③ Sum over $\vec{\gamma}$:

- locally j-bad:
- globally j-bad + $\deg(X_j) > 1$:
- globally j-bad + X_j is a plane:

Modifications to the polynomial sieve (Bonolis–Pierce '24)

Recall $\mathbb{1}_{\exists y: F(y,\vec{x})=0} \leq \mathbb{1}_{\exists y\in\mathbb{F}_p : F(y,\vec{x})=0\in\mathbb{F}_p}$

(cyclic + $p\equiv 1 \bmod d$) $\leq \#\{y\in\mathbb{F}_p : F(y,\vec{x})=0\in\mathbb{F}_p\}-1$

Suffices for $F(y,\vec{x})$ to be a polynomial in y^d

[Under GRH] for any $d\geq 1$ and as $\vec{x}$ varies in $[-B,B]^n$

$$\sum_{p\in[P,2P]} \#\{y\in\mathbb{F}_p : F(y,\vec{x})=0\}-1 = \begin{cases} \gg P/\log P, & \exists y\in\mathbb{Z},\ F(y,\vec{x})=0 \\ \\ \ll_\varepsilon P^{1/2+\varepsilon}, & \nexists y\in\mathbb{Z},\ F(y,\vec{x})=0 \end{cases}$$

Thus, (potentially under GRH) $\mathcal{P} = \{\text{primes in } [P, 2P]\}$

$$N(F,B) \leq \frac{1}{|\mathcal{P}|^2} \sum_{|\vec{x}| \leq B} \Big(\sum_{p \in \mathcal{P}} \nu_p(\vec{x}) - 1 \Big)^2$$

$\nu_p(\vec{x}) = \#\{y \in \mathbb{F}_p : F(y, \vec{x}) = 0\}$

$$\leq \frac{B^n}{|\mathcal{P}|} + \frac{1}{|\mathcal{P}|^2} \sum_{p \neq q \in \mathcal{P}} \sum_{|\vec{x}| \leq B} (\nu_p(\vec{x}) - 1)(\nu_q(\vec{x}) - 1)$$

(Poisson)

$$\ll \frac{B^n}{|\mathcal{P}|} + \frac{1}{|\mathcal{P}|^2} \sum_{p \neq q \in \mathcal{P}} \frac{B^n}{(pq)^n} \sum_{|\vec{\gamma}| \ll P^2/B} |S(p, \bar{q}\vec{\gamma})| \, |S(q, \bar{p}\vec{\gamma})|$$

where $$S(p, \vec{\gamma}) = \sum_{\vec{x} \in \mathbb{F}_p^n} (\nu_p(\vec{x}) - 1) \, e\Big(\frac{\vec{\gamma} \cdot \vec{x}}{p}\Big)$$

Let $F(Y, \vec{X})$ be an absolutely irreducible polynomial

PROOF OUTLINE

① Polynomial sieve + Poisson bound: (+GRH)

$$N(F,B) \ll \frac{B^n}{|P|} + \frac{1}{|P|^2} \sum_{p \neq q \in P} \frac{B^n}{(pq)^n} \sum_{1 \leq |\vec{\gamma}| \ll p^2/B} |S(p,\vec{\gamma})||S(q,\vec{\gamma})|$$

② Stratification of the exponential sum:

③ Sum over $\vec{\gamma}$:

- locally j-bad:
- globally j-bad + $\deg(X_j) > 1$:
- globally j-bad + X_j is a plane :

THE EXPONENTIAL SUM

$$S(p,\vec{\gamma}) = \sum_{\vec{x}\in\mathbb{F}_p^n} (v_p(\vec{x})-1)\, e\left(\frac{\vec{\gamma}\cdot\vec{x}}{p}\right)$$

$$= \begin{cases} \#\{(y,\vec{x})\in\mathbb{F}_p^{n+1} : F(y,\vec{x})=0\} - p^n, & \vec{\gamma}=0\in\mathbb{F}_p^n \\ \displaystyle\sum_{F(y,\vec{x})=0} e\left(\frac{\vec{\gamma}\cdot\vec{x}}{p}\right), & \vec{\gamma}\neq 0\in\mathbb{F}_p^n \end{cases}$$

If $\vec{\gamma}=0\in\mathbb{F}_p^n$ and p suff large,

$$|S(p,\vec{\gamma})| \ll p^{n-1/2}$$

[$F(y,\vec{x})$ is absolutely irred]

STRATIFICATION PART II

<u>Thm</u> (Katz-Laumon '85) homogeneous (Fouvry-Katz '01)

There exists constants $C, N \geq 1$ and varieties

$$\mathbb{A}^n = X_0 \supset X_1 \supset \ldots \supset X_n \qquad \operatorname{codim}(X_j) \geq j$$

such that $\forall p \nmid N$ and $\vec{\gamma} \in (\mathbb{A}^n - X_j)(\mathbb{F}_p)$

$$|S(p, \vec{\gamma})| \leq C p^{\frac{n}{2} + \frac{(j-1)}{2}}$$

Q: how does $\mathcal{X} = \{X_j\}, C, N$ depend on the polynomial $F(y, \vec{x})$?

Thm (Bonolis, Kowalski, W. '25)

Let $S(p, \vec{\gamma})$ be a family of exponential sums defined as above.

Then there exists a universal constant $N_{n, \deg F} \geq 1$

and a stratification $\mathcal{X} = \{X_j\}$ with $\operatorname{codim}(X_j) \geq j$

for $S(p, \vec{\gamma})$ satisfying that

- $X_j = V(G_{ij})$ where $\{G_{ij}\}$ are homogeneous polynomials
- $\log \| G_{ij} \| \ll_{n, \deg F} \log \| F \|$

"analytic uniformity"

algebraic uniformity $\Rightarrow$ analytic uniformity

$$\mathcal{M}_{n,D} = \left\{ \left((A_{\vec{e}})_{|\vec{e}| \le D},\, Y,\, \vec{X} \right) : \sum_{|\vec{e}| \le D} A_{\vec{e}}\, Y^{e_0} X_1^{e_1} \cdots X_n^{e_n} = 0 \right\}$$

parameterizes polynomials in $n+1$ variables of degree $\le D$

Then there exist polynomials $\{H_{ij}\}$ depending on n, D such that outside a codim one set

if $F(y, \vec{x}) = \sum_{|\vec{e}| \le D} a_{\vec{e}}\, y^{e_0} x_1^{e_1} \cdots x_n^{e_n}$

then $X_j = V(G_{ij})$ where $G_{ij}(\vec{\gamma}) = H_{ij}(\vec{\gamma}, (a_{\vec{e}})_{\vec{e}})$

Let $F(Y, \vec{X})$ be an absolutely irreducible polynomial

PROOF OUTLINE

① Polynomial sieve + Poisson bound: (+GRH)

$$N(F,B) \ll \frac{B^n}{|P|} + \frac{1}{|P|^2} \sum_{p \neq q} \frac{B^n}{(pq)^n} \sum_{|\vec{\gamma}| \ll P^2/B} |S(p,\vec{\gamma})| \, |S(q,\vec{\gamma})|$$

② Stratification of the exponential sum: $\mathcal{X} = \{X_j\}$ with $\operatorname{codim}(X_j) \geq j$

$$|S(p,\vec{\gamma})| \ll p^{\frac{n}{2} + \frac{(j-1)}{2}} \quad \text{for } \vec{\gamma} \in (\mathbb{A}^n - X_j)(\mathbb{F}_p)$$

③ Sum over $\vec{\gamma}$:

- locally j-bad:
- globally j-bad + $\deg(X_j) > 1$:
- globally j-bad + X_j is a plane:

Let $F(Y, \vec{X})$ be an absolutely irreducible polynomial

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The new second moment bound:

Would suffice: for V a plane,

$$\sum_{\vec{\gamma} \in V(\mathbb{F}_p)} |S(p, \vec{\gamma})|^2 \ll_{n, \deg F} p^{2n-1}$$

Let's try it! $V = \{\gamma_1 = 0\}$

$$\sum_{\vec{\gamma} \in V(\mathbb{F}_p)} |S(p, \vec{\gamma})|^2 = \sum_{\vec{\gamma} \in V(\mathbb{F}_p)} \sum_{\vec{x}, \vec{x}'} (\nu_p(\vec{x}) - 1)(\nu_p(\vec{x}') - 1) e\left(\frac{\vec{\gamma} \cdot (\vec{x} - \vec{x}')}{p}\right)$$

$$= p^{n-1} \sum_{\vec{z} \in \mathbb{F}_p^{n-1}} \left(\#\{(y, x) \in \mathbb{F}_p^2 : F(y, x, \vec{z}) = 0\} - p \right)^2$$

Consider

$$\#\{(y,x)\in\mathbb{F}_p^2 : F(y,x,\vec{z})=0\} - p$$

Q: Is it true that for all but $O(p^{n-2})$ values of $\vec{z}\in\mathbb{F}_p^{n-1}$ satisfy that $F(y,x,\vec{z})$ is absolutely irreducible?

[claimed in Cohen '81 Lemma 4.2 (ii)]

Ex: $F(Y,X_1,X_2,X_3) = (Y-X_1)^2 - X_2^2 - X_3^2$

If $\underbrace{x_2^2+x_3^2 = \square \in \mathbb{F}_p}_{\text{occurs for } \gg p^2 \text{ tuples } (x_2,x_3)\,!} \Rightarrow \#\{(y,x)\in\mathbb{F}_p^2 : F(y,x_1,x_2,x_3)=0\} - p \geq p$

WHAT HAPPENED?

Noether's Lemma: there exists a polynomial G defined over $\mathbb{Z}$ in the coefficients of $F(\vec{z})$ that determines if $F(\vec{z})$ is absolutely irreducible + $G \bmod p$ determines if $F(\vec{z})$ is reducible over $\overline{\mathbb{F}}_p$

CLAIM
$$\left[\begin{array}{l} \text{If } F(Y,\vec{X}) = \sum a_{i_0 i_1}(X_2,\dots,X_n) Y^{i_0} X_1^{i_1} \\ \qquad\qquad \text{then } G\big((a_{i_0 i_1}(X_2,\dots,X_n))\big) \neq 0 \in \mathbb{Z}[X_2,\dots,X_n] \end{array}\right]$$

for $F(Y,\vec{X}) = (Y-X_1)^2 - X_2^2 - X_3^2$

$G(1,-2,1,0,0,X_2^2+X_3^2) = 0$

$$\boxed{\text{Lemma 4.2 (ii)}}$$

Def: An extension $M/\mathbb{Q}(\vec{x})$ is n-genuine if for every $G(Y,\vec{x})$ s.t.

$$M = \mathbb{Q}(\vec{x})[Y]/(G(Y,\vec{x}))$$

then $\deg_Y G \geq 1$, $\deg_{x_i} G \geq 1$ $\forall i \in \{1,\dots,n\}$

Def: An extension $M/\mathbb{Q}(\vec{x})$ is strongly n-genuine if $\forall$ subextensions $\mathbb{Q}(\vec{x}) \subsetneq M' \subset M$,

M' is a n-genuine extension.

If $F(Y,\vec{x})$ generates a strongly n-genuine extension then $G(a_{i_0,i_1}(x_2,\dots,x_n)) \neq 0 \in \mathbb{Z}[x_2,\dots,x_n]$

NEW SECOND MOMENT BOUND

<u>Lemma</u>: Let $V = \{\gamma_1 = 0\}$ defined over $\mathbb{Z}$.

If $F(Y, \vec{X})$ is a strongly n-genuine polynomial then

$\forall$ primes $p \notin \mathcal{E}$, (with $|\mathcal{E}| \ll \log\|F\| / \log\log\|F\|$)

$$\sum_{\vec{\gamma} \in V(\mathbb{F}_p)} |S(p, \vec{\gamma})|^2 \ll_{n, \deg F} p^{2n-1}$$

EXAMPLES

① $Y^2 - X_1 - X_2$ is strongly 2-genuine

$\mathbb{Q}(X_1, X_2, \sqrt{X_1 + X_2})$ is a 2-genuine extension

② $(Y - X_1)^2 - X_2^2 - X_3^2$ is not 3-genuine

$\mathbb{Q}(X_1, X_2, X_3, \sqrt{X_2^2 + X_3^2})$ is not 3-genuine

③ $(Y^2 - X_1)^2 - X_2 - X_3$ is 3-genuine but not strongly 3-genuine

$\mathbb{Q}(X_1, X_2, X_3, \sqrt{X_1 + \sqrt{X_2 + X_3}})$ is 3-genuine

$\rotatebox{90}{$\subseteq$}$

$\mathbb{Q}(X_1, X_2, X_3, \sqrt{X_2 + X_3})$ is not 3-genuine

Let $F(Y, \vec{X})$ be an absolutely irreducible polynomial

PROOF OUTLINE

① Polynomial sieve + Poisson bound: (+GRH)

$$N(F,B) \ll \frac{B^n}{|P|} + \frac{1}{|P|^2} \sum_{p \neq q} \frac{B^n}{(pq)^n} \sum_{|\vec{\gamma}| \ll p^2/B} |S(p,\vec{\gamma})||S(q,\vec{\gamma})|$$

② Stratification of the exponential sum: $\mathcal{X} = \{X_j\}$ with $\mathrm{codim}(X_j) \geq j$

$$|S(p,\vec{\gamma})| \ll p^{\frac{n}{2} + \frac{(j-1)}{2}} \quad \text{for } \vec{\gamma} \in (\mathbb{A}^n - X_j)(\mathbb{F}_p)$$

③ Sum over $\vec{\gamma}$:

- locally j-bad: savings due to only finitely many p s.t. $\vec{\gamma} \in X_j(\mathbb{F}_p)$
- globally j-bad + $\deg(X_j) > 1$: savings from determinant method/ dimension growth
- globally j-bad + X_j is a plane: second moment bound IF $F(y, L(\vec{x}))$ is strongly n-genuine $\forall L \in GL_n(\mathbb{Q})$

Thm (Bonolis, Pierce, W. '25)

Let $F(Y, \vec{X})$ be absolutely irreducible polynomial in Y^d.

For $d \geq 2$ (and $d=1$ under GRH), if

$F(Y, L(\vec{X}))$ is strongly n-genuine $\forall\ L \in GL_n(\mathbb{Q})$

"strongly n-allowable"

Then $\forall\, \varepsilon > 0$,

$$N(F, B) \ll_{n, \deg F} \log(\|F\| + 2)^c\, B^{n-1+\frac{1}{n+1}+\varepsilon}$$

★ If $F(Y, L(\vec{X}))$ is n-genuine $\forall\ L \in GL_n(\mathbb{Q})$, but not strongly n-genuine, $N(F, B) \ll_{F, \varepsilon} B^{n-1+\varepsilon}$

EXAMPLES:

- ★ If $F(Y, \vec{x})$ is weighted homogeneous + <u>nonsingular</u>

  $\Rightarrow F(Y, \vec{x})$ is strongly n-allowable

- ★ $F(Y, \vec{x}) = Y^d - H(\vec{x})$ where $H(L(\vec{x}))$ is a polynomial in n variables $\forall\ L \in GL_n(\mathbb{Q})$ (cyclic case)

- ★ strongly n-allowable polynomials are generic in
  - space of polynomials of degree $\leq D$
  - space of <u>singular</u> polynomials of degree $\leq D$

THANK YOU!

Any Questions?