COUNTING POINTS ON A FAMILY OF DEL PEZZO SURFACES OF DEGREE ONE

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Swinnerton-Dyer 2002:

"It is known that Del Pezzo surfaces of degree $d > 4$ satisfy the Hasse principle and weak approximation; indeed those of degree 5 or 7 necessarily contain rational points. Del Pezzo surfaces of degree 2 or 1 have no aesthetic merits and have attracted little attention; it seems sensible to ignore them until the problems coming from those of degrees 4 and 3 have been solved."

Manin's Conjecture (over $\mathbb{Q}$)

Let X be a Fano variety
h an anticanonical height

$$N(B) = \#\{x \in X(\mathbb{Q}) : h(x) \leq B\}$$

As $B \to \infty$, $\exists$ constants $a, b, C_X \geq 0$ s.t.

$$N(B) \sim C_X B^a (\log B)^{b-1}$$

Def: a del Pezzo surface S is a smooth projective geometrically integral surface with an ample anticanonical sheaf.

- can have degree $1 \leq d \leq 9$

Another viewpoint: every del Pezzo surface $/\bar{\mathbb{Q}}$ is

- $\mathbb{P}^1 \times \mathbb{P}^1$ (and degree 8)

OR

- a blowup of $\mathbb{P}^2$ at $9-d$ points in general position

Manin's Conjecture (for dPs) (over $\mathbb{Q}$)

Let S be a del Pezzo surface of degree d.
Assume $S(\mathbb{Q}) \neq \emptyset$.
Then there exists a Zariski-dense open set $U \subset S$ such that as $B \to \infty$

$$N(U,B) = \#\{x \in U(\mathbb{Q}) : h(x) \leq B\}$$
$$\sim c_S \, B (\log B)^{\rho - 1}$$

Picard rank of S

(over $\overline{\mathbb{Q}}$, $\mathrm{Pic}_{\overline{\mathbb{Q}}} S = 10-d$)

A Brief History:

$d \geq 6$: dPs are toric (Batyrev-Tschinkel '98)

$d = 4, 5$: certain classes are known
(de la Bretèche, Browning, Destagnol, Fouvry, Peyre, Tenenbaum, W.)

Ex: surfaces containing conic fibrations
(de la Bretèche - Browning '11)

$d = 2, 3$: examples are known

Ex: $x_1 x_2^2 + x_2 x_0^2 + x_3^3 = 0$ (de la Bretèche - Browning - Derenthal '07)

$x_0^2 + x_1 x_2^3 + x_1^3 x_3 = 0$ (Baier - Browning '13)

$d = 1$: $\emptyset$

Upper/Lower Bounds for dP1s:

- $y^2 = (x - e_1 Q(u,v))(x - e_2 Q(u,v))(x - e_3 Q(u,v))$

  $e_1 \in \mathbb{Z}$, $e_2 = \bar{e}_3 \in \mathbb{Z}[i]$, $N(B) \ll B^{5/4+\varepsilon}$ (Munshi '08)

  $e_1, e_2, e_3 \in \mathbb{Z}$ distinct, $N(B) \ll B^{1+\varepsilon}$ (Le Boudec '15)

- Frei-Loughran-Sofos '18 $\quad N(B) \gg B(\log B)^{\rho-1}$

  pass to an extension of degree ≤ 13820

- elliptic surfaces

  $N(B) \ll B^{2.87+\varepsilon}$ Bhargava-Shankar-Tsimerman-Thorne-Tsimerman-Zhao '20

  Helfgott-Venkatesh '06

Let $A, B \in \mathbb{Z}$ satisfy $4A^3 - 27B^2 \neq 0$

$Q(u,v)$ positive definite binary quadratic form

$$S_{A,B,Q} = \{y^2 = x^3 + Ax\,Q(u,v)^2 + B\,Q(u,v)^3\}$$

- sits in weighted projective space $\mathbb{P}(2,3,1,1)$
- singular del Pezzo surface of degree one

Def: A rational point $(x,y,u,v) \in S_{A,B,Q}(\mathbb{Z})$ is integral with respect to the singularity if

$$\gcd(x,y,Q(u,v)) = 1$$

① singularity of $S_{A,B,Q}$ is $\{x = y = Q(u,v) = 0\}$

② strict subset of all of the rational points

$((x,y,u,v) \in S_{A,B,Q}(\mathbb{Z})$ s.t. $\gcd(x,y,u,v) = 1)$

Point-counting function:

Def: $$h((x,y,u,v)) := \max\left(|x|^{1/2}, |y|^{1/3}, |Q(u,v)|^{1/2}\right)$$

Q is positive-definite

$$N(T) := \#\left\{ (x,y,u,v) \in S_{A,B,Q}(\mathbb{Z}) : \begin{array}{l} y \neq 0,\ Q(u,v) \neq 0 \\ \gcd(x,y,Q(u,v)) = 1 \\ h((x,y,u,v)) \leq T \end{array} \right\}$$

Thm (W. '25) Let $A, B \in \mathbb{Z}$ satisfy $4A^3 - 27B^2 \neq 0$

$Q(u,v)$ a positive-definite quadratic form

Assume that $(A, B, Q(u,v))$ is disassociated.

Then as $T \to \infty$,

$$N(T) = C_{A,B,Q} T + O\left(T (\log T)^{-10^{-10}}\right)$$

* If $\{(x,y,u,v) \in S_{A,B,Q} : \gcd(x,y,Q(u,v)) = 1\} \subset U(Q)$ then this gives a lower bound for $N(U,T)$

Corollary: (of the method)

For the del Pezzo surface

$$y^2 = x^3 - x(u^2+v^2)^2$$

$$N(T) \gg T(\log T)^2$$

Manin Conjecture: $N(U,B) \asymp T(\log T)^4$

SYZYGYS OF BINARY QUARTIC FORMS

$$F(x,y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$$

The space of integral binary quartics is parametrized by a, b, c, d, e w/ an action of $SL_2(\mathbb{Z})$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot F(x,y) = F(\alpha x + \beta y, \gamma x + \delta y)$$

$\rightsquigarrow$ induces an action on $\mathbb{Z}[a,b,c,d,e]$

Invariants of binary quartics

$$\mathbb{Z}[a,b,c,d,e]^{SL_2(\mathbb{Z})} = \mathbb{Z}[I,J]$$

with

$$I = ae - 4bd + 3c^2$$

$$J = ace + 2bcd - b^2e - d^2a - c^3$$

Example: the discriminant

$$\Delta(F) = I(F)^3 - 27J(F)^2$$

Covariants:

<u>Def</u>: The Hessian of $F(x,y)$:

$$H(x,y) = \frac{1}{44} \det \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix}$$

<u>Def</u>: The Jacobian covariant of $F(x,y)$:

$$T(x,y) = \frac{-1}{8} \det \begin{pmatrix} F_x & F_y \\ H_x & H_y \end{pmatrix}$$

The Syzygy: (Cayley 1855, Hermite 1848)

$$T(x,y)^2 = -4\,H(x,y)^3 + I(F)\,H(x,y)\,F(x,y)^2 - J(F)\,F(x,y)^3$$

$$y^2 = x^3 + A\,x\,n^2 - B\,n^3$$

- Observation by Mordell used to parameterize points on quadratic twists of elliptic curves

$$\mathcal{F}(A,B)^{*} = \left\{ \begin{array}{c} \text{integral binary}^{*} \\ \text{quartic } F(x,y) \end{array} : \begin{array}{l} I(F) = -4A \\ J(F) = -4B \\ F(1,0) > 0 \end{array} \right\}$$

* also want $F(x,y)$ admissible, i.e. $\gcd(a,b)=1$

- invariant under $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z} \right\}$

- <u>almost</u> invariant under $SL_2(\mathbb{Z})$

- congruence conditions given by Bhargava–Shankar '15 for when $\mathcal{F}(A,B) \neq \emptyset$

Thm (Duke '21) Let $A, B \in \mathbb{Z}$ satisfy $4A^3 - 27B^2 \neq 0$.
Then

$$\mathcal{F}(A,B)/\Gamma_\infty \to \left\{ (x,y,n) \in \mathbb{Z}^2 \times \mathbb{Z}_+ : \begin{array}{c} y^2 = x^3 + Axn^2 + Bn^3 \\ \gcd(x,n) = 1 \end{array} \right\}$$

$$F \longmapsto \left(-H(1,0), \frac{T(1,0)}{2}, F(1,0) \right)$$

is a bijection.

Consequence: $\#\left\{ (x,y) \in \mathbb{Z}^2 : \begin{array}{c} y^2 = x^3 + Axn^2 + Bn^3, \ \gcd(x,n) = 1 \\ |x|^{1/2}, |y|^{1/3} \leq T \end{array} \right\}$

$$= \sum_{\substack{F \in \mathcal{F}(A,B)/\Gamma_\infty \\ F(1,0) = n}} \mathbb{1}_{\left|\frac{T(1,0)}{2}\right|^{1/3} \leq T} \, \mathbb{1}_{|H(1,0)|^{1/2} \leq T}$$

Our surface: $y^2 = x^3 + AxQ(u,v)^2 + BQ(u,v)^3$

$$N(T) = \#\left\{(x,y,u,v) \in S_{A,B,Q}(\mathbb{Z}) : \begin{array}{l} \gcd(x,y,Q(u,v))=1 \\ |x|^{1/2}, |y|^{1/3}, |Q(u,v)|^{1/2} \leq T \end{array}\right\}$$

$\#\{u,v : Q(u,v)=n\}$

$$= \sum_{n \leq T} r_Q(n) \sum_{\substack{F \in \mathcal{F}(A,B)/\Gamma_\infty \\ F(1,0)=n}} \mathbb{1}_{\left|\frac{T(1,0)}{2}\right|^{1/3} \leq T} \, \mathbb{1}_{|H(1,0)|^{1/2} \leq T}$$

$$= \sum_{F \in \mathcal{F}(A,B)/\Gamma_\infty} r_Q(F(1,0)) \, \mathbb{1}_{\left|\frac{T(1,0)}{2}\right|^{1/3} \leq T} \, \mathbb{1}_{|H(1,0)|^{1/2} \leq T}$$

infinite!

Let $\mathcal{F}(A,B) = \left\{ \begin{array}{l} \text{integral} \\ \text{binary quartic } F(x,y) \end{array} : \begin{array}{l} I(F) = -4A \\ J(F) = -4B \end{array} \right\}$

- invariant under the action of $SL_2(\mathbb{Z})$

AND

$$\mathbb{F}(A,B) = \mathcal{F}(A,B)/SL_2(\mathbb{Z}) \quad \text{is finite!}$$

Idea:

$$\mathcal{F}(A,B)/SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})/\Gamma_\infty \rightsquigarrow \mathcal{F}(A,B)/\Gamma_\infty$$

$$\cong \left\{ \begin{pmatrix} m_1 & * \\ m_2 & * \end{pmatrix} \right\}$$

Lemma (Duke '21) Let $A, B \in \mathbb{Z}$ s.t. $4A^3 - 27B^2 \neq 0$

Let $\mathrm{Aut}(F) \subset SL_2(\mathbb{Z})$ denote the automorphism group of F.

Then $\#\left\{ (x,y) \in \mathbb{Z}^2 : \begin{array}{l} y^2 = x^3 + Axn^2 + Bn^3,\ \gcd(x,n) = 1 \\ |x|^{1/2}, |y|^{1/3} \leq T \end{array} \right\}$

$$= \sum_{F \in \mathbb{F}(A,B)} \frac{1}{\mathrm{Aut}(F)} \sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ \gcd(m_1,m_2)=1 \\ \overline{(m_1,m_2)} \in R_F \\ F(m_1,m_2) = n}} \mathbb{1}_{\left|\frac{T(\vec{m})}{2}\right|^{1/3} \leq T} \, \mathbb{1}_{|H(\vec{m})|^{1/2} \leq T}$$

where $R_F = \{ \vec{m} \bmod \Delta(F) : \forall p \mid \Delta(F),\ (F(\vec{m}), H(\vec{m})) \neq (0,0) \bmod p \}$

Hence,

$$N(T) = \#\left\{(x,y,u,v) \in S_{A,B,Q}(\mathbb{Z}) : \begin{array}{l} \gcd(x,y,Q(u,v))=1 \\ |x|^{1/2}, |y|^{1/3}, |Q(u,v)|^{1/2} \le T \end{array}\right\}$$

$$= \sum_{n \le T} r_Q(n) \sum_{F \in \mathbb{F}(A,B)} \frac{1}{\mathrm{Aut}(F)} \sum_{\substack{m_1,m_2 \in \mathbb{Z} \\ \gcd(m_1,m_2)=1 \\ \vec{m} \in R_F \\ F(m_1,m_2)=n}} \mathbb{1}_{|T(\frac{\vec{m}}{2})|^{1/3} \le T} \mathbb{1}_{|H(\vec{m})|^{1/2} \le T}$$

finite!

$$= \sum_{F \in \mathbb{F}(A,B)} \frac{1}{\mathrm{Aut}(F)} \sum_{\substack{m_1,m_2 \in \mathbb{Z} \\ \gcd(m_1,m_2)=1 \\ \vec{m} \in R_F}} r_Q(F(m_1,m_2)) \mathbb{1}_{|T(\frac{\vec{m}}{2})|^{1/3} \le T} \mathbb{1}_{|H(\vec{m})|^{1/2} \le T}$$

$T \mathcal{R}_F$ for a convex region $\mathcal{R}_F \subset \mathbb{R}^2$

The main correlation sum:

$$\sum_{\substack{x,y \le X \\ (x,y) \equiv \alpha \bmod M \\ \gcd(x,y)=1}} r_Q(F(x,y))$$

$Q(u,v)$ positive definite

$F(x,y)$ squarefree binary quartic

<u>Def</u>: $(A, B, Q(u,v))$ is disassociated if the following holds: $\forall\ F \in \mathbb{F}(A,B)$, let K_F be the splitting field of F.

Then $K_F \cap \mathbb{Q}[u]/Q(u,1) = \mathbb{Q}$.

Thm (W. '25)

Let $Q(u,v)$ be a positive definite binary quadratic form

$F(x,y)$ be a squarefree binary quartic form

If $K_F \cap \mathbb{Q}[u]/Q(u,1) = \mathbb{Q}$, then $\exists\ C_{F,Q} \geq 0$ s.t.

$$\sum_{x,y \leq X} r_Q(F(x,y)) = C_{F,Q} X^2 + O\left(X^2 (\log X)^{-10^{-10}}\right)$$

gives the condition that $(A,B,Q(u,v))$ is disassociated for the point-count for $S_{A,B,Q}$

Some (incomplete) history on $\sum_{x,y\leq X} r_Q(F(x,y))$:

- when $Q(u,v)=u^2+v^2$:
  - $F=L_1L_2L_3L_4$ Heath-Brown '03, de la Bretèche-Browning '08, Matthiesen '12
  - $F=L_1L_2Q$ de la Bretèche '08, Destagnol '16
  - $F=Q_1Q_2$ de la Bretèche-Tenenbaum '13
  - $F=$ irred de la Bretèche-Tenenbaum '13
- when $Q(u,v)=u^2+av^2$, $\mathbb{Q}(\sqrt{a})\cap K_F=\mathbb{Q}$ W. '24

Sum of two squares: $Q(u,v) = u^2 + v^2$

$$r_Q(n) = 4 \sum_{d \mid n} \chi_4(d)$$

So,

$$\sum_{x,y \le X} r_Q(F(x,y)) = 4 \sum_{x,y \le X} \sum_{d \mid F(x,y)} \chi(F(x,y))$$

$$\approx \sum_{d \le X^2} \chi(d) \underbrace{\sum_{x,y \le X} \mathbb{1}_{d \mid F(x,y)}}_{\substack{\text{estimate using "elementary" (i.e. minimal complex analysis)} \\ \text{techniques about lattices and} \\ \text{distribution of divisors}}}$$

For general pos. def $Q(u,v)$:

$$r_Q(n) = \#\{(u,v) : Q(u,v) = n\}$$

$$\neq \sum_{d \mid n} \chi(d)$$

Need extra structure from elsewhere!

Connection to modular forms:

$$\Theta(z) = \sum_n r_Q(n)\, e^{2\pi i n z}$$

modular form of weight one for $\Gamma_0(4\Delta(Q), \chi_Q)$

$$= E(z) + C(z)$$

Hence: $r_Q(n) = \lambda_E(n) + \lambda_C(n)$

AND "$\lambda_E(n) \approx 1 * \chi_Q(n)$" is similar to $r_{u^2+v^2}(n)$ structurally

$$\sum_{x,y\le X} r_Q(F(x,y)) = \sum_{x,y\le X} \lambda_E(F(x,y)) + \sum_{x,y\le X} \lambda_C(F(x,y))$$

Ramanujan-Petersson Conjecture

If $C(z)$ is a weight k cusp form, then

$$|\lambda_C(n)| \ll_\varepsilon n^{\frac{k-1}{2}+\varepsilon}$$

(Eisenstein coeff $\lambda_E(n) \asymp n^{k-1}$)

! our cusp form has weight one!

The Eisenstein contribution:

<u>Prop</u>

$$\sum_{x,y\le X} \lambda_E(F(x,y)) = c_{F,Q} X^2 (\log X)^{\beta_{F,Q}} + O\left(X^2 (\log X)^{\beta_{F,Q} - 10^{-10}}\right)$$

where $\beta_{F,Q} = \#\{i : \mathbb{Q}[u]/Q(u,1) \subset K_{F_i}\}$ $\quad F = \prod_i F_i$

- if $K_F \cap \mathbb{Q}[u]/Q(u,1) = \mathbb{Q} \Rightarrow \beta_{F,Q} = 0$
- no need for an assumption btw $\mathbb{Q}[u]/Q(u,1)$ and K_F in this part of the proof

The Cuspidal Contribution:

<u>Prop</u>: If $K_F \cap \mathbb{Q}[u]/Q(u,1) = \mathbb{Q}$ then

$$\sum_{x,y \le X} \lambda_c(F(x,y)) \ll \underbrace{X^2 (\log X)^{-0.066}}_{\text{Compare to the Eisenstein contribution of size } \asymp X}$$

Summing Hecke eigenvalues along polynomials:

Thm (W. '25)

Let π be a cuspidal aut. rep of $GL_2(\mathbb{A}_{\mathbb{Q}})$
satisfying the Ramanujan-Petersson conjecture

$P(x) \in \mathbb{Z}[x]$ irreducible and solvable

K splitting field of $P(x)$

If the base change π_K is a cuspidal aut. rep of $GL_2(\mathbb{A}_K)$

Then

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X (\log X)^{-0.066}$$

Consider the sum:

$$\sum_{n \le X} \lambda_\pi(|P(n)|) \quad \text{for } \pi \text{ cuspidal}$$

- $P(n)$ linear $\Rightarrow O(X^{1-\delta})$ for $\delta > 0$
- $P(n)$ quadratic
  Blomer '08
  Templier '11
  Templier–Tsimerman '13
- higher degree poly [beyond endoscopy]

Trick: absolute values

PRO: higher degree polynomials

CON: at best polylogarithmic savings

Some (incomplete) history:

- Elliott–Moreno–Shahidi '84: $\sum_{n\leq X} \tau(P(n))$
- Holowinsky '09: $P(x) = x(x+\ell)$ $\rightsquigarrow$ holomorphic analog of QUE + good dependence on ℓ
- Nelson '22: $P(x)$ = quad
- Chiriac–Yang '22: π is nonCM holomorphic newform

An Upper Bound Sieve:

$f(n) = |\lambda_\pi(n)|$

<u>Thm</u> (Nair '92) Let $f(n)$ be a nonnegative multiplicative function and $f(p^\nu) \le A^\nu$, $f(n) \le Bn^\varepsilon$. ← Ramanujan-Petersson conj

For any irreducible polynomial $P(x) \in \mathbb{Z}[x]$

$$\sum_{n \le X} f(|P(n)|) \ll_{P,A,B} X \exp\left(\sum_{p \le X} \frac{\rho(p)(f(p)-1)}{p} \right)$$

where $\rho(p) = \#\{x \bmod p : P(x) \equiv 0 \bmod p\}$

<u>Thm</u> (de la Bretèche–Tenenbaum '12)

For any irreducible binary form $G(x,y) \in \mathbb{Z}[x,y]$,

$$\sum_{x,y \le X} f(|G(x,y)|) \ll_{G,A,B} X^2 \exp\left(\sum_{p \le X} \frac{\rho(p)(f(p)-1)}{p} \right)$$

where $\rho(p) = \#\{x \bmod p : G(x,1) \equiv 0 \bmod p\}$

Corollary: Let $C(z)$ cusp. eigenform $\longleftrightarrow \pi$ of $GL_2(\mathbb{A}_{\mathbb{Q}})$

If π_{K_F} is a cuspidal automorphic rep of $GL_2(\mathbb{A}_{K_F})$

then

$$\sum_{x,y \leq X} |\lambda_c(F(x,y))| \ll X^2 (\log X)^{-0.066}$$

- Since $F(x,y)$ is quartic, K_F is solvable

Claim: $K_F \cap \mathbb{Q}[u]/\mathbb{Q}(u,1) = \mathbb{Q} \Rightarrow \pi_{K_F}$ is cuspidal

Thm (Langlands '80) builds on Saito and Shintani

If π_{K_F} is not cuspidal, then $\exists\ \mathbb{Q} \subseteq E \subset E' \subseteq K_F$

such that
$$\pi_E = \mathrm{Ind}_{W_{E'}}^{W_E} \theta \quad \text{for } \theta \text{ a quasicharacter of } \mathbb{A}_{E'}^\times / E'^\times$$

Already, if $K_Q = \mathbb{Q}[u]/Q(u,1)$,
$$\pi = \mathrm{Ind}_{W_{K_Q}}^{W_Q} \chi \quad \text{for a class group character } \chi$$

$\Rightarrow$ if π_{K_F} is not cuspidal, $K_Q \subset K_F$.

After applying the sieve:

$$\sum_{n\leq X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X \exp\left(\sum_{p\leq X} \frac{\rho(p)(|\lambda_\pi(p)|-1)}{p}\right)$$

At all but f. many primes, $\rho(p) = \#\{\mathfrak{p} \subset \mathcal{O}_K : N\mathfrak{p} = p\}$

$\rightsquigarrow$ want to upper bound

$$\sum_{N\mathfrak{p}\leq X} \frac{|\lambda_\pi(N\mathfrak{p})|-1}{N\mathfrak{p}} + O_K(1)$$

Appearance of base change:

for $\mathfrak{p} \subset \mathcal{O}_K$ s.t. $N\mathfrak{p} = p$,

$$\lambda_{\pi_K}(\mathfrak{p}) = \lambda_\pi(N\mathfrak{p})$$

So, want to estimate

$$\sum_{N\mathfrak{p} \le X} \frac{|\lambda_{\pi_K}(\mathfrak{p})| - 1}{N\mathfrak{p}}$$

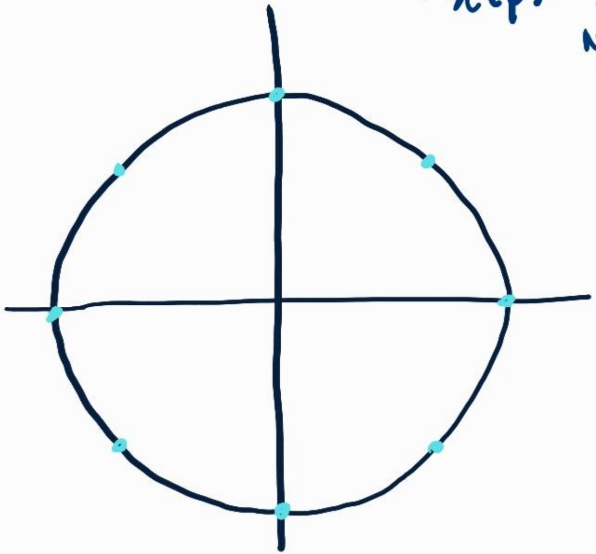
comes from the distribution of $\lambda_{\pi_K}(\mathfrak{p})$

Example: Let $L/\mathbb{Q}$ be quadratic

ψ unitary Hecke character of L

$$\lambda_\pi(p) = \sum_{N\beta = p} \psi(\beta) = \psi(\beta) + \overline{\psi(\beta)}$$

* when p splits in L

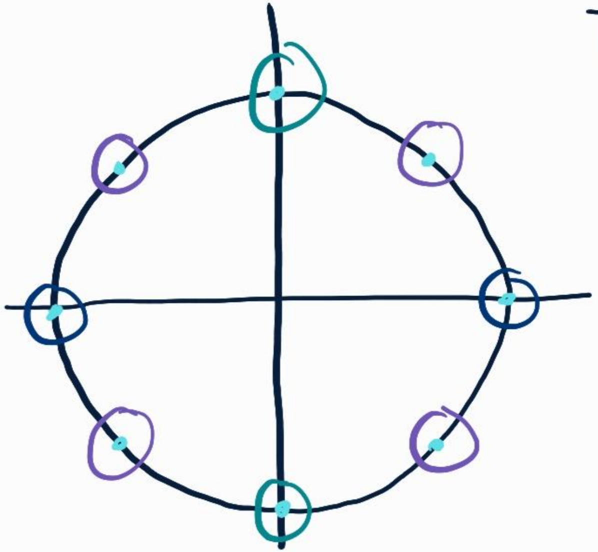


Then if $\psi(\beta) = e^{2\pi i \theta_p}$,

$$|\lambda_\pi(p)| = 2|\cos(2\pi\theta_p)|$$

Assume as $p \to \infty$,

$2\pi\theta_p$ follows the • distribution



Then $\displaystyle\sum_{p \le X} \frac{|\lambda_\pi(p)|}{p}$

$= \frac{1}{2} \times \frac{1}{4} \times 2 \log\log X$

$+ \frac{1}{2} \times \frac{1}{2} \times \sqrt{2} \log\log X$

$+ \frac{1}{2} \times \frac{1}{4} \times 0 \log\log X + O(1)$

$= \frac{1+\sqrt{2}}{4} \log\log X + O(1)$

$$\Rightarrow \sum_{p \le X} \frac{|\lambda_\pi(p)|-1}{p} = \frac{\sqrt{2}-3}{4} \log\log X + O(1)$$

$<1.$

In this case, we know

$$\sum_{n \le X} |\lambda_\pi(n)| \asymp \frac{X}{(\log X)^{\frac{3-\sqrt{2}}{4}}}$$

In general, if π is cuspidal,

$$\sum_{Np \le X} \frac{|\lambda_{\pi_K}(p)| - 1}{Np} \le -0.066 \log\log X + O_{\pi,K}(1)$$

$$\Rightarrow \sum_{n \le X} |\lambda_\pi(P(n))| \ll X (\log X)^{-0.066}$$

SUMMARY: ① $K_F \cap \mathbb{Q}[u]/Q(u,1) = \mathbb{Q} \Rightarrow \pi_{K_F}$ cuspidal

$$\Rightarrow \sum_{x,y \le X} |\lambda_C(F(x,y))| \ll X^2 (\log X)^{-0.066}$$

② $\displaystyle\sum_{x,y\le X} \lambda_E(F(x,y)) \sim c_{F,Q} X^2$

$$\Rightarrow \sum_{x,y \le X} r_Q(F(x,y)) = c_{F,Q} X^2 + O(X^2 (\log X)^{-10^{-10}})$$

③ $$N(T) = \sum_{F \in \mathbb{F}(A,B)} \frac{1}{\mathrm{Aut}(F)} \sum_{\substack{\vec{m} \in T^{1/2} R_F \\ \gcd(m_1,m_2)=1 \\ \overline{m} \in R_F}} r_Q(F(x,y))$$

$$= c_{A,B,Q}\, T + O_{A,B,Q}(T (\log T)^{-10^{-10}})$$

Other varieties and other syzygys:

Ex: $y^2 = x^3 - 27\Delta\, Q(u,v)^2$

can be parameterized by binary cubic forms of discriminant Δ

THANK YOU!

Any questions?