Prime number theorems for polynomials from homogeneous dynamics

Joint work with Giorgos Kotsovolis

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Motivating question

Given an irreducible polynomial $f(x) \in \mathbb{Z}[x]$, is $f(n)$ prime for infinitely many integers n?

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Bunyakovsky's conjecture

If $f(x) \in \mathbb{Z}[x]$ is a monic irreducible polynomial, then $f(n)$ is prime for infinitely many positive integers n.

The Bateman-Horn conjecture

A quantitative conjecture

Let $f(x)$ be a nonconstant irreducible monic polynomial. Then as $N \to \infty$:

$$\#\{n \in [1, N] : f(n) \text{ is prime}\} \sim \mathfrak{S} \cdot \int_0^N \frac{dx}{\log^+(f(x))},$$

where we write

$$\log^+(y) = \begin{cases} \max(2, \log(y)), & y > 0 \\ \infty, & y \leq 0, \end{cases}$$

and we define

$$\mathfrak{S} := \prod_p \left(\frac{1 - p^{-1}\#\{x \in \mathbb{F}_p : f(x) \equiv 0 \mod p\}}{(1 - 1/p)} \right).$$

The multivariate Bateman-Horn conjecture

Conjecture

Let $F(\mathbf{x})$ be a nonconstant irreducible polynomial over $\mathbb{Q}$ in n variables. Then as $T \to \infty$,

$$\#\{\mathbf{x} \in \mathbb{Z}^n : \|\mathbf{x}\|_\infty \leq T, F(\mathbf{x}) \text{ is prime}\} \sim \mathfrak{S} \cdot \int_{\|\mathbf{x}\|_\infty \leq T} \frac{d\mathbf{x}}{\log^+(F(\mathbf{x}))},$$

where the singular series $\mathfrak{S}$ is defined as a product of local densities:

$$\mathfrak{S} = \prod_p \left(\frac{1 - p^{-n}\#\{\mathbf{x} \in \mathbb{F}_p^n : F(\mathbf{x}) \equiv 0 \mod p\}}{(1 - 1/p)} \right).$$

History: the circle method

Let $F(\mathbf{x})$ be an irreducible polynomial in n variables of degree d. Then we have that

$$\begin{aligned}\#\{\mathbf{x} \in \mathbb{Z}^n : \|x\|_\infty \leq T, F(\mathbf{x}) \text{ is prime}\} \\ = \int_0^1 \Big(\sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \|\mathbf{x}\|_\infty \leq T}} e(F(\mathbf{x})\alpha)\Big)\Big(\sum_{\substack{p \leq c_F T^d \\ \text{prime}}} e(-p\alpha)\Big) d\alpha.\end{aligned}$$

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If n is sufficiently large in terms of d, then by evaluating this integral you can achieve a prime number theorem.

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- (Destagnol-Sofos, 2019): $n > (d-1)2^{d-1}$
- (Brüdern-Wooley, 2022): $n > \lceil d\log(d)\rceil + 5$ for F diagonal

History: the circle method

Number of variables	**Structure**
$(d-1)2^{d-1}$	any polynomial
$d\log(d)$	diagonal, homogeneous

Let $K/\mathbb{Q}$ be a degree d extension. Then if the ring of integers

$$\mathcal{O}_K = \mathbb{Z}[\alpha_1, \alpha_2, ..., \alpha_d],$$

the polynomial

$$N_{K/\mathbb{Q}}(\mathbf{x}) := N_{K/\mathbb{Q}}(x_1\alpha_1 + x_2\alpha_2 + ... + x_d\alpha_d)$$

is an irreducible homogeneous form of degree d in d variables.

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$N_{K/\mathbb{Q}}(\mathbf{x})$ is prime $\iff$ $\mathfrak{p} \subset \mathcal{O}_K$ is a prime ideal.

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Landau's prime ideal theorem

The number of prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ is asymptotic to

$$\#\{\mathfrak{p} \subset \mathcal{O}_K : N(\mathfrak{p}) \leq X\} \sim \int_0^X \frac{1}{\log^+(x)} dx.$$

Number of variables	Structure
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d	norm form of a degree d extension

Let $N_{K/\mathbb{Q}}(\mathbf{x})$ be a norm form for a degree d extension. Then an incomplete norm form is the polynomial

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Another important result:

- (Friedlander-Iwaniec, 1998): $x^2 + y^4$

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Consider the determinant on the space of $n \times n$ matrices:

$$\det(x_{11}, ..., x_{nn}).$$

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Theorem 2.1 (Kotsovolis-W., '23)

Define the following prime counting function:

$$\pi_{\det}(T) := \#\{A \in Mat_n(\mathbb{Z}) : \|A\|_\infty \leq T, \det(A) \text{ is prime}\}.$$

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For $d \geq 4$, the determinant polynomial is beyond the circle method.

Comparison with previous polynomials

Number of variables	**Structure**	**Irreducibility**
$(d-1)2^{d-1}$	any polynomial	any
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d	norm form	split over $\overline{\mathbb{Q}}$
$15d/22$	incomplete norm form	split over $\overline{\mathbb{Q}}$
d^2	determinant	$\overline{\mathbb{Q}}$

The determinant on symmetric matrices

Consider the determinant on the space of symmetric $n \times n$ matrices:

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Here dX is the Euclidean measure on $Sym_n(\mathbb{R}) \cong \mathbb{R}^{n(n+1)/2}$.

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A fact about the stabilizer

$\mathrm{SL}_n(\mathbb{R})$ is connected, simply connected, semisimple, and **has no compact factors**.

Key ingredient

For the determinant polynomial, we can count the number of integer points on the level sets V_m.

Let $\Omega \subset \mathrm{SL}_n(\mathbb{R})$ be a "nice" compact subset. Define the **cone of height** T as the set:

$$[0,T]\Omega = \{t\omega : t \in [0,T], \omega \in \Omega\}.$$

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Then, as $m \to \infty$:

$$\#V_m(\mathbb{Z}) \cap m^{1/n}\Omega \sim \prod_{j=2}^{n} \zeta(j)^{-1} \mathfrak{S}_n(m) \mu(\Omega),$$

where $\mathfrak{S}_n(m)$ is a singular series depending on m and μ is the measure on $\mathrm{SL}_n(\mathbb{R})$ induced by the Euclidean measure on $\mathrm{Mat}_n(\mathbb{R})$.

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- (Linnik-Skubenko, 1962): ergodic method
- (Sarnak, 1990): Hecke orbits

Vertical versus horizontal statistics

	Set up	**Limit**
Vertical statistics	$\#V_m(\mathbb{Z}) \cap m^{1/n}\Omega$	$m \to \infty$
Horizontal statistics	$\#V_m(\mathbb{Z}) \cap \{\|X\| \leq T\}$	$T \to \infty$

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(Duke-Rudnick-Sarnak, 1993): As $T \to \infty$,

$$\#V_m(\mathbb{Z}) \cap \{\|X\| \leq T\} \sim \prod_{j=2}^{n} \zeta(j)^{-1} C_{n,m} T^{n^2-n}$$

where

$$C_{n,m} = \frac{\pi^{n^2/2} m^{-n+1}}{\Gamma((n^2-n+2)/2)\Gamma(n/2)} \mathfrak{S}_n(m).$$

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- $\mathcal{O}_{p,q} \cong G/\mathrm{stab}(I_{p,q})$, where $\mathrm{stab}(I_{p,q}) \cong \mathrm{SO}(p,q)$.

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Condition on the stabilizer

If $p, q \neq 0$ then $\mathrm{SO}(p,q)$ has no compact factors, but $\mathrm{SO}(n,0)$ is compact!

The Linnik problem (indefinite orbits)

Theorem 3.1 (Oh, 2004)

Let $n \geq 3$, $p, q \neq 0$, and $\Omega \subset SL_n(\mathbb{R}) \cap \mathcal{O}_{p,q}(\mathbb{R})$ be a "nice" compact subset. As $m \to \infty$,

$$\#V_m(\mathbb{Z}) \cap m^{1/n}\Omega \sim \int_{m^{1/n}\Omega \times \prod_p V_m(\mathbb{Z}_p)} \delta(x) d\mu_m(x)$$

where $\delta : V_m(\mathbb{A}) \to \mathbb{R}$ is constant on the adelic orbits $\mathcal{O}(\mathbb{A})$ and

$$\delta(x) = \begin{cases} 2, & \mathcal{O}(\mathbb{A}) \text{ contains a } \mathbb{Q}\text{-point}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, μ_m is the Tamagawa measure on $V_m(\mathbb{A})$.

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Key ingredient

Ratner's Theorem!

The Linnik problem (positive-definite orbits)

Theorem 3.2 (Einsiedler-Margulis-Mohammadi-Venkatesh, 2020)

Suppose $\{Q_i\}_{i=1}^{\infty}$ varies through any sequence of pairwise inequivalent, integral, positive definite quadratic forms. Then the genus of Q_i, considered as a subset of $PGL_n(\mathbb{Z})\backslash PGL_n(\mathbb{R})/PO_n(\mathbb{R})$, equidistributes as $i \to \infty$.

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Suppose $\{Q_i\}_{i=1}^{\infty}$ *varies through any sequence of pairwise inequivalent, integral, positive definite quadratic forms. Then the genus of* Q_i*, considered as a subset of* $PGL_n(\mathbb{Z})\backslash PGL_n(\mathbb{R})/PO_n(\mathbb{R})$*, equidistributes as* $i \to \infty$*.*

As a consequence, if $\Omega \subset \mathcal{O}_{n,0}$, we have that as $m \to \infty$,

$$\#V_m(\mathbb{Z}) \cap m^{1/n}\Omega \sim h_n(m)\mu(\Omega),$$

where μ is the lift of the Haar measure and $h_n(m)$ is the class number of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy orbits of integral positive definite symmetric matrices of determinant m.

The Linnik problem (positive-definite orbits)

Theorem 3.2 (Einsiedler-Margulis-Mohammadi-Venkatesh, 2020)

Suppose $\{Q_i\}_{i=1}^{\infty}$ varies through any sequence of pairwise inequivalent, integral, positive definite quadratic forms. Then the genus of Q_i, considered as a subset of $PGL_n(\mathbb{Z})\backslash PGL_n(\mathbb{R})/PO_n(\mathbb{R})$, equidistributes as $i \to \infty$.

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Key ingredient

Theory of automorphic forms!

Upshot

Linnik problem

We have asymptotic formulas for $\Omega \subset \mathcal{O}_{p,q}$

$$\#V_m(\mathbb{Z}) \cap m^{1/n}\Omega$$

as $m \to \infty$.

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Linnik problem

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Still to be done

We want our formulas in terms of more familiar objects, i.e. in the same form as with the determinant polynomial.

Table of Contents

From now on, we look at the determinant of symmetric 3×3 matrices:

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$$F(a,b,c,d,e,f) = abc - af^2 - be^2 - cd^2 + 2def.$$

This polynomial is a **cubic in 6 variables**.

Oh's Theorem

$$\#V_m(\mathbb{Z}) \cap m^{1/3}\Omega \sim \int_{m^{1/3}\Omega \times \prod_p V_m(\mathbb{Z}_p)} \delta(x) d\mu_m(x)$$

What is happening with $\delta(x)$?

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What is happening with $\delta(x)$?

$$\delta(x) \neq 0 \iff \prod_p c_p(x_p) = 1,$$

where $c_p(x_p)$ are the Hasse-Minkowski invariants after viewing x_p as a p-adic quadratic form.

Indefinite orbits

Let $\mathcal{G}_p$ denote a $\mathrm{SL}_3(\mathbb{Z}_p)$ orbit of $V_m(\mathbb{Z}_p)$. If m is prime, then

$$\#V_m(\mathbb{Z}) \cap m^{1/3}\Omega \sim 2m \prod_{p \neq m} \mu_{m,p}(V_m(\mathbb{Z}_p)) \times \sum_{\substack{\mathcal{G}_2, \mathcal{G}_m \\ c_2(\mathcal{G}_2) c_m(\mathcal{G}_m) = c_\infty(\Omega)}} \mu_{m,2}(\mathcal{G}_2) \mu_{m,m}(\mathcal{G}_m).$$

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These p-adic densities appear in the Siegel mass formula!

Indefinite orbits

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Let $\Omega \subset \mathcal{O}_{p,q}$ for $(p,q) = (1,2)$. Then as $p \to \infty$

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Consequence of EMMV

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Class numbers

Siegel mass formula

$\downarrow$

Lemma 2 (Kitaoka, 1973)

As $p \to \infty$,

$$h_3(p) \sim p \cdot \pi^{-3}\zeta(2)\Gamma(1/2)\Gamma(3/2)$$

Relating the measures

Lemma 3

$$\mu(\Omega) = \pi^3 \zeta(2)^{-1} \zeta(3)^{-1} \Gamma(1/2)^{-1} \Gamma(3/2)^{-1} \mu_\infty(\Omega).$$

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- $\mu_\infty(\mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})) = \zeta(2)\zeta(3)$.
- The factor of $\pi^3\Gamma(1/2)^{-1}\Gamma(3/2)^{-1}$ comes from the fact we are actually working on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$.

Lemma 4 (Kitaoka, 1973)

As $p \to \infty$,

$$h_3(p) \sim p \cdot \pi^{-3}\zeta(2)\Gamma(1/2)\Gamma(3/2)$$

Lemma 5

$$\mu(\Omega) = \pi^3\zeta(2)^{-1}\zeta(3)^{-1}\Gamma(1/2)^{-1}\Gamma(3/2)^{-1}\mu_\infty(\Omega).$$

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This formula holds on both $\mathcal{O}_{3,0}, \mathcal{O}_{1,2}$.

Summing over primes

Define the counting function:

$$\pi(T\Omega) = \#\{A \in \mathrm{Sym}_3(\mathbb{Z}) \cap T\Omega : \det(A) \text{ is prime}\}.$$

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$$\begin{aligned}\pi(T\Omega) &= \sum_{p \le T^3} \#\{A \in \mathrm{Sym}_3(\mathbb{Z}) \cap T\Omega : \det(A) = p\} \\ &= \sum_{p \le T^3} \#V_p(\mathbb{Z}) \cap p^{1/3}\Omega.\end{aligned}$$

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We apply our solution to Linnik's problem:

$$\begin{aligned}\pi(T\Omega) &= (1 + o_\Omega(1))\mu_\infty(\Omega) \sum_{p \leq T^3} \zeta(3)^{-1} p \\ &= (1 + o_\Omega(1))\zeta(3)^{-1}\mu_\infty(\Omega)\frac{T^6}{6\log(T)}.\end{aligned}$$

Lemma 6

For $\Omega \subset \mathcal{O}_{3,0}$ *or* $\mathcal{O}_{1,2}$*,*

$$\pi(T\Omega) = (1 + o_\Omega(1))\zeta(3)^{-1} \int_{T\Omega} \frac{1}{\log^+(\det(X))} dX.$$

If $\Omega \subset \mathcal{O}_{2,1}$ *or* $\mathcal{O}_{0,3}$*, then* $\pi(T\Omega) = 0$.

Idea: approximate the box with cones!

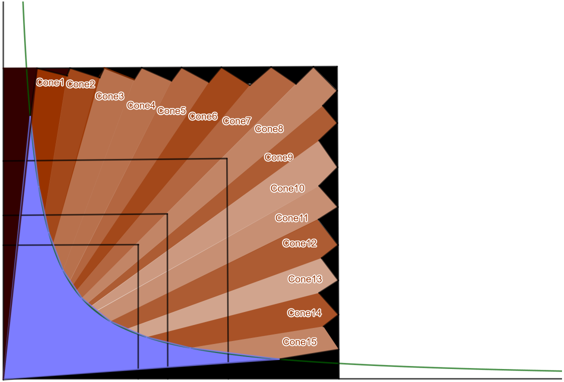

Figure: Approximating $[0,T]^2$ with cones from $xy = 1$

Image created by Giorgos Kotsovolis

DEFINITION 1

An ϵ-**cutting** of $\mathcal{R}_0$ is a finite set of disjoint "nice" connected compact subsets of $V_1(\mathbb{R})$, denoted as

$$\mathcal{C}_\epsilon = \{\Omega_i\}_{i=1}^{N(\epsilon)},$$

such that

$$\mathcal{R}_0 = \mathcal{E} \bigcup_{\Omega \in \mathcal{C}_\epsilon} [0, 1/\mathrm{ht}(\Omega)]\Omega,$$

where the exceptional set satisfies that $|\mathcal{E}| \leq \epsilon$.

Here, $\mathrm{ht}(\Omega) = \sup_{A \in \Omega} \|A\|$.

ϵ-CUTTING

Fix $\epsilon > 0$. If $\mathcal{C}_\epsilon$ is an ϵ-cutting of $\mathrm{Sym}_3(\mathbb{R}) \cap \{\|A\| \leq 1\}$, then

$$\mathrm{Sym}_3(\mathbb{R}) \cap \{\|A\| \leq T\} = T\mathcal{E} \cup_{\Omega \in \mathcal{C}_\epsilon} T/\mathrm{ht}(\Omega)\Omega,$$

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$$\begin{aligned} &\#\{A \in \mathrm{Sym}_3(\mathbb{Z}) : \|A\| \leq T, \det(A) \text{ is prime}\} \\ = &\#\{A \in \mathrm{Sym}_3(\mathbb{Z}) \cap T\mathcal{E} : \det(A) \text{ is prime}\} + \sum_{\Omega \in \mathcal{C}_\epsilon} \pi(T/\mathrm{ht}(\Omega)\Omega). \end{aligned}$$

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Counting primes on cones

$$\sum_{\Omega \in \mathcal{C}_\epsilon} \pi(T/\mathrm{ht}\Omega\Omega) = (1 + o_\epsilon(1))\zeta(3)^{-1} \int_{\cup T/\mathrm{ht}(\Omega)\Omega} \frac{1}{\log^+(\det(X))} dX.$$

Upper-bounding the exceptional set

Proposition 4.1

Let $\mathcal{R}$ be a convex region in $[0,T]^n$ and $F(\mathbf{x})$ a polynomial in n variables. Then there is a constant $c_F > 0$ such that

$$\#\{\mathbf{x} \in \mathcal{R}(\mathbb{Z}) : F(\mathbf{x}) \in \mathscr{P}\} \leq c_F \left(\frac{|\mathcal{R}|}{\log(T)} + T^{n-1/2} \right).$$

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- Upper bound holds for any polynomial F
- c_F is independent of $\mathcal{R}$
- Comes from an upper bound sieve + a level of distribution result

Upper-bounding the exceptional set

Applying the Proposition, we have

Lemma 7

$$\#\{A \in Sym_3(\mathbb{Z}) \cap T\mathcal{E} : \det(A) \textit{ is prime}\} \leq c_{\det}\frac{\epsilon T^6}{\log(T)} + T^{11/2}.$$

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So, we get that for any $\epsilon > 0$,

$$\#\{A \in \mathrm{Sym}_3(\mathbb{Z}) : \|A\| \leq T, \det(A) \text{ is prime}\}$$
$$= (1 + o_\epsilon(1))\zeta(3)^{-1} \int_{\|X\| \leq T} \frac{1}{\log^+(\det(X))} dX + O(\frac{\epsilon T^6}{\log(T)}).$$

Taking $\epsilon \to 0$, we get

THEOREM

$$\#\{A \in \mathrm{Sym}_3(\mathbb{Z}) : \|A\| \leq T, \det(A) \text{ is prime}\}$$
$$= (1 + o(1))\zeta(3)^{-1} \int_{\|X\| \leq T} \frac{1}{\log^+(\det(X))} dX.$$

The End

Thank you!