Distribution of primes in 4 term APs

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(Joint work with Mayank Pandey)

§1 Small scale inequidistribution of the primes

Denote $\pi(X) = \#\{p \leq X : p \text{ prime}\}$

$\sim X/\log X$ (Prime number theorem)

What about for short intervals?

Expect:

$$\pi(X+H) - \pi(X) \sim \frac{H}{\log(X)} \text{ for } H = o(X)$$

[Cramér's Model]

Under RH: For $H \gg X^{1/2+\varepsilon}$

$$\pi(X+H) - \pi(X) \sim \frac{H}{\log(X)}$$

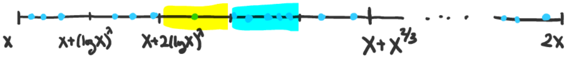
! this heuristic is not always true!

Thm (Maier, '85)

$$\limsup_{X\to\infty} \pi(X + (\log X)^{\lambda}) - \pi(X) > \log(X)^{\lambda-1}$$

$$\liminf_{X\to\infty} \pi(X + (\log X)^{\lambda}) - \pi(X) < \log(X)^{\lambda-1}$$

Picture:



Pf sketch:

Let $Q = P(z)$ to be determined later

Look at the table:

rows count # primes in intervals of length U

primes in APs

$$\left[\begin{array}{cccc}
\mathbb{1}_{P}(Qy+1) & \mathbb{1}_{P}(Qy+2) & \cdots\cdots & \mathbb{1}_{P}(Qy+U) \\
\mathbb{1}_{P}(Q(y+1)+1) & \mathbb{1}_{P}(Q(y+1)+2) & \cdots & \mathbb{1}_{P}(Q(y+1)+U) \\
\vdots & \vdots & \ddots & \vdots \\
\mathbb{1}_{P}(Q(2y)+1) & \mathbb{1}_{P}(Q(2y)+2) & \cdots & \mathbb{1}_{P}(Q(2y)+U)
\end{array}\right.$$

Consider: if $(a,Q) \neq 1$, $\mathbb{1}_{P}(Q(x+k)+a) = 0$

$\Rightarrow$ some of the columns are all zeros

How many columns are nonzero?

$\#\{a \in [1,U] : (a,Q) = (a,P(z)) = 1\}$ (# of rough numbers)

Def: (Buchstab function)

$$\omega : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 1} \quad \text{s.t.}$$

$$\omega(u) = \frac{1}{u}, \quad 1 \leq u \leq 2$$

$$\frac{d}{du}(u\,\omega(u)) = \omega(u-1), \quad u \geq 2$$

Fact:

$\omega(u) - e^{-\gamma}$ changes sign infinitely often

And:

$$\#\{n \in [1,U] : p \mid n \Rightarrow p > z\}$$

$$\sim \underbrace{e^{\gamma}\,\omega\left(\frac{\log U}{\log z}\right)} \cdot U \cdot \prod_{p \leq z}\left(1 - \frac{1}{p}\right) \longleftarrow$$

number of non zero columns

approaches 1 as $\log U / \log z \to \infty$

also is >1 and <1 infinitely often

How many nonzero elements are there in a "good" column?

$\#\{n \in [y,2y] : Qn+a \text{ is prime}\}$ (Primes in arithmetic progressions)

Thm (Gallagher, '70) no siegel zeros

For Q a good modulus, $x \geq Q^D$, $(a,Q)=1$,

$$\pi(2x, Q, a) - \pi(x, Q, a) \sim \frac{1}{\varphi(Q)} \frac{x}{\log(x)}$$

Fact: there are infinitely many z s.t. $P(z)$ is a good modulus.

So, for $y \geq Q^D$, $\#\{n \in [y, 2y] : Qn+a \text{ prime}\}$

$$\sim \frac{Q}{\varphi(Q)} \frac{y}{\log y}$$

So, the number of nonzero entries in the table is

$$\Sigma = \sum_{(a,Q)=1} \frac{1}{\varphi(Q)} \frac{y}{\log(y)} \sim e^{\gamma} \omega\left(\frac{\log U}{\log z}\right) \cdot \frac{yU}{\log(y)}$$

$$= \sum_{n \in [y, 2y]} \pi(Qn+U) - \pi(Qn)$$

So, $\exists n \in [y, 2y]$ s.t.

$$\pi(Qn+U) - \pi(Qn) \geq e^{\gamma} \omega\left(\frac{\log U}{\log z}\right) \frac{U}{\log(y)}$$

Now take

$$y = Q^D \Rightarrow X = Q^{D+1} = P(z)^{D+1}$$

$$U = (\log X)^{\lambda}$$

Since $e^{\gamma}\omega(u)$ changes sign ∞ often,
$\exists$ a sequence of z_i's and D_i's s.t.
$\leftarrow P(z_i)$ good

$$e^{\gamma} \omega\left(\frac{\log U_i}{\log z_i}\right) > 1. \qquad \square$$

History:

- sums of two squares (Balog-Wooley, '00)
- primes in Beatty sequences (Tongsomporn-Steuding, '20)

§2 The Case of 3-APs

Thm (Vinogradov)

$$\underbrace{\sum_{x,y\in[X,2X]} 1_p(x)1_p(x+y)1_p(x+2y)}_{\pi_3([X,2X]^2)} \sim \frac{2X^2}{(\log X)^3}\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^3}\right)$$

Thm (Zhan, '92)

For $H \gg X^{5/8+\varepsilon}$,

$$\pi_3\left([X,X+H]^2\right) \sim \frac{2H^2}{(\log X)^3}\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^3}\right)$$

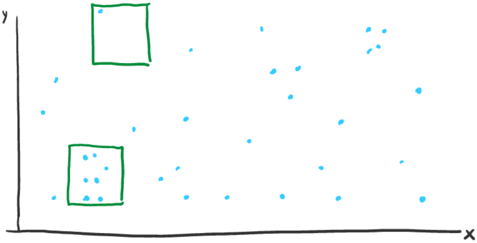
Thm (Pandey–W., '23+)

Let H denote $(\log X)^\lambda$. Then $\exists\, \delta_\lambda^+, \delta_\lambda^- > 0$ and $x_\pm, y_\pm \in [X,2X]$ such that

$$\pi_3\left([x_+,x_++H]\times[y_+,y_++H]\right) > (1+\delta_\lambda^+)\frac{H^2}{(\log X)^3}\, 2\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^3}\right)$$

$$\pi_3\left([x_-,x_-+H]\times[y_-,y_-+H]\right) < (1-\delta_\lambda^-)\frac{H^2}{(\log X)^3}\, 2\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^3}\right)$$

Picture:



Pf sketch:

For Maier's matrix method, we need

① Primes number theorems for good but large moduli

② Counts for the number of admissible residues

① For $Q = P(z)$ good:

$$\sum_{\substack{x,y\sim X\\ (x,y)\equiv(a,b) \bmod Q}} \Lambda(x)\Lambda(x+y)\Lambda(x+2y) \sim \frac{2X^2}{Q^2}\prod_{p\leq z}\left(1-\frac{1}{p}\right)^{-3}$$

Circle method $\Rightarrow$ suffices to get bounds on

$$\sum_{\substack{n\sim X\\ n\equiv a(Q)}} \left(\Lambda(n)-\frac{Q}{\varphi(Q)}\right)e(n\alpha) \quad \text{for all } \alpha\in\mathbb{R}$$

Major + minor arcs:

write $$\left|\alpha - \frac{a}{qQ}\right| \le \frac{Q(\log X)^{O(1)}}{X}, \quad (a,q)=1$$

If $q > (\log X)^B$, α is in the minor arc

else, α is in the major arc

① Minor arcs:

Use a Type I/II decomposition of Λ and proceed in standard way as from Vinogradov's proof

② Major arcs:

What does "good" mean?

Def: a modulus Q is good if $\forall\, \chi \bmod Q$, we have that

$$\sigma > 1 - \frac{c}{\log(Q|t|+1)} \Rightarrow L(\sigma+it,\chi) \neq 0.$$

Assume $Q = P(z)$ for $z \sim (\log X)^{1-\varepsilon}$

Claim: if $q < (\log X)^{\varepsilon/2}$, then $\underbrace{qQ}$ is still good.

$q \le z \Rightarrow$ zeros of the real character mod qQ

$\updownarrow$

zeros of the real character mod Q/q

Case: qQ is good $\Rightarrow$

$$\sum_{\substack{n\sim X\\ n\equiv b(Q)}} \mu(n)\, e\!\left(\frac{na}{qQ}\right) \ll \frac{X}{Q(\log X)^{O(1)}}$$

Case: qQ has a Siegel zero $\Rightarrow$

① $q > (\log X)^{\varepsilon/2}$

② there exists at most one $\hat{q}$ s.t. $\hat{q}Q$ (and multiples) will have a Siegel zero

Write: (assume $(q,Q)=1$)

$$\sum_{\substack{n\sim X\\ n\equiv b(Q)}} \left(\Lambda(n) - \frac{Q}{\varphi(Q)}\right) e\!\left(\frac{na}{qQ}\right) = \frac{1}{\varphi(Qq)} \sum_{\substack{\psi(Q)\\ \chi(q)}} \overline{\psi(b)} \left|\sum_{c(q)} \overline{\chi(c)}\, e\!\left(\frac{a(Qc+b)}{qQ}\right)\right|$$

$$\times \left|\sum_{n\sim X} \left(\Lambda(n) - \frac{Q}{\varphi(Q)}\right) \psi\chi(n)\right|$$

$$\ll \underset{\overset{\uparrow}{\text{from good characters } \psi, \chi}}{o(X/Q)} + \frac{X}{\varphi(Qq)} \Big| \underbrace{\sum_{c(q)} \chi(c)\, e\!\left(\tfrac{ac}{q}\right)}_{\text{Gauss sum } \ll q^{1/2}} \Big|$$

$$\ll o(X/Q) + \frac{X}{Q} q^{-1/2}$$

Since $q > (\log X)^{\varepsilon/2}$, this is $o(X)$.

② Admissible tuples estimate:

$$\sum_{\substack{x \sim X \\ y \sim Y}} \mathbb{1}_{(\cdot, P(z))=1}(x)\, \mathbb{1}_{(\cdot, P(z))=1}(x+y)\, \mathbb{1}_{(\cdot, P(z))}(x+2y)$$

$$\sim \frac{2XY}{(\log z)^3}\, \omega\!\left(\frac{\log X}{\log z}\right) \omega\!\left(\frac{\log (X+Y)}{\log z}\right) \omega\!\left(\frac{\log (X+2Y)}{\log z}\right) \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^3}\right)$$

③ Feeding into the Maier matrix method:

Take $U = z^{\alpha}$, $z \sim (\log X)^{1-\varepsilon}$

Consider the matrix

$$\mathcal{M} = \left\{ \mathbb{1}_p(r_1 + s_1 P(z))\, \mathbb{1}_p\big(r_1 + r_2 + (s_1+s_2)P(z)\big)\, \mathbb{1}_p\big(r_1 + 2r_2 + (s_1 + 2s_2)P(z)\big) \quad \begin{matrix} r_1, r_2 \in [1, U] \\ s_1, s_2 \in [X/P(z), 2X/P(z)] \end{matrix} \right\}$$

Count the number of nonzero entries:

$$\sum = \sum_{\substack{\text{admissible} \\ r_1, r_2}} \sum_{x, y \equiv (r_1, r_2)\, (P(z))} \mathbb{1}_p(x)\, \mathbb{1}_p(x+y)\, \mathbb{1}_p(x+2y)$$

$$= \frac{U^2 X^2}{P(z)^2}\, 2 \cdot \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^3}\right) \cdot \underbrace{e^{3\gamma} \omega(\alpha)^3}_{>1 \text{ and } <1\ \infty \text{ often}}$$

So, there exists a (s_1, s_2) where the box contains a larger than expected number of primes

§3 The case of 4-APs

Q: How often is

$$x,\ x+y,\ x+2y, \text{ and } x+3y \text{ prime?}$$

Heath-Brown '81:

There are infinitely many 4-term APs

$$\begin{array}{cccc} x, & x+y, & x+2y, & x+3y \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \mathcal{P} & \mathcal{P} & \mathcal{P} & \mathcal{P} \text{ or } \mathcal{P}\times\mathcal{P} \end{array}$$

Green-Tao, '10:

The number of 4-APs in primes $\leq N$ is

$$\sim \frac{3}{4} \prod_{p \geq 5} \left(1 - \frac{3p-1}{(p-1)^3}\right) \cdot \frac{N^2}{\log^4 N}$$

In fact, they show something stronger:

Def: Let $\Psi = (\psi_1, \dots, \psi_t)$ be a system of linear forms. Let $\hat{\psi}_i$ denote the homogeneous part of ψ_i. Then Ψ has finite complexity if $\forall i \in \{1, \dots, t\}$,

$\{\psi_1, \dots, \hat{\psi}_i, \dots, \psi_t\}$ can be partitioned into classes where $\hat{\psi}_i$ is not in the linear span of any of the classes.

Ex: k-AP

$x,\ x+y,\ x+2y, \dots\dots, x+(k-1)y$

Put each in its own class and $x+jy$ is independent from $x+ly$ for $i \neq j$

Non-ex: twin primes

$x,\ x+2$ ← homogeneous part is also x

Thm (Green-Tao, '10)

$$\sum_{\vec{x} \in [N]^d} \prod_{i=1}^{t} \Lambda(\psi_i(\vec{x})) = N^d \prod_p \beta_p \left(1 + o_\Psi(1)\right)$$

where
$$\beta_p = p^{-d} \sum_{\vec{x} \in \mathbb{F}_p^d} \prod_{i=1}^{t} \frac{p}{\varphi(p)} \mathbb{1}_{(\psi_i(\vec{x}), p)=1}$$

What is the analog of short intervals?

small boxes $x_1, \dots, x_d \sim X$

$$\vec{x} \in [x_1, x_1+H] \times \dots \times [x_d, x_d+H]$$

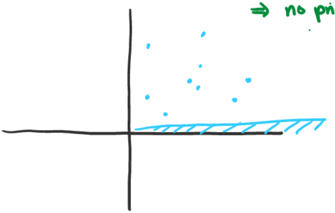
Why should we expect equidistribution?

because the forms are linear!

Ex: take $f(m,n) = m^2+n^2$ ← like lattice pts of $\mathbb{Z}[i]$

there are no integral solutions in $\arg(m+ni) \leq \frac{1}{X}$

⇒ no prime solutions either



Further evidence:

Thm (Matomäki, Shao, Tao, and Teräväinen, '22)

For $H \gg X^{5/8+\varepsilon}$ and Ψ a system of linear forms with finite complexity. Then

$$\sum_{\vec{x} \in [X, X+H]^d} \prod_{i=1}^{t} \Lambda(\psi_i(\vec{x})) = H^d \prod_p \beta_p \left(1 + o_\Psi(1)\right)$$

Thm (Pandey-W., '23+)

Let $\lambda > 1$ and write $H = (\log X)^\lambda$. Then $\exists\, \delta_\lambda^+, \delta_\lambda^- > 0$, and $\exists\, \vec{x}_\pm \in [X, 2X]^d$ s.t.

$$\sum_{\vec{x} \in \prod_{i=1}^{d} [x_i^+, x_i^+ + H]} \prod_{i=1}^{t} \mathbb{1}_{\mathcal{P}}(\psi_i(\vec{x})) \geq (1+\delta_\lambda^+) \frac{H^d}{(\log X)^t} \prod_p \beta_p$$

and

$$\sum_{\vec{x} \in \prod_{i=1}^{d} [x_i^-, x_i^- + H]} \prod_{i=1}^{t} \mathbb{1}_{\mathcal{P}}(\psi_i(\vec{x})) \leq (1-\delta_\lambda^-) \frac{H^d}{(\log X)^t} \prod_p \beta_p$$

Ex: the case of 4-APs

Let $\lambda > 1$ and $H = (\log X)^\lambda$. Then $\exists\, \delta > 0$, $\exists\, x^+, y^+ \sim X$ s.t.

$$\sum_{\vec{x} \in [x^+, x^+ + H] \times [y^+, y^+ + H]} \mathbb{1}_{\mathcal{P}}(x) \mathbb{1}_{\mathcal{P}}(x+y) \mathbb{1}_{\mathcal{P}}(x+2y) \mathbb{1}_{\mathcal{P}}(x+3y)$$

$$> (1+\delta) \frac{H^2}{(\log X)^4} \cdot \frac{3}{4} \prod_{p \geq 5} \left(1 - \frac{3p-1}{(p-1)^3}\right)$$

Ingredients of Maier matrix method:

① Good moduli: Take $Q = P(z)$

Call Q "good" if for all $q \leq (\log Q)^{\varepsilon}$, and characters χ mod qQ satisfy a good zero-free region.

Def: $\mathcal{A} = \{(a_1, a_2) \bmod Q : (a_1, Q) = (a_1 + a_2, Q) = (a_1 + 2a_2, Q) = (a_1 + 3a_2, Q) = 1\}$

"admissible residues"

Prop: For Q a good modulus, $\vec{a}$ an admissible residue

$$\sum_{\substack{\vec{x} \in [X]^2 \\ \vec{x} \equiv \vec{a}\,(Q)}} \prod_{i=1}^{t} \Lambda(\psi_i(\vec{x})) = \frac{1}{\#\mathcal{A}} X^2 \prod_p \beta_p (1 + o(1))$$

similar sums studied by Green and Tao for $\mathbb{1}_{\mathcal{P}}$ and by Matthiesen for various arithmetic functions

② Number of admissible tuples:

Prop:
$$\sum_{\vec{x} \in [U]^2} \prod_{i=1}^{t} \mathbb{1}_{(\cdot, P(z))=1}(\psi_i(\vec{x})) = \frac{\#\mathcal{A}}{P(z)^2} \left(e^{\gamma} \omega\left(\tfrac{\log U}{\log z}\right)\right)^4 U^2$$

Input into Maier matrix:

$$\mathcal{M} = \left\{ \mathbb{1}_{\mathcal{P}}(r_1 + s_1 Q)\, \mathbb{1}_{\mathcal{P}}(r_1 + r_2 + (s_1 + s_2)Q)\, \mathbb{1}_{\mathcal{P}}(r_1 + 2r_2 + (s_1 + 2s_2)Q)\, \mathbb{1}_{\mathcal{P}}(r_1 + 3r_2 + (s_1 + 3s_2)Q) \right\}_{\substack{r_1, r_2 \sim U \\ s_1, s_2 \sim X/Q}}$$

Then the # of nonzero entries:

$$\sum_{\substack{r_1, r_2 \sim U \\ \text{admissible}}} \sum_{\substack{x, y \sim X \\ (x, y) \equiv (r_1, r_2)(Q)}} \prod_{j=0}^{3} \mathbb{1}_{\mathcal{P}}(x + jy) \sim \frac{X^2 U^2}{Q^2 (\log X)^4} \left(e^{\gamma} \omega\left(\tfrac{\log U}{\log z}\right)\right)^4 \prod_p \beta_p$$

$\rightsquigarrow \exists\, s_1, s_2$ s.t.

$$\#\left\{ r_1, r_2 \sim U : \begin{matrix} Qs_1 + r_1 \\ Q(s_1 + s_2) + r_1 + r_2 \\ Q(s_1 + 2s_2) + r_1 + 2r_2 \\ Q(s_1 + 3s_2) + r_1 + 3r_2 \end{matrix} \text{ are prime} \right\}$$

$$\geq \left(e^{\gamma} \omega\left(\tfrac{\log U}{\log z}\right)\right)^4 \frac{U^2}{(\log X)^4} \prod_p \beta_p$$

□

§4 The Green-Tao machinery

Def: ν is a pseudorandom measure if

- $\mathbb{E}\nu = 1 + o(1)$
- For any set of linear forms of bounded height and linearly independent

$$\mathbb{E} \prod \nu(L_i(x)) = 1 + o(1)$$

- $\exists\ \tau : \mathbb{Z} \to \mathbb{R}^+$ s.t.

$$\mathbb{E}\, \tau^q(x) = O_q(1)$$

and for $h_1, \dots, h_m$,

$$\mathbb{E} \prod \nu(x+h_i) \le \sum_{i<j} \tau(h_i - h_j)$$

correlation condition can now be dropped

Motivation:

Want to look at $f : \mathbb{Z} \to \mathbb{R}$ that are unbounded

- lots of Cauchy-Schwartz is good when $|f| \le 1$
  but we want to substitute $|f| \le \nu$ since $f = \Lambda$ is unbounded

Def: Let $\Delta_h f(x) = f(x+h)\overline{f(x)}$.

The Gowers norm of f is

$$\|f\|_{U^s}^{2^s} := \mathbb{E}_{x, h_1, \dots, h_s} \Delta_{h_1} \dots \Delta_{h_s} f(x)$$

Von Neumann Thm: Let $\Psi = (x, x+y, x+2y, x+3y)$
ν a pseudorandom measure, $|f_i| \le \nu$.
If $\min \|f_j\|_{U^3} \le \delta$, then

$$\sum_{x,y \sim X} \prod_{i=0}^{4} f_0(x+iy) = O(\delta^c X^2)$$

Pf is lots of Cauchy-Schwartz + properties of ν

Motivation:

$$\|f\|_{U^2(\mathbb{Z}/N)} = \Big(\sum_{\xi \in \widehat{\mathbb{Z}/N}} |\hat{f}(\xi)|^4 \Big)^{1/4}$$

where $\hat{f}(\xi) = \mathbb{E}_{x \in \mathbb{Z}/N} f(x)\, e(-\xi(x))$

So, if $\|f\|_{U^2(\mathbb{Z}/N)} \gg 1$, then $\exists\ \xi$ s.t. $\hat{f}(\xi)$ is large.

$$\Rightarrow \sum_{n \sim N} f(n)\, e(n\alpha) \gg N \text{ for some } \alpha.$$

Recall for 3-APs, it came down to showing

$$\sum_{\substack{n \sim N \\ n \equiv a(Q)}} \Big(\Lambda(n) - \frac{Q}{\varphi(Q)} \Big) e(n\alpha) = o(N/Q) \ \forall\, \alpha \in \mathbb{R}.$$

What about for 4-APs?

Looking at U^3 norm $\rightsquigarrow$ need more than exponentials

Def: G is a simply connected Lie group

$\underset{\underset{G}{\parallel}}{G_0} \geq \underset{\underset{G}{\parallel}}{G_1} \geq G_2 \geq \dots$ a filtration. Then G is s-step nilpotent if $G_{s+1}=1$

Def: If $\Gamma \subset G$ is a discrete compact subgroup. Then G/Γ is a s-step nilmanifold.

Ex: $G=\mathbb{R}$, $\Gamma=\mathbb{Z}$ $\rightsquigarrow$ $\mathbb{R}/\mathbb{Z}$ torus is a 1-step nilmanifold

In fact all 1-step nilmanifolds are $\cong \mathbb{R}/\mathbb{Z}$

→ for the U^2 norm, it suffices to look at exponentials $e(n\alpha)$

Ex: $G=\begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \Gamma=\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$

Then $G/\Gamma \leftrightarrow \begin{pmatrix} 1 & \{x\} & \{y - x\lfloor z \rfloor\} \\ 0 & 1 & \{z\} \\ 0 & 0 & 1 \end{pmatrix}$ is a 2-step nilmanifold

Heisenberg

So for the U^3 norm, we also have to consider

$$e(\alpha n \lfloor \beta n \rfloor)$$

Inverse Thm for the Gowers Norm (Green-Tao '10, Manners '18)

Let ν be pseudorandom and $f \leq \nu$.

If $\|f\|_{U^{s+1}} \geq \delta$, then $\exists$ a s-step nilmanifold G/Γ of complexity $\ll \exp\exp O(\delta^{-O(1)})$, a Lipschitz function $F: G/\Gamma \to \mathbb{C}$ and a polynomial map $g: \mathbb{Z} \to G$ s.t.

$$\mathbb{E}\, f(n)\, \underbrace{F(g(n)\Gamma)}_{\uparrow \text{nilsequence}} \gg \exp\exp(-O(\delta^{-O(1)})).$$

In other words, if the U^3 norm is large, then f correlates with an exponential-like function.

Our two propositions reduce to:

$$(1) \quad \sum_{\substack{n \sim X \\ n \equiv b\,(Q)}} \left(\Lambda(n) - Q/\varphi(Q)\right) F(g(n)\Gamma) = o(X/Q)$$

for Q a good but large modulus.

$$(2) \quad \sum_{n \sim N} (\log z)\, \mathbb{1}_{(\cdot, P(z))=1}(n) - \omega\left(\frac{\log N}{\log z}\right) F(g(n)\Gamma) = o(N)$$

Notes about the proof of (1):

3-APs	4-APs
(1) Split into major minor arcs w/ $\left\lvert \alpha - \frac{a}{qQ} \right\rvert \leq \frac{Q(\log X)^{O(1)}}{X}$	Use Q-factor theorem on $F(g(n)\Gamma)$: We can write $g(n) = \xi(n) g'(n) \gamma(n)$ • $\xi(n)$ is close to constant as n-varies • $g'(n)$ is totally-equidistributed • $\gamma(n)$ is qQ periodic with $q < (\log X)^3$ Minor: g' is nontrivial Major: $g' = 1$
(2) Use a Type I/II decomposition for the minor arcs	Use Type I/II decomposition of Λ follows like in Green-Tao
(3) Major arcs when there is a good zero-free region	Since essentially we have $F(g(n)\Gamma) \approx F(\gamma(n)\Gamma)$ get $\sum_{n \equiv b (Q)} \left(\Lambda(n) - \frac{Q}{\varphi(Q)}\right) F(\gamma(n)\Gamma)$ $\approx \frac{1}{\varphi(qQ)} \sum_{\substack{\chi (q) \\ \varphi (Q)}} \overline{\varphi(b)} \sum_{a (q)} \bar{\chi}(a) F(\gamma(Qa+b)\Gamma)$ $\times \underbrace{\sum_{n \sim N} \left(\Lambda(n) - \frac{Q}{\varphi(Q)}\right) \chi\varphi(n)}$ for good $\chi\varphi$, get savings from this
(4) Saving from Gauss sum when there is a Siegel zero	Again have $q > (\log X)^2$. $A(\chi) = \sum_{a (q)} \chi(a) F(\gamma(a)\Gamma)$ ← analog of Gauss sum $\ll q^{1 - c 2^{-s}}$ Inverse theorem for Gowers norm over finite fields $\rightsquigarrow$ get saving of $(\log X)^{-\varepsilon c 2^{-s}}$