AN EXPONENTIAL SUM INVOLVING THE DIVISOR FUNCTION FOR $\mathbb{F}_q[T]$

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Abstract. In this note, we evaluate the weighted exponential sum

$$\sum_{\substack{F\in\mathbb{F}_q[T]\\ F \text{ monic}\\ \deg(F)=n}} d(F)e(F\theta),$$

where $e(\cdot)$ is the function field version of the exponential function, and θ a Laurent series in $1/T$. Using this formula, we find an expression for the L^s-norm of the above sum for $\Re(s) > 0$; for $s = 1$ and $s = 2$, we derive exact formulas for the L^s norm. We additionally obtain an exact formula for the following shifted divisor sum for $\deg(A) \leq n$:

$$\sum_{\substack{F\in\mathbb{F}_q[T]\\ F \text{ monic}\\ \deg(F)=n}} d(F)d(F+A).$$

1. Introduction

Let $\mathcal{M}_n$ be the set of monic polynomials in $\mathbb{F}_q[T]$ of degree n and let θ be a Laurent series in $1/T$. Specifically, θ is a series of the form

$$\theta = \sum_{n=-\infty}^{\infty} a_n T^n, \tag{1}$$

where a_n is nonzero for only finitely many positive n. In this note, we will find an exact evaluation of the sum

$$\sum_{F\in\mathcal{M}_n} d(F)e(F\theta), \tag{2}$$

where e is the exponential function defined in Definition 1.3. We will also look at some consequences of the expression. This sum is an analog of the following expression over the integers:

$$\sum_{n\leq X} d(n)e(n\alpha),$$

for $\alpha \in \mathbb{R}$ and $e(x) = \exp(2\pi i x)$. In [10], Wilton finds a similar expression for the above sum using Voronoi summation; however, our expression is simpler to due exact cancellation in our exponential sums from a lemma by Hayes [3]. First, we will set up the notation necessary from the function field setting.

1.1. **The function field setting.** We will first introduce some common notation, definitions, and functions in the function field setting; for more details, see [8] and [3]. We use $\mathcal{M}$ to denote the set of monic polynomials in $\mathbb{F}_q[T]$, and $\mathcal{M}_n$ the monic polynomials of degree n. For $F \in \mathbb{F}_q[T]$, we write $|F| := q^{\deg(F)}$ to be the norm of the polynomial (let $|0| := 0$). We will also work with the field of rational functions over $\mathbb{F}_q$, which we denote as

$$\mathbb{F}_q(T) = \left\{\frac{A}{B} : A, B \in \mathbb{F}_q[T], B \neq 0\right\}.$$

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We remark that elements of $\mathbb{F}_q(T)$ can be expressed as Laurent series in $1/T$, i.e. in the form of (1). We define the analogue of the unit interval $[0,1]$ in the function field setting as the following:

$$\mathscr{U} := \left\{ \sum_{i<0} a_i T^i : a_i \in \mathbb{F}_q \right\}. \tag{3}$$

We note that this is the natural analogue, as any Laurent series, including the elements in $\mathbb{F}_q(T)$, can be written as the sum of a polynomial and an element of the unit interval. We recall that there is a valuation on the Laurent series in $1/T$ given by:

$$v\left(\sum_{i\in\mathbb{Z}} a_i T^i\right) := -\left(\text{largest } i \text{ such that } a_i \neq 0\right).$$

Let us take $v(0) = \infty$. We note that v satisfies the following properties: for $A, B \in \mathbb{F}_q[T]$ and α, β Laurent series in $1/T$, we have that

- $v(A/B) = \deg(B) - \deg(A)$,
- $v(\alpha\beta) = v(\alpha) + v(\beta)$,
- $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$.

Finally, let $r \in \mathbb{R}_{\geq 0}$ and a be any Laurent series in $1/T$. Then, we define a ball of radius r around a to be $B_r(a) := \{\theta \text{ Laurent series in 1/T} : v(\theta - a) > r\}$. We take the Haar measure ϱ satisfying that for any a and $r \geq 0$, we have that $\varrho(B_r(a)) = q^{-r}$.

Next, we want the following analogue of Dirichlet's theorem on rational approximation.

Definition 1.1. For polynomials $G, H \in \mathbb{F}_q[T]$, the fraction G/H is *primordial* for n if

(1) $G/H \in \mathscr{U}$.
(2) $(G, H) = 1$.
(3) H is monic.
(4) $\deg(H) \leq n/2$.

We will denote the set of primordials for n as $\mathscr{V}_n$.

These primordial fractions give us a way of decomposing the unit interval into subintervals close to a rational fraction with small denominators.

Lemma 1.2 (Hayes, Theorem 4.3)**.** *For any $n \geq 0$, as G/H ranges in $\mathscr{V}_n$, $\{\mathscr{U}_{n,G/H}\}$ is a disjoint open cover of $\mathscr{U}$, where*

$$\mathscr{U}_{n,G/H} := \{\theta \in \mathscr{U} : v(\theta - G/H) > \deg(H) + \lfloor n/2 \rfloor\}.$$

Finally, we need to define the exponential function. Let us fix some nonprincipal additive character ψ of $\mathbb{F}_q$. We can then define the exponential function for function fields.

Definition 1.3 (The exponential function)**.** For $\sum_{i\in\mathbb{Z}} a_i T^i$ a Laurent series in $1/T$, our exponential function is defined as

$$e\left(\sum_{i\in\mathbb{Z}} a_i T^i\right) := \psi\left(a_{-1}\right).$$

1.2. **Statement of results.** We first calculate a nice expression for the exponential divisor sum above (2) using the hyperbola trick.

Theorem 1.4. *For $\theta \in \mathscr{U}_{n,G/H}$, where G/H is primordial for n,*

$$\sum_{F\in\mathcal{M}_n} d(F)e(F\theta) = \begin{cases} q^{n-\deg(H)} e(T^{n-\deg(H)} H\theta)(n + 1 - 2\deg(H)), & \textit{if } \theta \in B_n(G/H) \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 1.5. The expression above may be regarded as an analog of the Voronoi summation step applied in [7].

Next, we look at the L^s-norm of the divisor sum. Recently, Pandey [7] studied the L^s-norm of the integral version of the sum, showing that for any $0 < s < 2$, there exists an explicit $\epsilon_s > 0$ and C_s a positive constant such that

$$\int_0^1 \left| \sum_{n \leq x} d(n) e(n\alpha) \right|^s d\alpha = C_s x^{s/2} + O(x^{s/2 - \epsilon_s}).$$

In this note, we also compute the L^s-norm for $\Re(s) > 0$ in the function field setting. We note that this problem over integers is related to computing the L^s-moments for exponential sums weighted with coefficients of cusp forms; these were studied by Pandey in [7] using Jutila's circle method, by Jurkat and Van Horne in [5] using theta functions, and by Marklof in [6] using ergodic theory. We use the exact formula of Theorem 1.4 to derive a result for the L^s-norm in the function field setting.

Corollary 1.6. *For $s > 0$, we have that*

$$\int_{\mathscr{U}} \left| \sum_{F \in \mathcal{M}_n} d(F) e(F\theta) \right|^s d\varrho(\theta) = (n+1)^s q^{n(s-1)} + q^{n(s-1)} (1 - q^{-1}) \sum_{d=1}^{\lfloor n/2 \rfloor} \frac{(n+1-2d)^s}{q^{(s-2)d}}.$$

Consequently, for $s = 1$, we have that

$$\int_{\mathscr{U}} \left| \sum_{F \in \mathcal{M}_n} d(F) e(F\theta) \right| d\varrho(\theta) = \left(n - 2\lfloor n/2 \rfloor + \frac{q+1}{q-1} \right) q^{\lfloor n/2 \rfloor} - \frac{2}{q-1},$$

and for $s = 2$, we have that

$$\int_{\mathscr{U}} \left| \sum_{F \in \mathcal{M}_n} d(F) e(F\theta) \right|^2 d\varrho(\theta)$$
$$= q^n \cdot \left((n+1)^2 + (1 - q^{-1}) \left(-2n\lfloor n/2 \rfloor^2 + \frac{4}{3} \lfloor n/2 \rfloor^3 + n^2 \lfloor n/2 \rfloor - \frac{1}{3} \lfloor n/2 \rfloor \right) \right).$$

Remark 1.7. We note that as a consequence of the formula above, we know that for $s < 2$, the L^s-norm grows as $q^{sn/2}$ as $n \to \infty$. On the other hand, for $s > 2$, we get that the L^s-norm will grow as $n^s q^{n(s-1)}$ as $n \to \infty$.

Finally, we consider correlations of the divisor function sum. A computation of Ingham in [4] gives the asymptotic formula as $X \to \infty$:

$$\sum_{n \leq X} d(n) d(n+h) \sim \frac{6}{\pi^2} \sigma_{-1}(h) X \log(X)^2,$$

where $\sigma_{-1}(h) = \sum_{e|h} e^{-1}$. We consider the analogous sum over function fields. We note that both Conrey and Florea [1], and separately Gorodetsky [2], have derived exact formulas for this sum. In [2], Gorodetsky shows that for $A \in \mathbb{F}_q[T]$ with $\deg(A) < n$,

$$\frac{1}{q^n} \sum_{F \in \mathcal{M}_n} d(F) d(F+A) = (n+1)^2 + \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{(n-2i+1)^2}{q^i} \left(d_i(A) - d_{i-1}(A) \right),$$

where $d_i(A)$ denotes the number of monic divisors of A of degree i. Separately from Gorodetsky, Conrey and Florea also derive an exact formula for the above sum when $\deg(A) < n/2$ and an asymptotic with error term $O(q^{n/2+\epsilon})$ for $\deg(A) < n$.

Corollary 1.8. *Let $A \in \mathbb{F}_q[T]$ satisfy that either $\deg(A) < n$ or $\deg(A) = n$ and the leading coefficient of A is not -1. Then we have that*

$$\sum_{F \in \mathcal{M}_n} d(F)d(F+A) = n^2 q^n \sigma(A, n/2),$$

where $\sigma(A, n/2)$ is defined as

$$\sum_{\substack{B \in \mathcal{M}_{\lfloor n/2 \rfloor} \\ B|A}} \frac{(1+1/n-2\lfloor n/2 \rfloor/n)^2}{|B|} + \sum_{d=0}^{\lfloor n/2 \rfloor -1} \sum_{\substack{B \in \mathcal{M}_d \\ B|A}} \frac{1}{|B|} \cdot ((1+1/n-2d)/n)^2 - q^{-1}(1-1/n-2d/n)^2).$$

Remark 1.9. We can think of the factor $\sigma(A, n/2)$ as an analogue of the factor $\frac{\sigma_{-1}(h)}{\zeta(2)}$ from the integer setting. Recall that the zeta function over function fields is defined as:

$$\zeta_q(s) = \sum_{F \in \mathcal{M}} |F|^{-s}$$

For a fixed A, we have that as $n \to \infty$,

$$\sigma(A, n/2) \to (1 - q^{-1}) \sum_{\substack{B \in \mathcal{M} \\ B|A}} |B|^{-1} = \frac{\sigma_{-1}(A)}{\zeta_q(2)},$$

which parallels the asymptotic in the integer setting. Additionally, the above result matches the one of Gorodetsky in [2].

Remark 1.10. When considering A with $\deg(A) > n$, an issue arises from having to take different choices of primordial decompositions of $\mathscr{U}$ (one of degree n, and the other of $\deg(A)$). A similar problem appears when looking at replacing the divisor function in (2) with

$$d_k(F) = \#\{(G_1, ..., G_k) \in \mathbb{F}_q[T]^k : G_1 ... G_k = F, G_i \text{ monic}\}.$$

2. Proof of Theorem 1.4

We will invoke the following result of Hayes [3] during the calculation of the sum.

Lemma 2.1 (Hayes, Theorem 3.7)**.** *Let θ be a Laurent series in $1/T$ and $\theta' \in \mathscr{U}$ such that $\theta = \theta' + A$, for $A \in \mathbb{F}_q[T]$. Then for any positive integer s, we have that*

$$\sum_{F \in \mathcal{M}_s} e(F\theta) = \begin{cases} q^s e(T^s \theta), & v(\theta') > s \\ 0, & \textit{otherwise}. \end{cases}$$

Proof of Theorem 1.4. We first rewrite our desired sum:

$$\sum_{F \in \mathcal{M}_n} d(F) e(F\theta) = \sum_{AB \in \mathcal{M}_n} e(AB\theta) = 2 \sum_{\substack{\deg(A) < n/2 \\ A \text{ monic}}} \sum_{B \in \mathcal{M}_{n-\deg(A)}} e(AB\theta) + \sum_{A,B \in \mathcal{M}_{n/2}} e(AB\theta).$$

In the expression above, the second sum on the right hand side only exists if n is even. Let us denote the first sum as $S_1(\theta)$ and the second sum as $S_2(\theta)$.

First, we want to analyze $S_1(\theta)$. Since $\theta \in \mathscr{U}_{n,G/H}$, we have that $\theta = G/H + \beta$, where $(G, H) = 1$ and $v(\beta) > \lfloor n/2 \rfloor + \deg(H)$. Now, we can write for a fixed A,

$$A\theta = AG/H + A\beta.$$

Let us define $A' \in \mathbb{F}_q[T]$ with $\deg(A') < \deg(H)$ such that $AG \equiv A' \mod H$. Then we know that

$$e(AB\theta) = e(B(A'/H + A\beta)).$$

Furthermore, we can see that $A'/H + A\beta \in \mathscr{U}$, since $v(A\beta) = v(A) + v(\beta) > \deg(H)$.

We can now apply Lemma 2.1 to the inner sum of $S_1(\theta)$ to get the expression

$$S_1(\theta) = \sum_{\substack{\deg(A)<n/2 \\ A \text{ monic}}} \begin{cases} q^{n-\deg(A)} e(T^{n-\deg(A)} A\theta), & v(A'/H + A\beta) > n - \deg(A) \\ 0, & \text{otherwise.} \end{cases}$$

Now, we want to determine the A such that $v(A'/H + A\beta) > n - \deg(A)$. First, since $\deg(A) < n/2$, $v(A\beta) = v(A) + v(\beta) > \deg(H)$. If $A' \neq 0$, we will have that $v(A'/H) \leq \deg(H)$. In this case, we can see that $v(A\theta) \leq \deg(H) \leq n/2$. However, since $\deg(A) < n/2$, $n - \deg(A) > n/2$. So, these terms have no contribution to the sum since $v(A\theta) < n - \deg(A)$.

On the other hand, if $A' = 0$, we can see that $v(A'/H + A\beta) = v(A\beta) = v(\beta) - \deg(A)$. The condition that this is greater than $n - \deg(A)$ is exactly that $\theta \in B_n(G/H)$. So, we can see that the sum is 0 if $\theta \notin B_n(G/H)$. So, from now on, we will assume that $\theta \in B_n(G/H)$. Since $(G, H) = 1$, if $A' = 0$, then H divides A. So, we restrict the sum over A to a sum over multiples of H:

$$S_1(\theta) = \sum_{0 \leq d < n/2 - \deg(H)} q^{n-d-\deg(H)} \sum_{A'' \in \mathcal{M}_d} e(T^{n-d-\deg(H)} A'' H\theta).$$

We note that $v(X^{n-d-\deg(H)} H\theta) > d$ since $\theta \in B_n(G/H)$. So, applying Lemma 2.1 again, we get that the sum above becomes

(4)
$$\sum_{0 \leq d < n/2 - \deg(H)} q^{n-\deg(H)} e(T^{n-\deg(H)} H\theta) = q^{n-\deg(H)} e(T^{n-\deg(H)} H\theta) \cdot \begin{cases} n/2 - \deg(H), & n \text{ even} \\ \frac{n+1}{2} - \deg(H), & n \text{ odd.} \end{cases}$$

Next, we want to find a nice expression for $S_2(\theta)$ using similar reasoning. Let us again write $\theta = G/H + \beta$, for $\theta \in \mathscr{U}_{n,G/H}$, and that $A' \equiv AG \mod H$ for $\deg(A') < \deg(H)$. We again apply Lemma 2.1 to achieve

$$\sum_{A \in \mathcal{M}_{n/2}} \begin{cases} q^{n/2} e(T^{n/2} A\theta), & v(A'/H + A\beta) > n/2 \\ 0, & \text{otherwise.} \end{cases}$$

From the same analysis, we see that $v(A\beta) > \deg(H)$. We note similarly that $v(A'/H + A\beta) \leq \deg(H) \leq n/2$ if H does not divide A. Next, we can again note that when H divides A, the condition that $v(A\beta) > n/2$ is equivalent to $\theta \in B_n(G/H)$. So, we again have that the sum is 0 if $\theta \notin B_n(G/H)$. So from now on, we assume that $\theta \in B_n(G/H)$.

Now, we can simplify the sum to the following:

(5)
$$\sum_{A'' \in \mathcal{M}_{n/2-\deg(H)}} q^{n/2} e(T^{n/2} A'' H\theta) = q^{n-\deg(H)} e(T^{n-\deg(H)} H\theta).$$

Here, we have used Lemma 2.1 and that $\theta \in B_n(G/H)$ implies that $v(T^{n/2} H\theta) > n/2 - \deg(H)$.

Combining (4) and (5), we achieve that

$$\sum_{F \in \mathcal{M}_n} d(F) e(F\theta) = \begin{cases} q^{n-\deg(H)} e(T^{n-\deg(H)} H\theta)(n + 1 - 2\deg(H)), & \theta \in B_n(G/H) \\ 0, & \text{otherwise.} \end{cases}$$

□

3. Proof of Corollary 1.6

To calculate the integral, we invoke the following lemma of Hayes [3] about integrating the exponential function.

Lemma 3.1 (Hayes, Theorem 3.6)**.** *Let $a \in \mathscr{U}$ and $r \geq 0$. For any Laurent series b over $\mathbb{F}_q$, we have that*

$$\int_{B_r(a)} e(b\theta) d\varrho(\theta) = \begin{cases} q^{-r} e(ab), & v(b) > -r \\ 0, & \textit{otherwise.} \end{cases}$$

Proof of Corollary 1.6. Let us write

$$\Phi(H) = \left| (\mathbb{F}_q[T]/(H(T)))^{\times} \right|.$$

We note that for P a prime polynomial of degree d,

$$\Phi(P) = |\mathbb{F}_q[T]/(P(T))| - 1 = q^d - 1.$$

So, we have that $\Phi \star 1(P) = |P|$. Similarly, we can see this holds for prime powers; thus, from multiplicativity, we get the relation that $\Phi \star 1(A) = |A|$ for any $A \in \mathbb{F}_q[T]$. So, $\Phi = \mu \star |.|$.

Recalling Lemma 1.2, we re-express the integral over $\mathscr{U}$ as a sum over primordials $G/H \in \mathscr{V}_n$ (which do not necessarily have monic numerators) for n:

$$\begin{aligned} \int_{\mathscr{U}} \left| \sum_{F \in \mathcal{M}_n} d(F) e(F\theta) \right|^s d\varrho(\theta) &= \sum_{G/H \in \mathscr{V}_n} \int_{\mathscr{U}_{n,G/H}} \left| \sum_{F \in \mathcal{M}_n} d(F) e(F\theta) \right|^s d\varrho \\ &= \sum_{G/H \in \mathscr{V}_n} \frac{(n+1-2\deg(H))^s}{q^{s\deg(H) - n(s-1)}} \\ &= q^{n(s-1)} \sum_{d=0}^{\lfloor n/2 \rfloor} \frac{(n+1-2d)^s}{q^{sd}} \sum_{H \in \mathcal{M}_d} \Phi(H). \end{aligned}$$

Here we have used Theorem 1.4 to go from the first line to the second and the definition of ϱ and $\mathscr{U}_{n,G/H}$ to go from the second line to the third.

We calculate the inside sum using that $\Phi = \mu \star |.|$:

$$\sum_{H \in \mathcal{M}_d} \Phi(H) = q^d \sum_{\substack{A \in \mathcal{M} \\ \deg(A) \leq d}} \frac{\mu(A)}{|A|} \sum_{B \in \mathcal{M}_{d-\deg(A)}} 1 = q^{2d} \sum_{\substack{A \in \mathcal{M} \\ \deg(A) \leq d}} \frac{\mu(A)}{|A|^2}.$$

Since, we know that $\zeta_q(2) = \frac{q}{q-1}$, we have that

$$\sum_{A \in \mathcal{M}} \frac{\mu(A)}{|A|^2} = \zeta_q(2)^{-1} = 1 - q^{-1}.$$

By comparing coefficients of q^{-d} and noting that $\sum_{F \in \mathcal{M}_d} \mu(F) \ll q^{d/2}$ from Weil's theorem [9], we must have that

$$\sum_{A \in \mathcal{M}_d} \mu(A) = \begin{cases} 1, & d = 0 \\ -q, & d = 1 \\ 0, & d \geq 2. \end{cases} \tag{6}$$

So, we get that

$$\sum_{H \in \mathcal{M}_d} \Phi(H) = \begin{cases} 1, & d = 0 \\ q^{2d}(1 - q^{-1}), & d \geq 1. \end{cases}$$

We remark that this is also shown in Lemma 4.2 of Hayes [3]. So, our expression above becomes

$$\int_{\mathscr{U}} \left| \sum_{F \in \mathcal{M}_n} d(F) e(F\theta) \right|^s d\varrho(\theta) = (n+1)^s q^{n(s-1)} + q^{n(s-1)}(1 - q^{-1}) \sum_{d=1}^{\lfloor n/2 \rfloor} \frac{(n+1-2d)^s}{q^{(s-2)d}}.$$

We first evaluate when $s=1$ and $s=2$. When $s=1$, the above equation becomes

$$(n+1)+(1-q^{-1})\sum_{d=1}^{\lfloor n/2\rfloor}(n+1-2d)q^d=\left(n-2\lfloor n/2\rfloor+\frac{q+1}{q-1}\right)q^{\lfloor n/2\rfloor}-\frac{2}{q-1}.$$

When $s=2$, we get

$$(n+1)^2q^n+q^n(1-q^{-1})\sum_{d=1}^{\lfloor n/2\rfloor}(n+1-2d)^2=c_{q,n}q^n,$$

where

$$c_{q,n}=(n+1)^2+(1-q^{-1})\left(-2n\lfloor n/2\rfloor^2+\frac{4}{3}\lfloor n/2\rfloor^3+n^2\lfloor n/2\rfloor-\frac{1}{3}\lfloor n/2\rfloor\right)$$

□

4. Proof of Corollary 1.8

Proof of Corollary 1.8. Let us write $\deg(A)=m$. First we consider when $m<n$. Then we can write

$$\sum_{F\in\mathcal{M}_n}d(F)d(F+A)=\int_{\mathscr{U}}\left|\sum_{F\in\mathcal{M}_n}d(F)e(F\theta)\right|^2e(-A\theta)d\varrho(\theta).$$

Here we have applied Lemma 3.1 to obtain the orthogonality, as done by Hayes when applying the circle method in [3]. Applying Theorem 1.4, Lemma 1.2, and Lemma 3.1, we have that this sum is

$$\begin{aligned}&\sum_{G/H\in\mathscr{V}_n}q^{2(n-\deg(H))}(n+1-2\deg(H))^2\int_{B_n(G/H)}e(-A\theta)d\varrho(\theta)\\&=\sum_{G/H\in\mathscr{V}_n}q^{n-2\deg(H)}(n+1-2\deg(H))^2e(AG/H)\\&=q^n\sum_{\substack{H\in\mathcal{M}\\ \deg(H)\le n/2}}\frac{(n+1-2\deg(H))^2}{|H|^2}\sum_{G/H\in\mathscr{V}_n}e(AG/H).\end{aligned}$$

Let us temporarily fix a monic polynomial H. We recognize this internal sum as the function field version of Ramanujan's sum:

$$C_H(A):=\sum_{\substack{(G,H)=1\\ \deg(G)<\deg(H)}}e(AG/H).$$

We note that

$$\sum_{\substack{F\in\mathcal{M}\\ F|H}}C_F(A)=\sum_{G\in\mathbb{F}_q[T]/(H)}e(AG/H)=\begin{cases}|H|, & H\mid A\\ 0, & \text{otherwise.}\end{cases}$$

From Mobius inversion, we have that

$$C_H(A)=\sum_{B|H}\mu(H/B)\begin{cases}|B|, & B\mid A\\ 0, & \text{otherwise}\end{cases}=\sum_{\substack{B|H\\ B|A}}\mu(H/B)|B|.$$

Now, we return to summing over H. We have the following sum over $H\in\mathcal{M}_d$:

$$\sum_{H\in\mathcal{M}_d}C_H(A)=\sum_{B|A}|B|\sum_{\substack{H\in\mathcal{M}_d\\ B|H}}\mu(H/B)=\sum_{B|A}|B|\begin{cases}1, & \deg(B)=d\\ -q, & d-\deg(B)=1\\ 0, & \text{otherwise}\end{cases}=q^d\cdot\left(\sum_{\substack{B\in\mathcal{M}_d\\ B|A}}1-\sum_{\substack{B\in\mathcal{M}_{d-1}\\ B|A}}1\right).$$

Here we have used (6) to evaluate the Mobius sum. So, our original sum becomes

$$q^n \sum_{\substack{H\in\mathcal{M}\\ \deg(H)\le n/2}} \frac{(n+1-2\deg(H))^2}{|H|^2}\cdot C_H(A) = q^n \sum_{0\le d\le n/2} \frac{(n+1-2d)^2}{q^d}\Big(\sum_{\substack{B\in\mathcal{M}_d\\ B|A}} 1 - \sum_{\substack{B\in\mathcal{M}_{d-1}\\ B|A}} 1\Big).$$

We remark that the above expression is exactly the one from Gorodetsky [2]. We can also rewrite it as the following:

$$q^n \sum_{\substack{B\in\mathcal{M}_{\lfloor n/2\rfloor}\\ B|A}} \frac{(n+1-2\lfloor n/2\rfloor)^2}{|B|} + q^n \sum_{\substack{B\in\mathcal{M}\\ \deg(B)\le n/2-1\\ B|A}} \frac{1}{|B|}\cdot((n+1-2\deg(B))^2 - q^{-1}(n-1-2\deg(B))^2).$$

This is the formula claimed in Corollary 1.8.

Finally, we assume that $\deg(A)=n$ and $a_0\neq -1$ is the leading coefficient of A. Then, we consider

$$\sum_{F\in\mathcal{M}_n} d(F)d(F+A) = \int_U \left(\sum_{F\in\mathcal{M}_n} d(F)e(F\theta)\right)\left(\sum_{F\in(a_0+1)\mathcal{M}_n} d(F)e(-F\theta)\right) e(A\theta)d\varrho(\theta).$$

We again apply Theorem 1.4 to get the sum

$$\sum_{G/H\in\mathscr{V}_n} q^{2(n-\deg(H))}(n+1-2\deg(H))^2 \int_{B_n(G/H)} e((X^{n-\deg(H)}H - a_0X^{n-\deg(H)}H + A)\theta)d\varrho(\theta).$$

We can use Lemma 3.1 to see that this is

$$\sum_{G/H\in\mathscr{V}_n} q^{n-2\deg(H)}(n+1-2\deg(H))^2 e(FG/H),$$

where $F = X^{n-\deg(H)}H - a_0X^{n-\deg(H)}H + A$. However, we have that $e(FG/H)=e(AG/H)$, since the other terms of F are divisible by H. So, this becomes

$$q^n \sum_{d=0}^{n/2} \frac{(n+1-2d)^2}{q^{2d}} \sum_{H\in\mathcal{M}_d} C_H(A).$$

Thus, we are reduced to the same computation as before and derive the same formula. □

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