Application of the SUSY Non-linear Sigma Model

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1 Calculating the first quantum correction

Now that we have arrived at our non-linear sigma model, which was derived as a way to calculate Y^{00}, we will put it to a test. We will follow the derivation of Efetov [1] in Section 5.1.1 very closely, and reproduce the calculations step-by-step where needed. As a sanity check, we will try to re-obtain the semi-classical conductivity, and then its first quantum correction which was due to the Feynman diagrams called Cooperons. As might be expected, one can not simply calculate Y^{00} from the NLSM, instead, further approximations are needed to obtain a result. This is because the free energy functional has a term linear in Q, and integrals of this form do not converge. To perform calculations, we will need a parametrization of Q, the auxiliary supermatrix field introduced in Section 3.1 of the previous chapter, which will allow us to expand the free energy functional and the function $g^{\alpha\beta}$, the inverse of the effective lagrangian introduced in Section 3.2 of the previous chapter, about the main contributions (saddle points) in transversal direction (Goldstone modes), resulting in an exponent quadratic in the new parameter. This will also ensure, that we are indeed still in the saddle point approximation, which was used to derive the free energy functional in the first place. Furthermore, we will assume to be in the high-frequency limit, where characteristic deviations of Q from the mean-field solution Λ are small. A parametrization suitable for the calculation of Y^{00} in perturbation theory, is

$$Q = \Lambda(1+iP)(1-iP)^{-1} \quad , \quad P = \begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix} \quad , \quad \bar{B} = C_0 B^T C_0^T. \tag{1}$$

This means that B must be a 4×4 supermatrix-valued field, with the following structure

$$B = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}, \tag{2}$$

where a, b are complex valued 2×2 matrices and σ, ρ are 2×2 matrices consisting of Grassmann variables. As we are in the high-frequency limit the parameters P and B can be assumed to be small.
As discussed in Section 3.3 of [2], the parametrization should be of the form

$$Q = V\Lambda\bar{V},$$

as we are interested in the transverse fluctuations of Q around the mean-field solution Λ. Since the matrix V must also satisfy the condition $V\bar{V} = 1$ and therefore $\bar{V} = V^{-1}$, we find, using

$$P\Lambda = -\Lambda P, \tag{3}$$

that our parametrization is indeed of this form

$$Q = (1 - iP)\Lambda(1 - iP)^{-1}.$$

To retain the symmetries of Q, we can re-formulate the conditions on Q as conditions on P or B. The condition $\bar{Q} = KQ^{\dagger}K$, emerging from (**??**) in Section 3.3 of [2], becomes

$$P = KP^{\dagger}K, \tag{4}$$

and the constraint $Q^2 = 1$ is automatically satisfied because of (3).

This parametrization allows us to re express supertraces of Q in terms of supertraces of B without having to worry about the reparametrization of the integral, since the Jacobian of the substitution (1) can be shown to be equal to 1. In particular, it allows us to write the free energy functional as a function of B. The final expression, eq. (44) of the previous chapter, for the free energy functional was

$$F[Q] = \frac{\pi\nu}{8}\mathrm{str}\int\Big[D_0(\nabla Q)^2 + 2i(\omega + i\delta)\Lambda Q\Big]\mathrm{d}\mathbf{r}, \tag{5}$$

where we will now insert the parametrization (1). There are two terms we want to calculate

$$\mathrm{str}[(\nabla Q)^2] \tag{6}$$

and

$$\mathrm{str}[\Lambda Q]. \tag{7}$$

Let's start with (6), where we need an expression for ∇Q.

$$\nabla Q = \sum_i \partial_i Q$$

$$\begin{aligned}
\partial_i Q &= \partial_i\Big(\Lambda(1 + iP)(1 - iP)^{-1}\Big) \\
&= \Lambda\partial_i\Big((1 - iP)^{-1} + ((1 - iP)(iP)^{-1})^{-1}\Big) \\
&= \Lambda\partial_i\Big((1 - iP)^{-1} + \big((iP)^{-1} - 1\big)^{-1}\Big) \\
&= \Lambda\Big(- (1 - iP)^{-1}\partial_i(-iP)(1 - iP)^{-1} - \big((iP)^{-1} - 1\big)^{-1}\big(\partial_i(iP)^{-1}\big)\big((iP)^{-1} - 1\big)^{-1}\Big)
\end{aligned}$$

Using

$$\partial_i(P^{-1}) = -P^{-1}(\partial_i P)P^{-1},$$

we obtain

$$\partial_i Q = 2i\Lambda(1-iP)^{-1}\partial_i P(1-iP)^{-1}.$$

We can now insert this expression into the supertrace

$$\begin{aligned}\mathrm{str}(\partial_i Q)^2 &= -4\mathrm{str}\Big(\Lambda(1-iP)^{-1}\partial_i P(1-iP)^{-1}\Big)^2 \\ &= -4\mathrm{str}\Big(\underbrace{(1-iP)^{-1}\Lambda(1-iP)^{-1}}_{:=\mathrm{L}}\partial_i P(1-iP)^{-1}\Lambda(1-iP)^{-1}\partial_i P\Big).\end{aligned}$$

By the defining the "root" of $\Lambda = M^{-1}M^{-1}$, we can rewrite L

$$\begin{aligned}L &= (1-iP)^{-1}\Lambda(1-iP)^{-1} = (1-iP)^{-1}M^{-1}M^{-1}(1-iP)^{-1} \\ &= (M-iMP)^{-1}(M-iPM)^{-1} \\ &= \Big((M-iPM)(M-iMP)\Big)^{-1} \\ &= \Big(M^2 - iM^2P - iPM^2 - PM^2P\Big)^{-1} \\ &= \Big(\Lambda - i\Lambda P - iP\Lambda - P\Lambda P\Big)^{-1} \qquad P\Lambda = -\Lambda P \\ &= (\Lambda - P\Lambda P)^{-1} = (1+P^2)^{-1}\Lambda^{-1}.\end{aligned}$$

Inserting (1) we obtain

$$L = \begin{pmatrix}(1+B\bar{B})^{-1} & 0 \\ 0 & -(1+\bar{B}B)^{-1}\end{pmatrix}.$$

Finally, we arrive at

$$\begin{aligned}\mathrm{str}(\partial_i Q)^2 &= -4\mathrm{str}\Big(L(\partial_i P)L(\partial_i P)\Big) \\ &= 8\mathrm{str}\Big((1+B\bar{B})^{-1}\partial_i B(1+\bar{B}B)^{-1}\partial_i\bar{B}\Big).\end{aligned}$$

Next, we need to do the same thing for (7). Here, the following expansion will be useful

$$(1-iP)^{-1} = \sum_{n=0}^{\infty}(iP)^n. \tag{8}$$

It converges only for $||P|| < 1$, which in our case is a justified assumption, as we are in the high-frequency limit. With (8), we can simplify (7) in the following

way

$$\begin{aligned}\mathrm{str}[\Lambda Q] &= \mathrm{str}[(1+iP)(1-iP)^{-1}] \\ &= \mathrm{str}[(1+iP)\sum_{n=0}^{\infty}(iP)^n] \\ &= \mathrm{str}[\sum_{n=0}^{\infty}(iP)^n + \sum_{n=1}^{\infty}(iP)^n] \\ &= \mathrm{str}[1+2\sum_{n=1}^{\infty}(iP)^n].\end{aligned}$$

And, recalling that $\mathrm{str}(1) = 0$

$$\mathrm{str}[\Lambda Q] = 2\mathrm{str}[\sum_{n=1}^{\infty}(iP)^n].$$

A short reflection tells us that because of the form of P, only even powers will contribute to the supertrace. Neglecting the odd powers, the expression is now

$$\mathrm{str}[\Lambda Q] = 2\mathrm{str}[\sum_{n=1}^{\infty}(-1)^n(P)^{2n}],$$

where the even powers are of the form

$$P^{2n} = \begin{pmatrix}(B\bar{B})^{2n} & 0 \\ 0 & (\bar{B}B)^{2n}\end{pmatrix}.$$

To obtain an expression of the form (8), we add a neutral term $1-1$

$$\mathrm{str}[\Lambda Q] = 2\mathrm{str}[\sum_{n=0}^{\infty}(-1)^n(P)^{2n} - 1] = 2\mathrm{str}[\sum_{n=0}^{\infty}(-1)^n(P)^{2n}]$$

$$\begin{aligned}\mathrm{str}[\Lambda Q] &= 2\mathrm{str}\begin{pmatrix}\sum_{n=0}^{\infty}(-B\bar{B})^n & 0 \\ 0 & \sum_{n=0}^{\infty}(-\bar{B}B)^n\end{pmatrix} \\ &= 2\mathrm{str}\begin{pmatrix}(1+B\bar{B})^{-1} & 0 \\ 0 & (1+\bar{B}B)^{-1}\end{pmatrix} \\ &= 2\big(\mathrm{str}(1+B\bar{B})^{-1} + \mathrm{str}(1+\bar{B}B)^{-1}\big).\end{aligned}$$

We can now use the cyclicity of the supertrace to show

$$\begin{aligned}\mathrm{str}[(\bar{B}B)^n] &= \mathrm{str}[(\bar{B}B)^{n-1}\bar{B}B] \\ &= \mathrm{str}[B(\bar{B}B)^{n-1}\bar{B}] \\ &= \mathrm{str}[(B\bar{B})^n],\end{aligned}$$

and consequently

$$\mathrm{str}(1+\bar{B}B)^{-1} = \mathrm{str}(1+B\bar{B})^{-1}.$$

Finally, we obtain for (7)

$$\mathrm{str}[\Lambda Q] = 4\mathrm{str}(1 + B\bar{B})^{-1}.$$

We can now plug the expressions for (6) and (7) into (5) to arrive at the free energy functional as a function of B

$$F[B] = \pi\nu \int \mathrm{str}\Big[D_0(1 + B\bar{B})^{-1}\nabla B(1 + \bar{B}B)^{-1}\nabla\bar{B} + i\omega(1 + B\bar{B})^{-1}\Big]\mathrm{d}\mathbf{r}. \quad (9)$$

The aim of this whole calculation was to re-express the free energy functional in terms of a small parameter, such that an expansion can be performed. In our case, we would like to consider the expansion of $F[B]$ up to quartic order in B. Using (8) we expand up to quartic order

$$\begin{aligned}(1 + B\bar{B})^{-1}\nabla B(1 + \bar{B}B)^{-1}\nabla\bar{B} &\approx (1 - B\bar{B})\nabla B(1 - \bar{B}B)\nabla\bar{B} \\ &\approx \nabla B\nabla\bar{B} - B\bar{B}\nabla B\nabla\bar{B} - \nabla B\bar{B}B\nabla\bar{B}.\end{aligned}$$

And

$$(1 + B\bar{B})^{-1} \approx 1 - B\bar{B} + (B\bar{B})^2. \quad (10)$$

Inserting theses expansions into (9) and grouping the quadratic and quartic terms (whilst recalling that $\mathrm{str}(1) = 0$), we obtain

$$F_2 = \pi\nu \int \mathrm{str}\big(D_0\nabla B\nabla\bar{B} - i\omega B\bar{B}\big)\mathrm{d}\mathbf{r} \quad (11)$$

and

$$F_4 = \pi\nu \int \mathrm{str}\big[- D_0\big(\nabla\bar{B}\nabla B\bar{B}B + \nabla B\nabla\bar{B}B\bar{B}\big) + i\omega\big(B\bar{B}\big)^2\big]\mathrm{d}\mathbf{r}. \quad (12)$$

Now that we have found an expansion of the free energy functional, we would like to go back to our expression for the conductivity, and try to find a corresponding expansion for the pre-exponential terms of the integrand. We had started from the following expression for the correlation function

$$Y^{00}(\mathbf{r}, \epsilon, \omega) = \langle\Psi^I(\mathbf{r})\Psi^J(\mathbf{r})\Psi^K(\mathbf{r})\Psi^L(\mathbf{r})\rangle_{F_\omega[\Psi,Q]},$$

which we simplified using Wick's theorem, neglecting certain contractions in the limit $r >> l$, and introducing the matrix

$$k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (13)$$

The result was (**??**) in [2]

$$Y^{00}(\mathbf{r}, \epsilon, \omega) = 2\int g^{12}_{\alpha\beta}(\mathbf{r}, \mathbf{r})k_{\beta\beta}g^{21}_{\beta\alpha}(\mathbf{0}, \mathbf{0})\exp(-F[Q])DQ,$$

with no summation over either α or β, and where g is defined in (**??**) of [2]. Now, we can make use of the identity (**??**)[2], valid at saddle points of the free energy

$$Q(\mathbf{r}) = \frac{1}{\pi\nu} g(\mathbf{r}, \mathbf{r}) \tag{14}$$

to replace g with Q, since we are in the saddle point approximation. This yields

$$Y^{00}(\mathbf{r}, \epsilon, \omega) = 2(\pi\nu)^2 \int Q^{12}_{\alpha\beta}(\mathbf{r}) k_{\beta\beta} Q^{21}_{\beta\alpha}(\mathbf{0}) \exp(-F[Q]) DQ. \tag{15}$$

Exploiting the fact that the indices are arbitrary, we can re-write this expression as a sum over them with an appropriate pre-factor

$$Y^{00}(\mathbf{r}, \epsilon, \omega) = \frac{(\pi\nu)^2}{32} \int \mathrm{str}\big[k(1+\Lambda)(1-\tau_3)Q(\mathbf{r})(1-\Lambda)(1-\tau_3)kQ(\mathbf{0})\big] \exp(-F[Q]) DQ,$$

where

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the matrix living in the subspace of the matrices a,b etc. in

$$Q = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} \quad , \quad Q^{ij} = \begin{pmatrix} a^{ij} & \rho^{ij} \\ \sigma^{ij} & b^{ij} \end{pmatrix},$$

whereas the matrix k lives in the space of the matrices Q^{ij}. The appropriate tensor products with the identity matrix of the remaining subspaces are considered. This form seems more complicated than (15), but we will see that it is a convenient form for our parametrization, and furthermore, the breaking of certain symmetries is highlighted by the appearance of projectors, and therefore the reduction of our manifold. This is the form we want to insert our parametrization in. To re-express the pre-exponential term as a function of B, we notice that we don't have to worry about the factors $k(1-\tau_3)$because they act on each B inside Q separately. We are therefore mainly interested in the quantity

$$\mathrm{str}[(1+\Lambda)Q(\mathbf{r})(1-\Lambda)Q(\mathbf{0})].$$

So we insert our parametrization (1)

$$\begin{aligned} \mathrm{str}[(1+\Lambda)Q(\mathbf{r})(1-\Lambda)Q(\mathbf{0})] = \mathrm{str}[&(1+\Lambda)(1+iP(\mathbf{r}))\sum_0^\infty (iP(\mathbf{r}))^n \\ &\times(\Lambda-1)(1+iP(\mathbf{0}))\sum_0^\infty (iP(\mathbf{0}))^n]. \end{aligned}$$

This is the product of two similar terms

$$(1+\Lambda)(1+iP(\mathbf{r}))\sum_0^\infty (iP(\mathbf{r}))^n$$

and

$$(\Lambda-1)(1+iP(\mathbf{0}))\sum_{0}^{\infty}(iP(\mathbf{0}))^n.$$

For $\sum_0^\infty (iP)^n$ we only keep terms up to third order

$$\sum_{0}^{\infty}(iP)^n \approx \begin{pmatrix} (1-B\bar{B}) & iB(1+\bar{B}B) \\ i\bar{B}(1+B\bar{B}) & (1+\bar{B}B) \end{pmatrix}.$$

Together with

$$(1+\Lambda)(1+iP(\mathbf{r})) = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & iB(\mathbf{r}) \\ i\bar{B}(\mathbf{r}) & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & iB(\mathbf{r}) \\ 0 & 0 \end{pmatrix}$$

and

$$(\Lambda-1)(1+iP(\mathbf{0})) = -2\begin{pmatrix} 0 & 0 \\ i\bar{B}(\mathbf{0}) & 1 \end{pmatrix},$$

we arrive at

$$\begin{aligned} Y^{00}(\mathbf{r},\omega) =& \frac{(\pi\nu)^2}{2}\int \mathrm{str}\big[k(1-\tau_3)\big(B(\mathbf{r})-B(\mathbf{r})\bar{B}(\mathbf{r})B(\mathbf{r})\big)k(1-\tau_3) \\ &\times\big(\bar{B}(\mathbf{0})-\bar{B}(\mathbf{0})B(\mathbf{0})\bar{B}(\mathbf{0})\big)\big]\exp(-F_2)\exp(-F_4)DBD\bar{B}. \end{aligned} \tag{16}$$

Upon inspection, the calculation splits into two parts: the integral with F_2, which is gaussian and can be calculated directly, and the quantum correction F_4 which is quartic and cannot be calculated directly. We start by calculating the lowest order term

$$\begin{aligned} Y_2^{00}(\mathbf{r},\omega) &= \frac{(\pi\nu)^2}{2}\int \mathrm{str}\big[k(1-\tau_3)B(\mathbf{r})k(1-\tau_3)\bar{B}(\mathbf{0})\big]\exp(-F_2)DBD\bar{B} \\ &= \frac{(\pi\nu)^2}{2}\int \big[\sum_{i=1}^{i=2} k_{ij}(1-\tau_3)_{jk}B(\mathbf{r})_{kl}k_{lm}(1-\tau_3)_{mn}\bar{B}(\mathbf{0})_{ni} \\ &-\sum_{i=3}^{i=4} k_{ij}(1-\tau_3)_{jk}B(\mathbf{r})_{kl}k_{lm}(1-\tau_3)_{mn}\bar{B}(\mathbf{0})_{ni}\big]\exp(-F_2)DBD\bar{B} \\ &= \frac{(\pi\nu)^2}{2}\int \big[\sum_{i=1}^{i=2}(1-\tau_3)_{ik}B(\mathbf{r})_{kl}k_{lm}(1-\tau_3)_{mn}\bar{B}(\mathbf{0})_{ni} \\ &+\sum_{i=3}^{i=4}(1-\tau_3)_{ik}B(\mathbf{r})_{kl}k_{lm}(1-\tau_3)_{mn}\bar{B}(\mathbf{0})_{ni}\big]\exp(-F_2)DBD\bar{B}, \end{aligned} \tag{17}$$

where we abused notation by defining

$$(1-\tau_3) = \begin{pmatrix} (1-\tau_3) & 0 \\ 0 & (1-\tau_3) \end{pmatrix} = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we have already inserted the matrix (13). We can now insert the matrix $(1-\tau_3)$ into (17)

$$\begin{aligned} Y_2^{00}(\mathbf{r},\omega) &= \frac{(\pi\nu)^2}{2} 2 \int \big[B(\mathbf{r})_{2l} k_{lm} (1-\tau_3)_{mn} \bar{B}(\mathbf{0})_{n2} \\ &\quad + B(\mathbf{r})_{4l} k_{lm} (1-\tau_3)_{mn} \bar{B}(\mathbf{0})_{n4}\big] \exp(-F_2) DBD\bar{B} \\ &= 2(\pi\nu)^2 \int \big[B(\mathbf{r})_{22}\bar{B}(\mathbf{0})_{22} - B(\mathbf{r})_{24}\bar{B}(\mathbf{0})_{42} \\ &\quad + B(\mathbf{r})_{42}\bar{B}(\mathbf{0})_{24} - B(\mathbf{r})_{44}\bar{B}(\mathbf{0})_{44}\big] \exp(-F_2) DBD\bar{B}. \end{aligned} \tag{18}$$

At this point it will be useful to explicitly calculate a few quantities, as the symmetries of B will have an important impact on our results. We start by explicitly writing out the form of B, which has the symmetries (4) inherited from Q

$$B = \begin{pmatrix} a_1 & a_2 & i\sigma_1 & i\sigma_2 \\ -\bar{a}_2 & \bar{a}_1 & -i\bar{\sigma}_2 & -i\bar{\sigma}_1 \\ \bar{\rho}_1 & -\rho_2 & ib_1 & ib_2 \\ \bar{\rho}_2 & -\rho_1 & i\bar{b}_2 & i\bar{b}_1 \end{pmatrix}.$$

Using the definition of the charge conjugation operation we calculate

$$\begin{aligned} \bar{B} &= C_0 B^T C_0^T \\ &= \begin{pmatrix} -i\sigma_{p,2} & 0 \\ 0 & \sigma_{p,1} \end{pmatrix} \begin{pmatrix} a^T & -\rho^T \\ \sigma^T & b^T \end{pmatrix} \begin{pmatrix} -i\sigma_{p,2}^T & 0 \\ 0 & \sigma_{p,1}^T \end{pmatrix} \\ &= \begin{pmatrix} -i\sigma_{p,2} & 0 \\ 0 & \sigma_{p,1} \end{pmatrix} \begin{pmatrix} a^T & -\rho^T \\ \sigma^T & b^T \end{pmatrix} \begin{pmatrix} i\sigma_{p,2} & 0 \\ 0 & \sigma_{p,1} \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{p,2} a^T \sigma_{p,2} & i\sigma_{p,2}\rho^T\sigma_{p,1} \\ i\sigma_{p,1}\sigma^T\sigma_{p,2} & \sigma_{p,1} b^T \sigma_{p,1} \end{pmatrix} \\ &= \begin{pmatrix} \bar{a}_1 & -a_2 & -\rho_1 & -\rho_2 \\ \bar{a}_2 & a_1 & -\bar{\rho}_2 & -\bar{\rho}_1 \\ i\bar{\sigma}_1 & i\sigma_2 & i\bar{b}_1 & ib_2 \\ i\bar{\sigma}_2 & i\sigma_1 & i\bar{b}_2 & ib_1 \end{pmatrix} \end{aligned},$$

where $\sigma_{p,i}$ are the Pauli matrices. Using these explicit forms of B, we can calculate in a straight-forward manner

$$\mathrm{str}(B\bar{B}) = 2\big(\bar{a}_1 a_1 + \bar{a}_2 a_2 + \bar{\sigma}_1\sigma_1 + \bar{\sigma}_2\sigma_2 + \bar{\rho}_1\rho_1 + \bar{\rho}_2\rho_2 + \bar{b}_1 b_1 + \bar{b}_2 b_2\big). \tag{19}$$

With this, (18) becomes

$$Y_2^{00}(\mathbf{r},\omega) = 2(\pi\nu)^2 \int \big[a_1\bar{a}_1 + \sigma_1\bar{\sigma}_1 + \rho_1\bar{\rho}_1 + b_1\bar{b}_1\big] \exp(-F_2) DBD\bar{B}.$$

Now that we have put the pre-exponential term in a form we can handle, we should also do some work on F_2. The idea is to handle the matrix entries B_{ij} and as components of a vector $\Psi = \big(B\big)$ and to bring F_2 into a form $\langle \Psi, A\Psi\rangle$. This

type of scalar product between two matrices is called Frobenius inner product, and is given by $\langle A, B\rangle = \mathrm{tr}(A^\dagger B)$. Since our operation of charge conjugation $\bar{B}$ is related by an additional pre-factor to the operation of hermitian conjugation, we can deduce that the term $\mathrm{str}(B\bar{B})$ is such a Frobenius product. However, this argumentation has to be taken with a pinch of salt, as our "Frobenius product" doesn't have a definite sign, since the supertrace introduces a minus sign. The minus sign is however not important, because Gaussian integrals over Grassmann numbers always "converge", and the fermionic part of the inner product does not have to be positive definite. Thus, terms of the form

$$\int B_{ij}\bar{B}_{ji}\mathrm{exp}\Big(-\frac{1}{2}C\mathrm{str}(B\bar{B})\Big)DBD\bar{B} \tag{20}$$

can be handled via the formula for gaussian integrals

$$\int \Psi_i\Psi_j^*\mathrm{exp}\big(-\Psi^*A\Psi\big)\mathrm{d}\Psi\mathrm{d}\Psi^* = \big(A^{-1}\big)_{ij}, \tag{21}$$

where

$$A = C\cdot\mathbb{1}. \tag{22}$$

So we start by going to momentum space, where we need the following identity

$$\begin{aligned}
\int \nabla B(\mathbf{r})\nabla\bar{B}(\mathbf{r})\mathrm{d}\mathbf{r} &= \int \nabla\Big(\int e^{i\mathbf{k}\mathbf{r}}B(\mathbf{k})\mathrm{d}\mathbf{k}\Big)\nabla\Big(\int e^{i\mathbf{k}'\mathbf{r}}\bar{B}(\mathbf{k}')\mathrm{d}\mathbf{k}'\Big)\mathrm{d}\mathbf{r}\\
&= \int\int (i\mathbf{k})e^{i\mathbf{k}\mathbf{r}}B(\mathbf{k})\mathrm{d}\mathbf{k}\int (i\mathbf{k}')e^{i\mathbf{k}'\mathbf{r}}\bar{B}(\mathbf{k}')\mathrm{d}\mathbf{k}'\mathrm{d}\mathbf{r}\\
&= \int\int (i\mathbf{k})B(\mathbf{k})\mathrm{d}\mathbf{k}\int (i\mathbf{k}')\bar{B}(\mathbf{k}')e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}}\mathrm{d}\mathbf{k}'\mathrm{d}\mathbf{r}\\
&= \int (i\mathbf{k})B(\mathbf{k})\mathrm{d}\mathbf{k}\int (i\mathbf{k}')\bar{B}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}')\mathrm{d}\mathbf{k}'\\
&= \int \mathbf{k}^2B(\mathbf{k})\bar{B}(-\mathbf{k})\mathrm{d}\mathbf{k}.
\end{aligned}$$

Using this identity, we write down the exponential in momentum space, and explicitly write out the supertrace

$$\begin{aligned}
F_2 &= \pi\nu\int \mathrm{str}\big(D_0\nabla B\nabla\bar{B} - i\omega B\bar{B}\big)\mathrm{d}\mathbf{r}\\
&= \pi\nu\int \mathrm{str}\big(D_0\mathbf{k}^2B\bar{B} - i\omega B\bar{B}\big)\mathrm{d}\mathbf{k}\\
&= \pi\nu\int \big(D_0\mathbf{k}^2 - i\omega\big)\mathrm{str}\big(B\bar{B}\big)\mathrm{d}\mathbf{k}\\
&= 2\pi\nu\int \big(D_0\mathbf{k}^2 - i\omega\big)\big(\bar{a}_1a_1 + \bar{a}_2a_2 + \bar{\sigma}_1\sigma_1 + \bar{\sigma}_2\sigma_2 + \bar{\rho}_1\rho_1 + \bar{\rho}_2\rho_2 + \bar{b}_1b_1 + \bar{b}_2b_2\big)\mathrm{d}\mathbf{k}.
\end{aligned}$$

Using the formula for gaussian integrals over real vectors (because we have split the complex conjugates into separate components), we can calculate integrals

of the form

$$\int B_{ij}\bar{B}_{lm}\exp\big(-F_2\big)DBD\bar{B} = \pm\delta_{im}\delta_{jl}\frac{1}{2\pi\nu\big(D_0\mathbf{k}^2 - i\omega\big)}, \tag{23}$$

where the sign depends on the index j. Using (23) we arrive at the semi-classical result, equivalent to the Drude-Boltzmann conductivity

$$\begin{aligned} Y_2^{00}(\mathbf{r},\omega) &= 2(\pi\nu)^2\int\big[a_1\bar{a}_1 + \sigma_1\bar{\sigma}_1 + \rho_1\bar{\rho}_1 + b_1\bar{b}_1\big] \\ \times\exp\Big(-2\pi\nu\int\big(D_0\mathbf{k}^2 - i\omega\big)\big(\bar{a}_1a_1 + \bar{a}_2a_2 + \bar{\sigma}_1\sigma_1 + \bar{\sigma}_2\sigma_2 + \bar{\rho}_1\rho_1 + \bar{\rho}_2\rho_2 + \bar{b}_1b_1 + \bar{b}_2b_2\big)\mathrm{d}\mathbf{k}\Big)DBD\bar{B} \\ &= 2(\pi\nu)^2 4\frac{1}{2\pi\nu\big(D_0\mathbf{k}^2 - i\omega\big)} \\ &= \frac{4\pi\nu}{D_0\mathbf{k}^2 - i\omega}. \end{aligned} \tag{24}$$

Now, we are interested in the first quantum correction, and whether we are able to recover the Cooperon contribution to the conductivity using the SUSY NLSM formalism. As mentioned previously, the first quantum correction contains quartic terms of B in the exponential, and can therefore not be calculated without some additional work. A standard approach in these cases is doing a mean-field approximation. The goal is to reduce any higher order terms in B to quadratic terms. This means we replace all higher order terms by quadratic terms multiplied with the average of the remaining terms over the "non-interacting" Lagrangian. For example

$$\nabla\bar{B}\nabla B\bar{B}B$$

becomes

$$\begin{aligned} &\nabla\bar{B}\nabla B\langle\bar{B}B\rangle_{F_2} + \langle\nabla\bar{B}\nabla B\rangle_{F_2}\bar{B}B + \underbrace{\nabla\bar{B}\nabla B\bar{B}}_{\langle\nabla\bar{B}M\bar{B}\rangle_{F_2}}B + \underbrace{\nabla\bar{B}\nabla B\bar{B}B}_{\langle\nabla\bar{B}M_1M_2B\rangle_{F_2}} \\ &+\nabla\bar{B}\underbrace{\nabla B\bar{B}B}_{\langle\nabla BMB\rangle_{F_2}} + \bar{B}\langle\nabla B\bar{B}\rangle_{F_2}B, \end{aligned}$$

where $\langle\cdot\cdot\rangle_A$ denotes the average $\int\cdot\cdot\exp(-A)$ and the matrices M are not being averaged, i.e. considered fixed while averaging. It is therefore useful to calculate these averages beforehand, and insert them afterwards. As an example, we

calculate the average

$$\begin{aligned}
\langle B\bar{B}\rangle^{ij}_{F_2} &= \sum_l \langle B_{il}\bar{B}_{lj}\rangle_{F_2} \\
&= \sum_l \int B_{il}\bar{B}_{lj}\exp\Big(-\pi\nu\int\big(D_0\mathbf{k}^2 - i\omega\big)\mathrm{str}\big(B\bar{B}\big)\mathrm{d}\mathbf{k}\Big)DBD\bar{B} \\
&= \sum_l \int B_{il}\bar{B}_{lj}\exp\Big(-\pi\nu\int\big(D_0\mathbf{k}^2 - i\omega\big)\sum_m\Big[\sum_{n=1}^{n=2}B_{nm}\bar{B}_{mn} - \sum_{n=3}^{n=4}B_{nm}\bar{B}_{mn}\Big]\mathrm{d}\mathbf{k}\Big)DBD\bar{B} \\
&= \delta_{ij}\Big(\frac{2}{\pi\nu(D_0\mathbf{k}^2 - i\omega)} - \frac{2}{\pi\nu(D_0\mathbf{k}^2 - i\omega)}\Big) \\
&= 0.
\end{aligned}$$

To solve the integral, we applied the identity for gaussian integrals in the sixth line. We see that the average vanishes. This result is a consequence of the symmetry in our theory, where we have the same number of bosonic and fermionic variables. The supertrace introduces opposite signs for the two, and the two contributions cancel each other out. To explain this further one should realize, that for these types of averages, the fermionic variables will have the same order $(a\bar{a})$ in the pre-exponent and in the exponent. Looking back at the formula for gaussian integrals (23) we see that we require the opposite orders in the two terms. Therefore, this will introduce an additional minus sign from the anti-commutation and the result is that the contributions cancel out. Similarly, we find that

$$\langle\bar{B}B\rangle_{F_2} = \langle\nabla B\nabla\bar{B}\rangle_{F_2} = \langle\nabla\bar{B}\nabla B\rangle_{F_2} = 0.$$

Further, we can argue that because of the translational invariance of our system, the averages should not depend on coordinates, and thus, these averages with negative parity

$$\langle\bar{B}\nabla B\rangle_{F_2} = \langle\nabla\bar{B}B\rangle_{F_2} = \langle\nabla B\bar{B}\rangle_{F_2} = \langle B\nabla\bar{B}\rangle_{F_2} = 0$$

must vanish. The remaining two possible parings are

$$\langle B\bar{M}B\rangle_{F_2}$$

and

$$\langle\bar{B}M\bar{B}\rangle_{F_2},$$

where, again, the matrices M are fixed while averaging, and should have the same structure as the matrices B and $\bar{B}$ have. As an example, we calculate

$$\begin{aligned}
\langle B\bar{M}B\rangle_{F_2} &= \int B(\mathbf{r})\bar{M}B(\mathbf{r})\exp(-F_2)DBD\bar{B} \\
&= \int\Big[\int B(\mathbf{k})\bar{M}B(-\mathbf{k})\exp(-F_2)DBD\bar{B}\Big]\mathrm{d}\mathbf{k} \\
&= \int A(\mathbf{k})\mathrm{d}\mathbf{k}.
\end{aligned}$$

Let's consider an entry A_{ij} of this matrix, omitting the variable $\mathbf{k}$, and using the identity $\bar{M} = C_0 M^T C_0^T$

$$\begin{aligned} A_{ij} &= \int B_{il}\bar{M}_{lm}B_{mj}\exp(-F_2)DBD\bar{B} \\ &= \int B_{il}(C_0)_{lm}M_{nm}\bar{B}_{rn}(C_0)_{jr}\exp(-F_2)DBD\bar{B} \end{aligned}.$$

To further simplify things let's consider $j = 1$ and use the explicit form of

$$C_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{25}$$

to calculate

$$\begin{aligned} A_{i1} &= \int B_{il}(C_0)_{lm}M_{nm}\bar{B}_{rn}(C_0)_{1r}\exp(-F_2)DBD\bar{B} \\ &= -\int B_{il}(C_0)_{lm}M_{nm}\bar{B}_{2n}\exp(-F_2)DBD\bar{B}. \end{aligned}$$

Now we can use the fact that (23) contains the factor $\delta_{im}\delta_{jl}$ to replace the summation over l with the condition $l = 2$ and also insert the condition $n = i$

$$\begin{aligned} A_{i1} &= -\int B_{i2}(C_0)_{2m}M_{im}\bar{B}_{2i}\exp(-F_2)DBD\bar{B} \\ &= -\int B_{i2}M_{i1}\bar{B}_{2i}\exp(-F_2)DBD\bar{B} \\ &= -\int B_{i2}M_{i1}\bar{B}_{2i}\exp\Big(\pi\nu\int\left(D_0\mathbf{k}^2 - i\omega\right)\sum_m\Big[\sum_{n=1}^{n=2}B_{nm}\bar{B}_{mn} - \sum_{n=3}^{n=4}B_{nm}\bar{B}_{mn}\Big]\Big)DBD\bar{B}. \end{aligned}$$

We see that for $i = 3, 4$ the matrix elements B_{i2} and M_{i1} anti-commute, and we obtain an additional minus sign when we pull M_{i1} out of the integral. However, this additional sign is compensated when we evaluate the gaussian integral, which then has the form (23), and we obtain in both cases

$$A_{i1} = -M_{i1}\frac{1}{2\pi\nu}\frac{1}{D_0\mathbf{k}^2 - i\omega}. \tag{26}$$

If we define

$$I_1 = \frac{1}{2\pi\nu}\int\frac{1}{D_0\mathbf{k}^2 - i\omega}\mathrm{d}\mathbf{k}, \tag{27}$$

we can write

$$\langle B\bar{M}B\rangle_{F_2}^{i1} = -I_1 M_{i1}. \tag{28}$$

This generalizes to all entries, such that we arrive at

$$\langle B\bar{M}B\rangle_{F_2} = -I_1 M.$$

Similarly, we obtain

$$\langle \bar{B} M \bar{B} \rangle_{F_2} = -I_1 \bar{M}.$$

Therefore, the only contribution in this mean-field scheme, when applied to (12), comes from the term

$$\pi\nu \int \mathrm{str}\big[i\omega\big(B\bar{B}\big)^2\big]\mathrm{d}\mathbf{r} \quad \to \quad -2iI_1\omega\pi\nu \int \mathrm{str}\big(B\bar{B}\big)\mathrm{d}\mathbf{r}.$$

In this mean field scheme, the contribution from F_4 can be included into F_2 resulting in a change of the coefficient multiplying ω in (24). We are not done yet, as the mean field technique allows us to handle the higher order terms in the pre-exponential as well. In (16) the terms of the form

$$B(\mathbf{r}) - \underbrace{B(\mathbf{r})\bar{B}(\mathbf{r})B(\mathbf{r})}_{\langle B\bar{M}B\rangle_{F_2}}$$

become

$$B(\mathbf{r}) + I_1 B(\mathbf{r}) = (1 + I_1)B(\mathbf{r}).$$

So we obtain an additional prefactor to (24) if we apply the MFS to both terms of that form, so

$$\mathrm{str}\big[k(1-\tau_3)\big(B(\mathbf{r}) - B(\mathbf{r})\bar{B}(\mathbf{r})B(\mathbf{r})\big)k(1-\tau_3)\big(\bar{B}(\mathbf{0}) - \bar{B}(\mathbf{0})B(\mathbf{0})\bar{B}(\mathbf{0})\big)\big],$$

becomes

$$(1 + I_1)^2 \mathrm{str}\big[k(1-\tau_3)B(\mathbf{r})k(1-\tau_3)\bar{B}(\mathbf{0})\big].$$

These two changes gives us the corrected correlation function

$$Y_4^{00}(\mathbf{k},\omega) = \frac{4\pi\nu(1+I_1)^2}{D_0\mathbf{k}^2 - i(1+2I_1)\omega}.$$

Since the term I_1 is a quantum correction, it is considered small and it is therefore reasonable to neglect I_1^2. Following this logic we can massage our new result into the same form as the second order result

$$\frac{4\pi\nu(1+I_1)^2}{D_0\mathbf{k}^2 - i(1+2I_1)\omega} \approx \frac{4\pi\nu}{\frac{D_0}{(1+2I_1)}\mathbf{k}^2 - i\omega}.$$

This gives us the first quantum correction to the diffusion coefficient

$$D_4 = \frac{D_0}{(1+2I_1)} \approx D_0(1-2I_1) = D_0\bigg(1 - \frac{1}{\pi\nu}\int \frac{1}{D_0\mathbf{k}^2 - i\omega}\mathrm{d}\mathbf{k}\bigg), \tag{29}$$

and corresponds to the weak localization result which can be obtained via the Cooperon as well.

References

[1] Konstantin Efetov. *Supersymmetry in disorder and chaos (Chapter 4)*. Cambridge Univ. Press, Cambridge, UK, 2012.

[2] Jeremy Mann. A non linear sigma model from conductivity : Supersymmetry method. 2018.