GR — HW #8 — 8/11/2017

Q1 Energy Conditions

Let $\mathcal{M}$ be a 4-dim pseudo-Riem. mnfld. w/ metric g of signature $+1,-1,-1,-1$.
Let $p\in\mathcal{M}$ be given, and $\{e_\alpha\}_\alpha$ a basis of $T_p\mathcal{M}$ with $g_p(e_\alpha,e_\beta)=\eta_{\alpha\beta}$ with $\eta=\operatorname{diag}(1,-1,-1,-1)$.
The 4-velocity of an observer at rest is e_0.

i) **Def.:** The cone of future-oriented, timelike or light-like vectors is $\overline{V^+} := \{ v\in T_p\mathcal{M} \mid g_p(v,v)\geq 0 \wedge v_0\geq 0\}$

where $v_0\in\mathbb{R}$ is the component of v in $\{e_\alpha\}_\alpha$: $v_0 = v(e_0{}^*)$.

Cl.: $\forall\, v\in T_p\mathcal{M}, \left\{ v\in\overline{V^+} \Leftrightarrow \left[g_p(v,w)\geq 0 \;\; \forall\, w\in\overline{V^+}\right]\right\}$

Pf.: $\boxed{\Rightarrow}$ [redacted]

By Cauchy-Schwarz, [redacted]

$\boxed{\Leftarrow}$ Let $v\in T_p\mathcal{M}$: $g_p(v,w)\geq 0 \;\forall\, w\in\overline{V^+}$. Want $g_p(v,v)\geq 0 \wedge v_0\geq 0$.
Define $w\in T_p\mathcal{M}$ with components $\left(1, \frac{v_i}{(v_1{}^2+v_2{}^2+v_3{}^2)^{1/2}}\right)$.

Cl.: $w\in\overline{V^+}$

Pf.: [redacted]

[redacted]

ii) Let $(a,b)\in\mathbb{R}^2$. $u\in T_p\mathcal{M}$: $g_p(u,u)>0$ (timelike).
Define $T := a\, u^b\otimes u^b - b\, g_p(u,u)\, g$.

Cl.: $\left[\forall\, v\in T_p\mathcal{M},\; g_p(v,v)\geq 0 \Rightarrow g_p(v,(T v^\#))\geq 0\right] \Leftrightarrow a\geq 0 \wedge b\leq a$.

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Let $v \in T_pM$ be given s.t. $g_p(v,v) \geq 0$.

$$Tv = a \underbrace{u^\flat(v)}_{g_p(u,v)} u^\flat - b\, g_p(u,u)\, g_p(v,\cdot)$$

$$(Tv)^\# = a\, g_p(u,v)\, u - b\, g_p(u,u)\, v$$

$\Rightarrow g_p(v,(Tv)^\#) =$ [redacted]

Cl.: $g_p(u,v)^2 \geq g_p(u,u)\, g_p(v,v)$

Pf.: WLOG we may assume $\{e_\alpha\}_\alpha$ has been chosen s.t. $u = u_0 e_0$ $\exists\, u_0 \in \mathbb{R}$.

[redacted]

iii) Cl.: $T_{00} \geq 0$ in every basis (for some $(0,2)$-tensor T) iff

$\forall\, v \in T_pM : g_p(v,v) \geq 0$, $g_p(v,(Tv)^\#) \geq 0$ $= T(v,v)$

Pf.: [redacted]

iv) Let T be a $(0,2)$-tensor.

Cl.: $T_{00} + \sum_{i=1}^{3} T_{ii} \geq 0$ in every basis iff $T(v,v) - \frac{1}{2}\, tr(T) g(v,v) \geq 0$

for every $v \in T_pM : g_p(v,v) \geq 0$.

Pf.: Recall that since T is a $(0,2)$-tensor, its trace needs to be defined by first converting one of its slots to a covector using the metric.

In a basis $\{e_\alpha\}_\alpha$ as above,

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$$T = T(e_\alpha, e_\beta)\, e_\alpha^* \otimes e_\beta^* \quad \text{where} \quad e_\alpha^* \equiv e_\alpha^\flat$$

$$\Rightarrow \quad T(g_p(\cdot,-),\cdot) = T(e_\alpha, e_\beta)\, e_\alpha \otimes e_\beta^*$$

$$tr(T) = T(e_\alpha, e_\beta)\, tr(e_\alpha \otimes e_\beta^*) = T(e_\alpha, e_\beta)\, g_p(e_\alpha, e_\beta)$$
$$= T_{00} - T_{11} - T_{22} - T_{33}$$

OTOH, $T(e_0, e_0) - \frac{1}{2} tr(T)\, g(e_0, e_0) =$

$=$ [redacted]

Cl.: Let T be the $(0,2)$-energy momentum tensor. Then

$$\left[T(v,v) - \tfrac{1}{2} tr(T)\, g(v,v) \geq 0 \quad \forall\, v \in T_pM : g_p(v,v) \geq 0\right]$$

$$\Leftrightarrow$$

$$Ricci(v,v) \geq 0$$

Pf.: Recall the Einstein field eq-ns: [redacted]

$$\boxed{G = 8\pi T} \quad ⊛ \qquad \rho(X,Y) \equiv \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

where G is the Einstein $(0,2)$-tensor: $\quad tr(Z \mapsto \rho(Z,X)Y) \equiv Ricci(X,Y)$

$$G \equiv Ricci - \tfrac{1}{2} R g \quad , \quad R \equiv tr(Ricci)$$

Take the trace of ⊛ to get:

[redacted]

$\Rightarrow \quad -R =$ [redacted]

$\Rightarrow \quad Ricci =$ [redacted]

$\boxed{Ricci = \text{[redacted]}}$ Alternative formulation of E.F.E.

1b) Cl.: If T is a $(0,2)$-tensor then the condition $T_{00} \geq 0$ and $g_p(T(e_0,\cdot), T(e_0,\cdot)) \geq 0$ in every basis $\{e_\alpha\}_\alpha$ is equivalent to

$$\left.\begin{aligned} &g_p(v,(Tv)^\#) = T(v,v) \geq 0 \\ &g_p((Tv)^\#,(Tv)^\#) \geq 0 \end{aligned}\right\} \forall\ v \in T_pM:\ g_p(v,v) \geq 0$$

Pf.: [redacted]

Note: The above cond. is also equivalent to $T(e_0,\cdot) \in \overline{V}^+$.

Cl.: The above cond. is also equivalent to $T_{00} \geq |T_{\alpha\beta}|$ $\forall\, \alpha,\beta$, in <u>any</u> basis.

Pf.: Let $v \in T_pM$ be given with $g_p(v,v)$.

Then we assume $T(v,\cdot) \in \overline{V}^+$.

By (i), we know that equivalent to $g_p(T(v,\cdot)^\#, w) \geq 0$ $\forall w \in \overline{V}^+$.

$\boxed{\Rightarrow}$ Pick $v_0 = w_0 = 1$ (so $\|\vec{v}\| \leq 1$, $\|\vec{w}\| \leq 1$). Then

$g_p(T(v,\cdot)^\#, w) = T(v,w) = T_{\alpha\beta} v_\alpha w_\beta \geq 0$ +

With $\vec{v} = \vec{w} = 0$, we get [redacted]

with $v = \pm e_i$, $w = 0$ we get [redacted]

with $v = \pm e_i$, $w = \pm e_k$ we get [redacted]

$\overset{+}{\Rightarrow}$ [redacted]

$\boxed{\Leftarrow}$ We may perform Euclidean on $\vec{e}$ to get $T_{01} = T_{02} = 0$ WLOG. Then [redacted]

vi) <u>Perfect Fluid</u> $T \equiv (\varepsilon + p)\, u^\flat \otimes u^\flat - p\, \underbrace{g_p(u,u)}_{=1}\, g$ $\quad tr(T) = \varepsilon + p - 4p = \varepsilon - 3p$

$\longrightarrow$ T as in (ii) with $a =$ [redacted] $b =$ [redacted] (p pressure, ε energy density).

⊛ (iii) implies (by (ii)) that [redacted] and [redacted] $\Leftrightarrow$ [redacted]

$\longrightarrow$ $T - \frac{1}{2} tr(T) g =$ [redacted]

$=$ [redacted] as in (ii) with $a =$ [redacted], $b =$ [redacted]

⊛ (iv) implies (by (ii)) that [redacted] ≥ 0 and [redacted] ≥ 0 $\Leftrightarrow$ $\boxed{[redacted] \geq 0}$

$$\longrightarrow \quad g((Tu)^{\#}(Tu)^{\#}) = u_\alpha T_{\alpha\beta}\, g_{\beta\gamma}\, T_{\gamma\delta}\, u_\delta\, g_{\alpha\delta}$$
$$= g(u, (``T^T T" u)^{\#})$$

But T is symmetric, so $T^T T = T^2$.

$$(T^2)_{\alpha\beta} \equiv T_{\alpha\gamma}\, g_{\gamma\delta}\, T_{\delta\beta}$$

If $\quad T = a u^\flat \otimes u^\flat - b\, g(u,u) g$

$T(g^{-1}(\cdot,-),\cdot) = a u \otimes u^\flat - b\, g(u,u) \mathbb{1}$

$T^2 =$ [redacted]

$=$ [redacted]

⊛ (iv) implies (by (ii)) that [redacted] ≥ 0

[redacted]

Since $\varepsilon \geq 0$, this implies $\boxed{\text{[redacted]}}$

The other condition from (ii) is [redacted] ≥ 0 which is always true. ✓

Electromagnetism $\quad T \equiv$ something

Note $tr(T) = 0$ (HW6Q1) so both (iii) and (iv) imply the same condition, [redacted] ≥ 0.

We know for EM, $T_{00} = \frac{1}{2}(\|\vec{E}\|^2 + \|\vec{B}\|^2)$, so [redacted]

(iv) is also satisfied:

$T_{0i} = (\vec{E} \wedge \vec{B})_i$ and [redacted] $\leq T_{00}$ ✓

The Vacuum with the Cosmological Term $\quad T = \Lambda g, \Lambda \geq 0$

$T_{00} = \Lambda \geq 0$ ✓ $\Rightarrow$ ([redacted]) is satisfied.

$T - \frac{1}{2} tr(T) g =$ [redacted] does not have its [redacted] component positive! $\Rightarrow$ ([redacted]) is not satisfied!

$T^2 = \Lambda^2 g \Rightarrow$ ([redacted]) is satisfied. ✓

6 Q2

Bound on the Cosmological Const

E.F.E: $\boxed{G = 8\pi T}$ (5.11)

with a cosmological const, we get:

$$\boxed{G = 8\pi T + \Lambda g}$$

(i) $G \equiv \text{Ricci} - \frac{1}{2} R g$

$\Rightarrow \text{tr}(G) = $ [redacted] $\Rightarrow -R = $ [redacted]

$\Rightarrow \text{Ricci} +$ [redacted] $=$ [redacted]

$\boxed{\text{Ricci} = \text{[redacted]}}$

We saw in (5.16) that $(\text{Ricci})(0,0) \sim \Delta\varphi$, φ being the grav. pot.

from the E.F.E. we get [redacted]

$(\text{Ricci})_{00} =$ [redacted]

$=$ [redacted]

$\Rightarrow \boxed{\Delta\varphi = 4\pi\rho - \text{[redacted]}}$

ii) For a point mass $\rho = M\delta$ (at the origin).

Then one can verify it is solved by

$\varphi(x) =$ [redacted] (as $\Delta \|x\|^2 =$ [redacted])

iii) $-(\nabla\varphi)(x) =$ [redacted] is the gravitational accel.

Additional term negligible as long as

[redacted]

Estimate with $M =$ sun mass
$\|x\| =$ Pluto's orbit radius

$\Leftrightarrow$ $\boxed{\text{[redacted]}}$