

Q1 Energy Conditions

Let M be a 4-dim pseudo-Riem. manifold w/ metric g of signature $+1, -1, -1, -1$.

Let $p \in M$ be given, and $\{e_\alpha\}_\alpha$ a basis of $T_p M$ with $g_p(e_\alpha, e_\beta) = \eta_{\alpha\beta}$ with $\eta = \text{diag}(1, -1, -1, -1)$.

The 4-velocity of an observer at rest is e_0 .

i) Def: The cone of future-oriented, timelike or light-like vectors is $\bar{V}^+ := \{v \in T_p M \mid g_p(v, v) \geq 0 \wedge v_0 \geq 0\}$

where $v_0 \in \mathbb{R}$ is the component of v in $\{e_\alpha\}_\alpha$: $v_0 = v(e_0^*)$.

Cl: $\forall v \in T_p M, \{v \in \bar{V}^+ \iff [g_p(v, w) \geq 0 \forall w \in \bar{V}^+]\}$

Pf: \Rightarrow



By Cauchy-Schwarz,



\Leftarrow Let $v \in T_p M$: $g_p(v, w) \geq 0 \forall w \in \bar{V}^+$. Want $g_p(v, v) \geq 0 \wedge v_0 \geq 0$. Define $w \in T_p M$ with components $(1, \frac{v_i}{(v_1^2 + v_2^2 + v_3^2)^{1/2}})$.

Cl: $w \in \bar{V}^+$



ii) Let $(a, b) \in \mathbb{R}^2$. $u \in T_p M$: $g(u, u) > 0$ (timelike).

Define $T := a u^b + b u^a - b g_p(u, u) g$.

Cl: $[\forall v \in T_p M, g_p(v, v) \geq 0 \Rightarrow g_p(v, T v) \geq 0] \iff a > 0 \wedge b \leq a$.

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Let $v \in T_p M$ be given s.t. $g_p(v, v) \geq 0$.

$$Tv = a \underbrace{u^b}_{g_p(u, v)} u^b - b g_p(u, u) g_p(v, \cdot)$$

$$(Tv)^\# = a g_p(u, v) u - b g_p(u, u) v$$

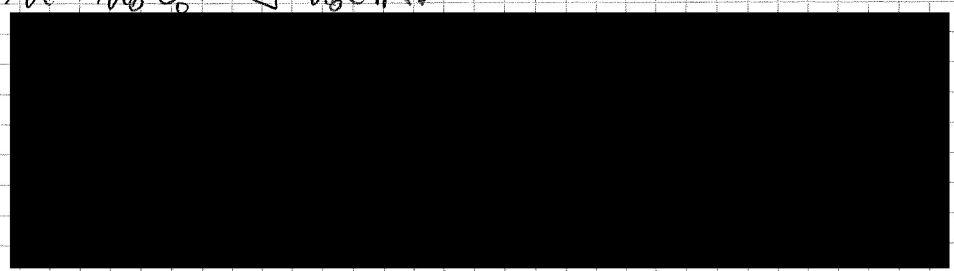
$$\Rightarrow g_p(v, (Tv)^\#) =$$

$$=$$

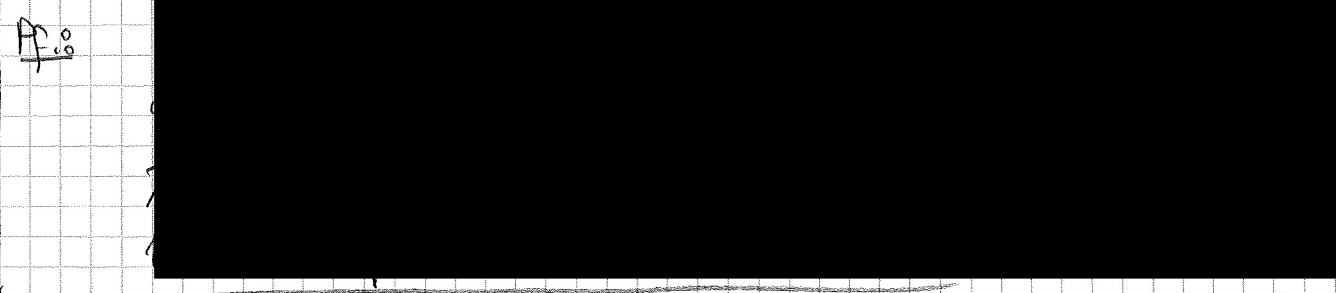
$$=$$

Cl: $g_p(u, v)^2 \geq g_p(u, u) g_p(v, v)$

Pr: WLOG we may assume $\{e_\alpha\}_\alpha$ has been chosen s.t. $u = u_0 e_0 \quad \exists u_0 \in \mathbb{R}$.



iii) Cl: $T_{00} \geq 0$ in every basis (for some (0,2)-tensor T) iff $\forall v \in T_p M : g_p(v, v) \geq 0, \quad g_p(v, (Tv)^\#) \geq 0 \stackrel{T(v, v)}{=}$



iv) Let T be a (0,2)-tensor.

Cl: $T_{00} + \sum_{i=1}^3 T_{ii} \geq 0$ in every basis iff $T(v, v) - \frac{1}{2} \text{tr}(T) g(v, v) \geq 0$ for every $v \in T_p M : g_p(v, v) \geq 0$.

Pr: Recall that since T is a (0,2)-tensor, its trace

needs to be defined by first converting one of its slots to a covector using the metric. 3

In a basis $\{e_\alpha\}$ as above,

$$T = T(e_\alpha, e_\beta) e_\alpha^* \otimes e_\beta^* \quad \text{where } e_\alpha^* \equiv e_\alpha^b$$

$$\Rightarrow T(g_p(\cdot, \cdot), \cdot) = T(e_\alpha, e_\beta) e_\alpha \otimes e_\beta^*$$

$$\begin{aligned} \text{tr}(T) &= T(e_\alpha, e_\beta) \text{tr}(e_\alpha \otimes e_\beta^*) = T(e_\alpha, e_\beta) g_p(e_\alpha, e_\beta) \\ &= T_{00} - T_{11} - T_{22} - T_{33} \end{aligned}$$

$$\text{OTOH, } T(e_0, e_0) - \frac{1}{2} \text{tr}(T) g(e_0, e_0) =$$

$$= g_{00}^{-1} \dots$$

Cl₁₀ Let T be the $(0,2)$ -energy momentum tensor. Then

$$\left[T(e_0, e_0) - \frac{1}{2} \text{tr}(T) g(e_0, e_0) \geq 0 \quad \forall T \in T_p M : g_p(e_0, e_0) \geq 0 \right]$$

\Leftrightarrow

$$\text{Ricci}(e_0, e_0) \geq 0$$

PP₁₀ Recall the Einstein field eq-ns:

$$G = 8\pi T \quad (*)$$

$$g(X, Y) \equiv \nabla_X Y - \nabla_Y X - \nabla_{[X, Y]}$$

where G is the Einstein $(0,2)$ -tensor:

$$\int \text{tr}(Z \mapsto g(Z, X) Y) \equiv \text{Ricci}(X, Y)$$

$$G \equiv \text{Ricci} - \frac{1}{2} R g, \quad R \equiv \text{tr}(\text{Ricci})$$

Take the trace of $(*)$ to get:

$$\Rightarrow -R = \dots$$

$$\Rightarrow \text{Ricci} = \dots$$

$$\boxed{\text{Ricci} = \dots}$$

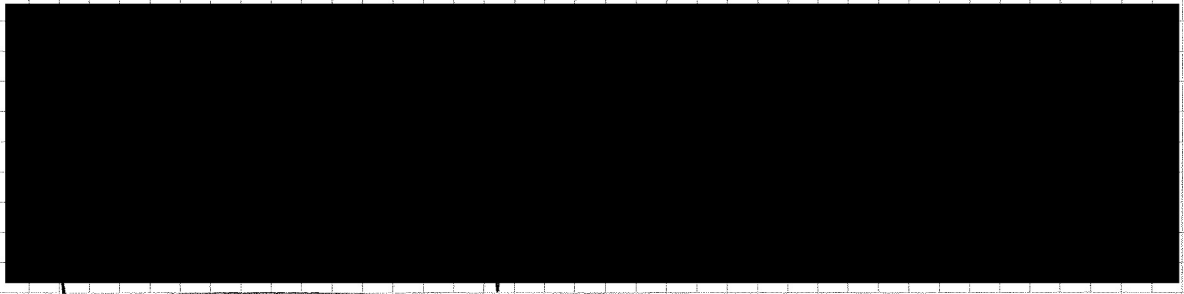
Alternative formulation of E.F.E.

10) Cl₁₀ If T is a $(0,2)$ -tensor then the condition $T_{00} \geq 0$ and $g_p(T(e_0, \cdot), T(e_0, \cdot)) \geq 0$ in every basis $\{e_\alpha\}$ is equivalent to

$$g_p(v, (v)^\#) = T(v, v) \geq 0 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \forall v \in T_p M : g_p(v, v) \geq 0$$

$$g_p((v)^\#, (v)^\#) \geq 0$$

P. 30



Note: The above cond. is also equivalent to $T(e_0, \cdot) \in \bar{V}^+$.

Q. 10: The above cond. is also equivalent to $T_{00} \geq |T_{0\beta}| \quad \forall \alpha, \beta$, in any basis.

P. 3: Let $v \in T_p M$ be given with $g_p(v, v) \geq 0$.

Then we assume $T(v, \cdot) \in \bar{V}^+$.

By (i), we know that equivalent to $g_p(T(v, \cdot)^\#, w) \geq 0 \quad \forall w \in \bar{V}^+$.

\Rightarrow Pick $v_0 = w_0 = 1$ (so $\|v\| \leq 1, \|w\| \leq 1$). Then $g_p(T(v, \cdot)^\#, w) = T(v, w) = T_{\alpha\beta} v_\alpha w_\beta \geq 0$

with $v = w = 0$, we get [redacted]

with $v = \pm e_i, w = 0$ we get [redacted]

with $v = \pm e_i, w = \pm e_k$ we get [redacted]

\Leftarrow We may perform Euclidean on \vec{e} to get $T_{01} = T_{02} = 0$
 $\forall \alpha \neq 0, 1$. Then [redacted]

(ii) Perfect Fluid $T \equiv (\epsilon + p) u^\alpha u^\beta - p g_{\alpha\beta}$ $\text{tr}(T) = \epsilon + p - 4p = \epsilon - 3p$
 $\rightarrow T$ as in (ii) with $a = \epsilon, b = -p$ (p pressure, ϵ energy density).

\otimes (iii) implies (by (ii)) that [redacted] and [redacted]

$$\rightarrow T - \frac{1}{2} \text{tr}(T) g = [redacted]$$

as in (ii) with $a = [redacted], b = [redacted]$

\otimes (iv) implies (by (ii)) that [redacted] > 0 and [redacted] $\geq 0 \Leftrightarrow [redacted] \geq 0$

$$\rightarrow g((Tu)(Tu)) = u_\alpha T_{\alpha\beta} g_{\beta\gamma} T_{\gamma\delta} u^\delta g_{\delta\epsilon} \\ = g(u, (T^T T) u)$$

But T is symmetric, so $T^T T = T^2$.

$$(T^2)_{\alpha\beta} = T_{\alpha\gamma} g_{\gamma\delta} T_{\delta\beta}$$

If $T = a u^a u^b - b g_{ab} g$

$$T(g^i(\cdot, -), \cdot) = a u^i u^b - b g_{ab} g^i(\cdot, a)$$

$$T^2 =$$

$$=$$

(*) (i) implies (by (ii)) that $\dots > 0$

Since $\epsilon > 0$, this implies \dots

The other condition from (ii) is $\dots > 0$ which is always true. \checkmark

Electromagnetism

$T \equiv$ something

Note $\text{tr}(T) = 0$ (HW6Q1) so both (ii) and (i) imply the same condition, $\dots \geq 0$.

We know for EM, $T_{00} = \frac{1}{2} (\|\vec{E}\|^2 + \|\vec{B}\|^2)$, so \dots

(i) is also satisfied:

$$T_{0i} = (\vec{E} \wedge \vec{B})_i \quad \text{and} \quad \dots \leq T_{00} \quad \checkmark$$

The Vacuum with the Cosmological Term

$$T = \Lambda g, \Lambda > 0$$

$$T_{00} = \Lambda > 0 \quad \checkmark \Rightarrow \dots \text{ is satisfied.}$$

$T - \frac{1}{2} \text{tr}(T) g = \dots$ does not have its \dots component positive! $\Rightarrow \dots$ is not satisfied!

$$T^2 = \Lambda^2 g \Rightarrow \dots \text{ is satisfied. } \checkmark$$

Q2

Bound on the Cosmological Const

$$E.F.E: \quad G_i = 8\pi T \quad (5.11)$$

with a cosmological const, we get:

$$G_i = 8\pi T + \Lambda g$$

(i) $G_i \equiv \text{Ricci} - \frac{1}{2} R g$

$$\Rightarrow \text{tr}(G_i) = \dots \Rightarrow -R = \dots$$

$$\Rightarrow \text{Ricci} + \dots = \dots$$

$$\text{Ricci} = \dots$$

We saw in (5.16) that $(\text{Ricci})(0,0) \sim \Delta\varphi$, φ being the grav. pot.

from the E.F.E. we get

$$(\text{Ricci})_{00} = \dots$$

$$= \dots$$

$$\Rightarrow \Delta\varphi = 4\pi\rho - \dots$$

ii) For a point mass $\rho = M\delta$ (at the origin).

Then one can verify it is solved by

$$\varphi(x) = \dots \quad (\text{as } \Delta \|x\|^{-2} = \dots)$$

iii) $-(\nabla\varphi)(x) = \dots$ is the gravitational accel.

Additional term negligible as long as

$$\dots$$

$$\Leftrightarrow \dots$$

Estimate with $M = \text{sun mass}$
 $\|x\| = \text{Pluto's orbit radius}$