2. $\boxed{P^2}$

On Conservation Laws

Goal: See how infinitesimal conservation eq-ns on j and T lead to global conserved quantities as one goes from $SR \to GR$.

$tr(\nabla j) = 0$ leads to charge conservation ✓ (as we shall see promptly)

$tr(\nabla T) = 0$ does <u>not</u>.

Def.: A 3-dim sub manifold of M, Σ, is called spacelike iff $\forall\ (p,q) \in \Sigma^2$, p and q are spacelike separated, that is $g(p,q) > 0$.

Gauss' Theorem: pp. 15 $\displaystyle\int_M \underbrace{(\operatorname{div}_\eta X)\eta}_{\parallel} = \int_{\partial M} i_X \eta$

η is the volume n-form $\qquad d(i_X \eta) = \mathcal{L}_X \eta$

Let W be a vector field

Let $\varphi : U \to \mathbb{R}^n$ be a chart

Cl.: $tr(\nabla W)\sqrt{-\det(g^\varphi)} = \partial_i \sqrt{-\det(+g^\varphi)}\, W_i^\varphi$

Pf.: $tr(\nabla W) = \partial_i W_i^\varphi + \Gamma_{iik} W_k^\varphi$

$$\partial_i \sqrt{-\det(g^\varphi)}\, W_i^\varphi = \sqrt{-\det(g^\varphi)}\, \partial_i W_i^\varphi - \frac{1}{2}\frac{1}{\sqrt{-\det(g^\varphi)}} \times$$

$$\times(\partial_i \det(g^\varphi))\, W_i^\varphi$$

By Jacobi's formula, $\partial_i \det(g^\varphi) = \det(g^\varphi)\, tr(g^{\varphi^{-1}} \partial_i g^\varphi)$

$$\Rightarrow \partial_i \sqrt{-\det(g^\varphi)}\, W_i^\varphi = \sqrt{-\det(g^\varphi)}\Big(\partial_i W_i^\varphi + \frac{1}{2}\underbrace{tr(g^{\varphi^{-1}}\partial_i g^\varphi)}_{\Gamma_{kki}} W_i^\varphi\Big)$$

General relativity. Problem set 7.

HS 17 *Due:* Th, November 9, 2017

1. Charged dust

Consider a charged dust consisting of particles of mass m and electric charge e.

i) Derive the equations of motion for $\rho(x)$ (mass density in the local rest frame) and $u^\mu(x)$ (4-velocity) in an electromagnetic field $F_{\mu\nu}(x)$. Show that the electric 4-current $j^\mu(x)$ satisfies

$$j^\mu{}_{;\mu} = 0$$

without making use of the Maxwell equations.

Hint: The equation of motion of charged particle is (4.15).

ii) Let $T^{\mu\nu}_{\mathrm{em}}$, $T^{\mu\nu}_{\mathrm{d}}$ be the energy-momentum tensors of the electromagnetic field, resp. of the charged dust. Show that

$$(T^{\mu\nu}_{\mathrm{em}} + T^{\mu\nu}_{\mathrm{d}})_{;\nu} = 0\ .$$

Hint: Apply the "comma $\rightsquigarrow$ semicolon" rule to the special relativitic equation $T^{\mu\nu}_{\mathrm{em},\nu} = -c^{-1}F^{\mu\nu}j_\nu$.

2. On conservation laws

In special relativity the fact that currents and energy-momentum tensors are divergence-free, $j^\nu{}_{,\nu} = 0$, $T^{\mu\nu}{}_{,\nu} = 0$, implies akin integral formulations as conservation laws: The total charge, resp. energy-momentum vector

$$Q(t) = \int_{x^0=ct} j^0 d^3x\ , \qquad P^\mu(t) = \int_{x^0=ct} T^{\mu 0} d^3x$$

are independent of time t, assuming fields decaying at spatial infinity (see Electrodynamics). Not so in general relativity. Among the equations

$$j^\nu{}_{;\nu}\ , \qquad T^{\mu\nu}{}_{;\nu} = 0$$

only the first one admits such a formulation: The charge

$$Q(\Sigma) = \int_\Sigma (j,n)\sqrt{-g_\Sigma}\, d^3x \tag{1}$$

is independent of the spacelike 3-surface $\Sigma \subset M$ extending to spatial infinity (see below for notation).

i) Derive (1) using Gauss' theorem in the form ($D \subset M$ a bounded domain, W a vector field)

$$\int_D W^\nu{}_{;\nu}\sqrt{-g}\, d^4x = \int_{\partial D} (W,n)\sqrt{\mp g_{\partial D}}\, d^3x\ , \tag{2}$$

where $(\cdot,\cdot) = g(\cdot,\cdot)$ is the spacetime metric; $g(x) = \det(g_{\mu\nu}(x))$, and likewise for the induced metric $g_{\partial D}$: $(X,Y)_{\partial D} = (X,Y)$ for $X, Y \in T_p(\partial D)$, $(p \in \partial D)$; and n is the

ii) Then we get, for $D \subseteq M$ bdd. domain

$$\int_D \mathrm{tr}(\nabla W)\sqrt{-\det(g^{\varphi})} = \int_D \partial_i \sqrt{-\det(g^{\varphi})}\, W_i^{\varphi}$$

$$\overset{\text{Gauss on } \mathbb{R}^4}{=} \int_{\partial D} \sqrt{-\det(g^{\varphi})}\, W_i^{\varphi}\, n_i$$

n_i normal to ∂D

In appropriate choice of coordinates where

$$g = \begin{bmatrix} g_{00} & 0 \\ 0 & g_{\partial D} \end{bmatrix}$$

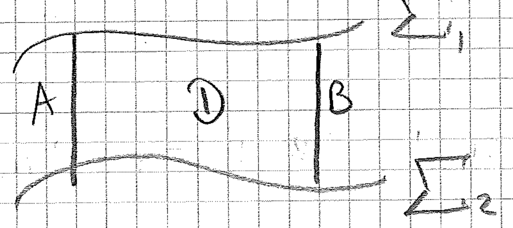
We find $\int_{\partial D} g(W,n)\sqrt{\mp g_{\partial D}}$ sign dependent on where $g(n,n) = \pm 1$.

The total charge on Σ is given by:

$$Q(\Sigma) := \int_{\Sigma} g(j,n)\sqrt{-g_{\Sigma}}$$

__Cl.:__ $Q(\Sigma)$ is indep. of Σ.

__Pf.:__



D regularizing cutoff bounded domain between them.

$$Q(\Sigma_1) - Q(\Sigma_2) = \int_{\Sigma_1} g(j,n)\sqrt{-g_{\Sigma}} - \int_{\Sigma_2} g(j,n)\sqrt{-g_{\Sigma_2}}$$

$$\overset{\text{Gauss}}{=} \int_D \underbrace{\mathrm{tr}(\nabla j)}_{=0}\sqrt{-g} \underbrace{- \int_A - \int_B (\ ) g(j,n)\sqrt{g_{A,B}}}_{j \to 0 \text{ at } \infty}$$

$\boxed{4}$

ii) For a $(0,2)$ tensor, $\mathrm{tr}(\nabla T)_j \sqrt{-\det(g^\varphi)} \neq \partial_i \sqrt{-\det(g^\varphi)}\, T_{ij}$

Hence this fails!

Indeed, we find

$$\boxed{\begin{aligned}\sqrt{-\det(g^\varphi)}\, \mathrm{tr}(\nabla T)_j &= \partial_i\left(\sqrt{-\det(g^\varphi)}\, T_{ij}\right) + \\ &\quad + \sqrt{-\det(g^\varphi)}\, \Gamma_{j\ell k} T_{\ell k}\end{aligned}}$$