GR — Homework #4 — 11/10/2017

$\boxed{\text{Q1}}$ Affine Connections

Let $\nabla : \Gamma(TM)^2 \longrightarrow \Gamma(TM)$ be a given affine connection (so it satisfies the axioms on the bottom of pp. 18 in the lecture notes). (Recall $\Gamma(TM)$ is the $\mathcal{F}(M)$-module of sections on TM, or of vector fields on M).

Let $\delta : \Gamma(TM)^2 \rightarrow \Gamma(TM)$ be another given map (not necessarily an affine connection.

Define $B : \Gamma(TM)^2 \rightarrow \Gamma(TM)$ via $\boxed{B := \nabla - \delta}$.

Note this makes sense as $\Gamma(TM)$ is an $\mathcal{F}(M)$-module, so that its points may be added or subtracted.

Define a map $Q : \Gamma(T^*M) \times \Gamma(TM)^2 \longrightarrow \mathcal{F}(M)$ via

$$(\omega, X, Y) \longmapsto \omega(B(X,Y))$$

Cl.: δ is an affine connection $\boxed{\Longleftrightarrow}$ $Q \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$, that is, Q is a tensor field of type $(1,2)$.

Pf.: Recall $Q \in \Gamma(TM \otimes T^*M \otimes T^*M)$ iff it is an $\mathcal{F}(M)$-multilinear map in each of its slots.

For Q to be $\mathcal{F}(M)$-linear in its 1st slot we should have:

① $Q(f\omega, X, Y) \overset{?}{=} f\, Q(\omega, X, Y) \quad \forall\ f \in \mathcal{F}(M)$.

But
$$\begin{aligned} Q(f\omega, X, Y) &\equiv (f\omega)(B(X,Y)) \\ &= f\, \omega(B(X,Y)) \\ &\equiv f\, Q(B(X,Y)) \end{aligned}$$

② $Q(\omega_1 + \omega_2, X, Y) \overset{?}{=} Q(\omega_1, X, Y) + Q(\omega_2, X, Y)$

But
$$\begin{aligned} Q(\omega_1 + \omega_2, X, Y) &\equiv (\omega_1 + \omega_2)(B(X,Y)) \\ &\equiv \omega_1(B(X,Y)) + \omega_2(B(X,Y)) \\ &\equiv Q(\omega_1, X, Y) + Q(\omega_2, X, Y) \end{aligned}$$

So apparently Q is always $\mathcal{F}(M)$-linear in its first slot, regardless of what B is. ✓

Next note that Q is $\mathcal{F}(M)$-linear in its 2nd slot iff

(by $\mathcal{F}(M)$-linearity of ω).

Since ∇ is $\mathcal{F}(M)$-linear in its 1st slot and $\nabla = B + \delta$,

iff

which is the axioms for δ to be an affine connection.

Finally, Q is $\mathcal{F}(M)$-linear in its 2^{nd} slot iff

But note that

$$\nabla_X Y - \delta_X Y \equiv B(X,Y)$$

$$\Rightarrow \begin{cases} \quad = fB(X,Y) \\ \quad = B(X,fY) \end{cases}$$

But we know that ∇ is an affine connection, so from its -axiom

$$\Rightarrow \begin{cases} \quad = fB(X,Y) \\ \quad = B(X,fY) \end{cases}$$

$$\Leftrightarrow \quad B(X,fY) - fB(X,Y) =$$

$$\Leftrightarrow \quad \text{iff} \quad \underbrace{\text{R.H.S} = 0}_{\delta \text{ obeys axiom (iii)}}$$

We find (separately for each axiom) that Q is $\mathcal{F}(M)$-linear in its 2^{nd} & 3^{rd} slots iff δ is also an affine connection.

An equivalent claim:

Cl.: The space of affine connections on M forms an affine space over the vector space of $\mathcal{F}(M)$-bilinear maps $\Gamma(TM)^2 \to \Gamma(TM)$.

Pf.: Recall $\mathcal{A}$ is an affine space over the v/sp. V iff $\exists$ gp. morphism $t: V \to \mathrm{Sym}(\mathcal{A})$ (where V is viewed as a gp. w.r.t. its additive structure and $\mathrm{Sym}(\mathcal{A})$ is the gp. of bijections $\mathcal{A} \circlearrowleft$) s.t. $\forall\, a \in \mathcal{A}$, $V \ni v \mapsto t(v)a \in \mathcal{A}$ is bijective.

Let $\mathcal{A}(M)$ be the sp. of affine connections on M, and $V(M)$ be the v/sp. of $\mathcal{F}(M)$-bilinear maps $\Gamma(TM)^2 \to \Gamma(TM)$.

Define $t: V(M) \to \mathrm{Aut}(\mathcal{A}(M))$ via

$$B \mapsto (\cdot \mapsto \cdot - B)$$

Verify it's a gp. morphism, that it's well-defined essentially by the proof above and it is bijective again by the same proof.

Cl.: If $(\nabla, \delta) \in \mathcal{A}(\mathcal{M})^2$ then $((1-\alpha)\nabla + \alpha\delta) \in \mathcal{A}(\mathcal{M})$ $\forall\ \alpha \in (0,1)$.

Pf.: Let B be as before w.r.t. ∇ and δ.

Note $\equiv \alpha B$

Since $B \in \mathcal{V}(\mathcal{M}) \Leftrightarrow \alpha B \in \mathcal{V}(\mathcal{M})$ (with $\mathcal{V}(\mathcal{M})$ as above) $\forall\ \alpha \in \mathbb{R}$, we apply the 1st claim to get the result.

Let $\varphi : \mathcal{U} \to \mathbb{R}^n$ be a chart. Recall we found an expression for the Christoffel symbols w.r.t. φ of the parallel transport induced by a $\nabla \in \mathcal{A}(\mathcal{M})$ to be:

$$\Gamma_{\nabla}{}^{\varphi}{}_{ijk} = e_i^{\varphi}(\nabla_{d_j^{\varphi}} d_k^{\varphi})$$

where $\{e_i^{\varphi}\}_i$ was the chart-basis for $\Gamma(T^*\mathcal{M}|_{\mathcal{U}})$, and $\{d_i^{\varphi}\}_i$ ―― $\Gamma(T\mathcal{M}|_{\mathcal{U}})$.

Cl.: The components of Q, $Q^{\varphi}{}_{ijk}$, satisfy:

$$Q^{\varphi}{}_{ijk} = \Gamma_{\nabla}{}^{\varphi}{}_{ijk} - \Gamma_{\delta}{}^{\varphi}{}_{ijk}$$

Pf.: $\Gamma_{\nabla}{}^{\varphi}{}_{ijk} - \Gamma_{\delta}{}^{\varphi}{}_{ijk} =$

$\overset{e_i^{\varphi} \text{ linear}}{=}$

$\overset{\text{def. of } B}{=}$

$=$

We see again that the difference of two Christoffel symbols of two different connections behaves like a tensor, just like the difference of the symbols in two different charts.

4 Q2 <u>Euclidean Metric in Polar Coord.</u>

$\mathcal{M} = \mathbb{R}^2$ with the chart $\varphi = \mathbb{1}_{\mathcal{M}}$.

The metric g is a point in $\Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ which obeys certain axioms.

Thus we may specify it as the components of a $(0,2)$-tensor in a chart-basis $\{e_i^\varphi\}_i$ as:

$$\{g(d_i^\varphi, d_j^\varphi)\}_{i,j=1}^2$$

So, a 2×2 matrix.

The axioms are that g should be symmetric & non-deg., for that components matrix that means it should be symmetric and invertible.

Define $\{g(d_i^\varphi, d_j^\varphi)\}_{i,j} := \mathbb{1}_{2\times 2}$, which satisfies these requirements.

Recall from the previous HW the polar coord. chart

$$\psi : \mathbb{R}^2 \setminus \{0\} \longrightarrow (0,\infty)\times[0,2\pi)$$
$$x \longmapsto \begin{bmatrix} \|x\| \\ \arctan(x_2/x_1) \end{bmatrix}$$

To compute the metric in polar coord. we apply the usual coord. transf.:

$$\underbrace{g(d_i^\psi, d_j^\psi)}_{g_{ij}^\psi} = N^{\psi\varphi}_{ii'} N^{\psi\varphi}_{jj'} \underbrace{g(d_{i'}^\varphi, d_{j'}^\varphi)}_{=\delta_{i',j'}} =$$

$(|A|^2 = A^T A)$

$= ($

$j =$

We found then

$$N^{\psi\varphi}(\psi^{-1}(r,\varphi)) = \Big[\qquad \Big]$$

$$\Rightarrow g^\psi(\psi^{-1}(r,\varphi)) =$$

The Christoffel symbols of the g-Levi-Civita connection $\boxed{5}$
are given by the formula

$$\Gamma^{\varphi}_{ijk} = \frac{1}{2} (g^{\varphi})^{-1}{}_{il} \left(g^{\varphi}{}_{jl,k} + g^{\varphi}{}_{kl,j} - g^{\varphi}{}_{jk,l} \right)$$

 Q3

Torsion & the Hessian

$\nabla \in \mathcal{A}(\mathcal{M})$ affine connection

Cl.: $df = \nabla f \qquad \forall \; f \in \mathcal{F}(\mathcal{M})$

Pf.:

Def.: $\forall \; f \in \mathcal{F}(\mathcal{M})$, the Hessian H of f is

$$H(f) := \nabla(\nabla f) \qquad \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M}).$$

Recall the def. of torsion: (2.8)

Def.: Torsion T corresp. to $\nabla \in \mathcal{A}(\mathcal{M})$ is a map

$$\Gamma(T\mathcal{M})^2 \longrightarrow \Gamma(T\mathcal{M}) \quad \text{defined by}$$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

Cl.: $H(f)$ is symmetric $\iff T = 0$

Pf.: $(H(f))(X,Y) \equiv (\nabla(df))(X,Y)$

$$\underset{\substack{\uparrow \\ \text{slot of } \nabla. \\ \text{always takes} \\ \text{last argument}}}{=} (\nabla_Y df)(X) \underset{\substack{\uparrow \\ (2.8)}}{=}$$

$$=$$

$$\Rightarrow (H(f))(X,Y) - (H(f))(Y,X) =$$

$$=$$

$$\equiv (T(X,Y))(f)$$

Q4 # Geodesics in the Hyperbolic Plane

$\mathcal{M} := \{x \in \mathbb{R}^2 \mid \pi_2(x) > 0\}$

Chart $\varphi := \mathbb{1}_{\mathcal{M}}$ — $\pi_n : \mathbb{R}^m \to \mathbb{R}$ proj. onto nth coord.

Metric $g^\varphi(x) = \pi_2(x)^{-2}\, \mathbb{1}_{2\times 2} \equiv (x_2)^{-2}\, \mathbb{1}_{2\times 2}$

Recall that a curve $\gamma : \mathbb{R} \to \mathcal{M}$ is called a g-geodesic iff $\boxed{\nabla \dot\gamma = 0}$ where ∇ is the g-Levi-Civita connection.

That implies in a chart $\nabla\dot\gamma = 0$ becomes ((3.10)):

$$\boxed{\ddot\gamma^\varphi_i + \Gamma^\varphi_{ijk}\, \dot\gamma^\varphi_j\, \dot\gamma^\varphi_k = 0}$$

Γ evaluated on γ!

We start by calculating Γ^φ:

$$\Gamma^\varphi{}_{ijk} = \frac{1}{2}((g^\varphi)^{-1})_{il}\left[(g^\varphi)_{jl,k} + (g^\varphi)_{kl,j} - (g^\varphi)_{jk,l}\right]$$

$g^\varphi(x) = (x_2)^{-2}\,\mathbb{1}_{2\times 2}$, $(g^\varphi)^{-1}(x) = (x_2)^{2}\,\mathbb{1}_{2\times 2}$

$\Gamma^\varphi{}_{1jk}(x) =$

$=$

$\boxed{\Gamma^\varphi{}_{1\cdot\cdot}(x) = }$

$\Gamma^\varphi{}_{2jk}(x) =$

$=$

$\boxed{\Gamma^\varphi{}_{2\cdot\cdot}(x) = }$

8

We go on to calculate the geodesic eq-n:

$$\ddot{\gamma}^{\varphi}_1 + \begin{bmatrix}\dot{\gamma}^{\varphi}_1 & \dot{\gamma}^{\varphi}_2\end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}\dot{\gamma}^{\varphi}_1 \\ \dot{\gamma}^{\varphi}_2\end{bmatrix} = 0$$

① (We omit the φ superscript...)

$$\ddot{\gamma}_2 + \begin{bmatrix}\dot{\gamma}_1 & \dot{\gamma}_2\end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}\dot{\gamma}_1 \\ \dot{\gamma}_2\end{bmatrix} = 0$$

②

Cl.: If $\dot{\gamma}_1 \neq 0$ then ① $\wedge$ ② $\Rightarrow$ $\gamma_1^2 + \gamma_2^2 - a\gamma_1 = b$ $\exists (a,b) \in \mathbb{R}^2$.

Pf.:

$$\boxed{\dot{\gamma}_1 = D\,\gamma_2^{\,2}}$$

Let $f: \mathbb{R} \to \mathbb{R}_{>0}$ be given by $\gamma_1(t) \mapsto \gamma_2(t)$

(We know we can find one such map for all t by reparametrizing γ_2 to be w.r.t. γ_1 instead of t)

Cl.: ② implies $f f'' + (1+f')^2 = 0$

Pf.:

$\Rightarrow$

However $ff'' + 1 + f'^2 = 0$ may be solved via:

$$f(x) = \pm\sqrt{a^2 - (x - x_0)^2} \quad \text{(the } - \text{ solution is irrelevant)}$$

Indeed, $f'(x) =$

$f''(x) =$

$(ff'' + 1 + f'^2)(x) =$ ✓

We find the desired result.

This eq-n describes a semi-circle whose center is on the horizontal axis.
Note that if $\dot{\gamma}_1 = 0$, we find γ describes a vertical line.

Note that except for $\frac{\dot{\gamma}_1}{\gamma_2^2}$ being a constant there is another constant of motion:

Cl.: $\frac{\|\dot{\gamma}\|^2}{\gamma_2^2}$ is a constant of motion.

Pf.:

Later on you'll see another way to find conserved quantities via "Killing fields".