GR – HW #2 Solutions – 27/9/2017

Q1 The Jacobi Identity (Wald Ch. 2 Ex. 3)

Let A, B, C be vector fields on the manifold $\mathcal{M}$.

Repetition of script:

That means they are sections in the tangent bundle $T\mathcal{M}$.
Let $f \in \mathcal{F}(\mathcal{M})$. For any $p \in \mathcal{M}$, $A(p) \in T_p\mathcal{M}$, so $A(p): \mathcal{F}_p(\mathcal{M}) \to \mathbb{R}$. Hence $(A(p))(f) \in \mathbb{R}$. Thus $p \mapsto (A(p))(f)$ is again in $\mathcal{F}(\mathcal{M})$.

Thus it makes sense to compose vector fields together, as $A \circ B \equiv A(p \mapsto (B(p))(\cdot))$ [A acts on the map whose argument is p]

However note that $A \circ B$ is not a tangent field since it doesn't have the Leibnitz property:

$$
\begin{aligned}
(AB)(fg) &\equiv A(p \mapsto (B(p))(fg)) \quad \text{B Leibnitz} \\
&= A(p \mapsto (B(p))(f)\,g(p) + f(p)(B(p))(g)) \\
&= A((p \mapsto (B(p))(f)\,g(p)) + (p \mapsto f(p)(B(p))(g))) \quad \text{A linear} \\
&= A(p \mapsto (B(p))(f)\,g(p)) + A(p \mapsto f(p)(B(p))(g)) \\
&\equiv A((B(\cdot))(f)g) + A(f(B(\cdot))(g)) \\
&= ((AB)(\cdot))(f)\,g(\cdot) + (B(\cdot))(f)(A(\cdot))(g) + (A(\cdot))(f)\,(B(\cdot))(g) \\
&\quad + f(\cdot)((AB)(\cdot))(g)
\end{aligned}
$$

However the commutator, $[A,B] \equiv AB - BA$, does:

$$
\begin{aligned}
[A,B](fg) &= (AB)(f)\,g + \cancel{B(f)A(g)} + \cancel{A(f)B(g)} + f(AB)(g) \\
&\quad - (BA)(f)\,g - \cancel{A(f)B(g)} - \cancel{B(f)A(g)} - f(BA)(g) \\
&= ([A,B])(f)\;g + f\,([A,B])(g) \quad \checkmark
\end{aligned}
$$

2 (i) **Cl.:** $[A,[B,C]] + (\text{cyclic permutations}) = 0$

Pf.: $[A,[B,C]] \equiv [A, BC - CB] = ABC - ACB - BCA + CBA$

$[A,[B,C]] + (\text{cyc. perm.}) \equiv [A,[B,C]] + [B,[C,A]] + [C,[A,B]] =$

$=$

$-$

$-$ $= 0$ ✓

(ii) Let $\{Y_i\}_{i=1}^{n} \subseteq \Gamma(\mathcal{M})$ be n-vector fields s.t. $\forall p \in \mathcal{M}$, $\{Y_i(p)\}_{i=1}^{n}$ is a basis of $T_p\mathcal{M}$.

Note $[Y_i, Y_j] \in \Gamma(\mathcal{M})$, so we may expand it at each point $p \in \mathcal{M}$ using the basis $\{Y_i(p)\}_{i=1}^{n}$:

$[Y_i, Y_j] =: C^{ij}_{k} Y_k$ (this eq-n defines the expansion coefficients C^{ij}_{k}).

Cl.: $C^{ij}_{k} = -C^{ji}_{k}$

Pf.: $C^{ij}_{k} \equiv [Y_i, Y_j]_k = (-[Y_j, Y_i])_k \overset{\text{Component map is linear}}{=} -[Y_j, Y_i]_k$

$\equiv -C^{ji}_{k}$

Note that since everything is a function of the point $p \in \mathcal{M}$, C^{ij}_{k} are also p-dependent and so they are maps $\mathcal{M} \to \mathbb{R}$.

Cl.: $C^{ij}_{k} C^{\cdot\cdot}_{\cdot} + C^{\cdot\cdot}_{\cdot} C^{\cdot\cdot}_{\cdot} + C^{\cdot\cdot}_{\cdot} = Y_{\cdot} C^{\cdot\cdot}_{\cdot} + Y_{\cdot} C^{\cdot\cdot}_{\cdot} + Y_{\cdot} C^{\cdot\cdot}_{\cdot}$

Pf.: By the Jacobi identity we have for any (i,j,k) in $\{1,\dots,n\}^3$:

$[Y_i, [Y_j, Y_k]] + (\text{cyclic perm.}) = 0$

$$\Leftrightarrow \quad [Y_i, C^{jk}_{l} Y_l] + \text{cyc. perm of } (i,j,k) = 0$$

HW2 Q2

September 29, 2017

1 Notation

Let $\varphi : U_\varphi \to \mathbb{R}^n$ and $\psi : U_\psi \to \mathbb{R}^n$ be two charts near some $p \in \mathcal{M}$.

Then we define basis vectors of $T_p\mathcal{M}$ corresponding to these charts as $d_i^\varphi := \left[\partial_i \left(\cdot \circ \varphi^{-1}\right)\right] \circ \varphi$. Note that this is really a vector *field* defined in a neighborhoud of p. In a point $q \in \mathcal{M}$ it is a tangent vector: d_i^φ at q is $\partial_i|_{\varphi(q)} \left(\cdot \circ \varphi^{-1}\right)$. There are analogous definitions for ψ. We define the expansion coefficients of a vector field X in the basis corresponding to φ as X_i^φ:

$$X = X_i^\varphi d_i^\varphi$$

so that $X_i^\varphi \equiv X(\varphi_i)$ with $\varphi_i := \pi_i \circ \varphi$ and $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the natural projection. The transition rule (going from φ to ψ) for the expansion coefficients may be derived easily as

$$\begin{aligned} X_i^\psi &\equiv X(\psi_i) \\ &= X_j^\varphi d_j^\varphi (\psi_i) \end{aligned}$$

so that we define

$$M_{ij}^{\psi\varphi} := d_j^\varphi (\psi_i)$$

and get

$$X_i^\psi = M_{ij}^{\psi\varphi} X_j^\varphi$$

Similarly, we can move the basis vectors themselves:

$$\begin{aligned} d_i^\psi &= d_i^\psi (\varphi_j)\, d_j^\varphi \\ &= M_{ji}^{\varphi\psi} d_j^\varphi \\ &=: N_{ij}^{\psi\varphi} d_j^\varphi \end{aligned}$$

We also have a natural basis for $(T_p\mathcal{M})^*$, given by the dual of d_i^φ. Explicitly it is given by

$$e_i^\varphi := \cdot\,(\varphi_i)$$

That is, given any tangent vector X, $e_i^\varphi (X) \equiv X(\varphi_i) = X_i^\varphi$. The expansion coefficients of a 1-form ω are given by

$$\omega_i^\varphi = \omega\left(d_i^\varphi\right)$$

so that

$$\omega = \omega_i^\varphi e_i^\varphi$$

and the transformation rule for the expansion coefficients is

$$\begin{aligned} \omega_i^\psi &\equiv \omega\left(d_i^\psi\right) \\ &= \omega_j^\varphi e_j^\varphi \left(d_i^\psi\right) \end{aligned}$$

But $e_j^\varphi \left(d_i^\psi\right) \equiv d_i^\psi (\varphi_j) = N_{ij}^{\psi\varphi}$ so that we get

$$\omega_i^\psi = N_{ij}^{\psi\varphi} \omega_j^\varphi$$

and of course the dual basis vectors transform again in the opposite way compared to the expansion coefficients:

$$\begin{aligned} e_i^\psi &= e_i^\psi \left(d_j^\varphi\right) e_j^\varphi \\ &= d_j^\varphi (\psi_i)\, e_j^\varphi \\ &= M_{ij}^{\psi\varphi} e_j^\varphi \end{aligned}$$

We find that the expansion coefficients of a general $(k,\, l)$ tensor T transform as

$$T^\psi_{i_1 \cdots i_k j_1 \cdots j_l} = M^{\psi\varphi}_{i_1 i_1'} \cdots M^{\psi\varphi}_{i_k i_k'} N^{\psi\varphi}_{j_1 j_1'} \cdots N^{\psi\varphi}_{j_l j_l'} T^\varphi_{i_1' \cdots i_k' j_1' \cdots j_l'}$$

2 Properties of the Transition Matrices

1 Claim. We have $N_{ij}^{\psi\varphi} M_{ik}^{\psi\varphi} = \delta_{jk}$ and $N_{ij}^{\psi\varphi} M_{kj}^{\psi\varphi} = \delta_{ik}$.

Proof. We start by plugging in the definitions

$$N_{ij}^{\psi\varphi} M_{ik}^{\psi\varphi} \equiv d_i^{\psi}(\varphi_j)\, d_k^{\varphi}(\psi_i)$$

we swap out φ_j and ψ_i for e_j^{φ} and e_i^{ψ} respectively, because it is more transparent then that these are dual vectors to the d's. We get

$$\begin{aligned} N_{ij}^{\psi\varphi} M_{ik}^{\psi\varphi} &= d_i^{\psi}\left(e_j^{\varphi}\right) d_k^{\varphi}\left(e_i^{\psi}\right) \\ &= d_j^{\varphi *}\left(d_i^{\psi}\right) d_i^{\psi *}\left(d_k^{\varphi}\right) \\ &= \left\langle d_j^{\varphi},\, d_i^{\psi}\right\rangle \left\langle d_i^{\psi},\, d_k^{\varphi}\right\rangle \\ &= \left\langle d_j^{\varphi},\, d_i^{\psi} \otimes d_i^{\psi *} d_k^{\varphi}\right\rangle \end{aligned}$$

Now we use the fact that $d_i^{\psi} \otimes d_i^{\psi *} = \mathbb{1}$ because $\left\{ d_i^{\psi} \right\}_{i=1}^{n}$ is an ONB of $T_p\mathcal{M}$ for each p in the domain of that basis. Thus

$$N_{ij}^{\psi\varphi} M_{ik}^{\psi\varphi} = \left\langle d_j^{\varphi},\, d_k^{\varphi}\right\rangle$$

and again using the fact that $\{ d_i^{\varphi} \}_{i=1}^{n}$ is a basis one obtains the proper result. The other result is obtained by repeating the argument with $\varphi \leftrightarrow \psi$. □

2 Corollary. *We have* $d_l^{\varphi}\left(N_{ij}^{\psi\varphi}\right) M_{ik}^{\psi\varphi} = -N_{ij}^{\psi\varphi} d_l^{\varphi}\left(M_{ik}^{\psi\varphi}\right)$.

Proof. Apply d_l^{φ} on the foregoing equation. Since δ_{ik} is a constant scalar function, we get zero on the left hand side (as a tangent vector working on any scalar function is zero). On the right hand side we use the Leibniz property of d_l^{φ}. □

3 Claim. We have $d_k^{\varphi}\left(M_{ii'}^{\psi\varphi}\right) = d_{i'}^{\varphi}\left(M_{ik}^{\psi\varphi}\right)$.

Proof. If we expand out the definitions we will find that this boils down to the fact that $[d_i^{\varphi},\, d_k^{\varphi}] = 0$, which is always true for basis tangent vectors which correspond to charts, which is what d_i^{φ} is. Indeed,

$$\begin{aligned} M_{ii',\,k} - M_{ik,\,i'} &\equiv d_k^{\varphi}(M_{ii'}) - d_{i'}^{\varphi}(M_{ik}) \\ &= d_k^{\varphi}(d_{i'}^{\varphi}(\psi_i)) - d_{i'}^{\varphi}(d_k^{\varphi}(\psi_i)) \\ &= [d_k^{\varphi},\, d_{i'}^{\varphi}](\psi_i) \end{aligned}$$

and $\left[d_i^{\varphi},\, d_j^{\varphi}\right] = 0$ because

$$\begin{aligned} \left(\left[d_i^{\varphi},\, d_j^{\varphi}\right]\right)(f) &\equiv d_i^{\varphi} d_j^{\varphi} f - (i \leftrightarrow j) \\ &= \left[\partial_i\left(d_j^{\varphi} f \circ \varphi^{-1}\right)\right] \circ \varphi - (i \leftrightarrow j) \\ &= \left[\partial_i\left(\left[\partial_j\left(f \circ \varphi^{-1}\right)\right] \circ \varphi \circ \varphi^{-1}\right)\right] \circ \varphi - (i \leftrightarrow j) \\ &= \left[\partial_i\left(\partial_j\left(f \circ \varphi^{-1}\right)\right)\right] \circ \varphi - (i \leftrightarrow j) \\ &= 0 \end{aligned}$$

as $\partial_i\partial_j = \partial_j\partial_i$. □

3 Some short hand notation to make the calculation lighter

From this point onwards, since the charts φ and ψ are fixed, we omit them from the notation. Thus φ is considered the "original" chart and ψ the "new" chart. Consequently, all expansion coefficients in the original chart φ will have φ simply dropped expansion coefficients in the new chart ψ will be denoted by a bar above. We also abbreviate $M_{ij}^{\psi\varphi}$ simply as M_{ij} and the same for N. Finally we also abbreviate $d_i^{\varphi}(O) \equiv O_{,\,i}$ for any object O (typically O is an expansion coefficient in φ or ψ carrying itself some indices, but the application of d_i^{φ} always will be noted with a comma after all other indices).

Hence the transformation law for a vector's expansion coefficients

$$\overline{X}_i = M_{ij} X_i$$

The transformation law for a dual vector's expansion coefficients

$$\overline{\mu}_i = N_{ij} \mu_i$$

Transformation law for a (1, 1) tensor's expansion coefficients

$$\overline{T}_{ij} = M_{ii'} N_{jj'} T_{i'j'}$$

Transformation law for a basis vector

$$\overline{d}_i = N_{ij} d_j$$

In the exercise, we "define" the Lie derivative along a vector field X of the (1, 1) tensor T via its components as

$$(L_X T)_{ij} = T_{ij,\,k} X_k - T_{kj} X_{i,\,k} + T_{ik} X_{k,\,j}$$

To see how it transforms, we must see how its constituent parts transform:

$$\begin{aligned} \overline{X}_{i,\,j} &\equiv \overline{d}_j \left(\overline{X}_i \right) \\ &= N_{jj'} d_{j'} \left(M_{ii'} X_{i'} \right) \\ &= \\ &= \end{aligned}$$

So that $\overline{X}_{i,\,j}$ does *not* transform like a (1, 1) tensor, due to the extra first term (the second term alone is how it should have transformed had it been a (1, 1) tensor).

We have also

$$\begin{aligned} \overline{T}_{ij,\,k} &\equiv \overline{d}_k \left(\overline{T}_{ij} \right) \\ &= \\ &= \end{aligned}$$

So that $\overline{T}_{ij,\,k}$ does *not* transform like a (1, 2) tensor, due to the extra first two terms (the third term alone is how a (1, 2) tensor should have transformed).

We check however the transformation law of $\overline{(L_X T)}_{ij}$:

$$\begin{aligned} \overline{(L_X T)}_{ij} &= \overline{T}_{ij,\,k} \overline{X}_k - \overline{T}_{kj} \overline{X}_{i,\,k} + \overline{T}_{ik} \overline{X}_{k,\,j} \\ &= \end{aligned}$$

(Regroup to terms containing derivatives of N and M and those that don't)

$$=$$

We know what the answer *should* be:

$$\begin{aligned} \overline{(L_X T)}_{ij} &\stackrel{?}{=} M_{ii'} N_{jj'} \left(L_X T \right)_{i'j'} \\ &= M_{ii'} N_{jj'} T_{i'j',\,k} X_k - M_{ii'} N_{jj'} T_{kj'} X_{i',\,k} + M_{ii'} N_{jj'} T_{i'k} X_{k,\,j'} \end{aligned}$$

So we identify those terms in $\overline{(L_X T)}_{ij}$ as C (for "correct", the last two lines) and R (for "rest", the first two lines):

$$C :=$$

and

$$R :=$$

We want to show that

$$C \stackrel{?}{=} M_{ii'}N_{jj'}T_{i'j',\,k}X_k - M_{ii'}N_{jj'}T_{kj'}X_{i',\,k} + M_{ii'}N_{jj'}T_{i'k}X_{k,\,j'} \tag{1}$$

and that $R = 0$.

We start with the first task. In order to do that we must we must "cancel out" factors of M and N. Take for instance the first term in C:

$$N_{kk'}M_{ii'}N_{jj'}M_{kk''}T_{i'j',\,k'}X_{k''} = (M_{ii'}N_{jj'}T_{i'j',\,k'})\,(N_{kk'}M_{kk''}X_{k''})$$

Using 1 we find for that term

$$N_{kk'}M_{ii'}N_{jj'}M_{kk''}T_{i'j',\,k'}X_{k''} =$$
$$=$$
$$=$$

so we get the first term on the RHS of (1) correctly. We proceed similarly using 1 twice more to find that (1) is correct.

We go on to prove that $R = 0$: We use 1 three more times to find:

$$R \equiv$$
$$=$$

for the first line, in the second term we relabel as $i' \leftrightarrow k$ to get

But now use 3 so the first line of our most recent expression for R is zero.

We go on to the next line. We relabel in the second term $j' \leftrightarrow k'$ and $k \leftrightarrow k''$ to get

Now we deal with the term $N_{k''j'}N_{jk'}M_{k''k,\,k'}$. In fact we can rewrite it as

$$N_{k''j'}N_{jk'}M_{k''k,\,k'} =$$
$$=$$
$$=$$
$$=$$

so that we really get zero. In the last expression, used again the fact that $M_{ij,\,k} = M_{ik,\,j}$ (in 3) as proven above already, as well as 2. The proof is finally complete.