General relativity. Problem set 1.

HS 17 *Due:* Tue, September 26, 2017

1. The sphere as a manifold

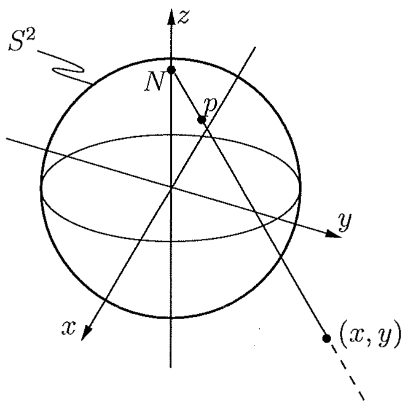
Consider the sphere

$$S^2 = \{p = (x_1, x_2, x_3) \,|\, x_1^2 + x_2^2 + x_3^2 = 1\}$$

and its covering $S^2 = U_+ \cup U_-$ by the two open sets

$$U_\pm = S^2 \setminus \{(0, 0, \pm 1)\}$$

obtained by removing the north, resp. south pole from S^2. The stereographic projection $p \mapsto (x, y)$ shown in the figure provides a chart for U_+ with coordinate neighborhood $\mathbb{R}^2 \ni (x, y)$; similarly there is one for U_- with neighborhood $\mathbb{R}^2 \ni (\bar{x}, \bar{y})$. On which subset of $\mathbb{R}^2$ is the transition function $(x, y) \mapsto (\bar{x}, \bar{y})$ defined? Compute that function.



2. Tensors

a) Show that not all tensors in

$$\begin{aligned} V \otimes V &= \{T \,|\, T \text{ is a bilinear form over } V^* \times V^*\} \\ &= \{\text{linear combinations of tensors } v_1 \otimes v_2 \,|\, v_1, v_2 \in V\} \end{aligned}$$

are of the form $v_1 \otimes v_2$ (simple tensors).

b) Identify $V \otimes W^*$ with the linear space $\mathcal{L}(W, V)$ of linear maps $W \to V$.

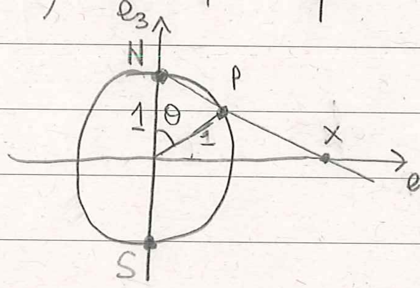
GR – HW # 1 Solutions

Q1 The Sphere as a Manifold

$$S^2 := \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \} = \underbrace{(\|\circ\| - 1)^{-1}(\{0\})}_{\text{zero set}}$$

Define $\mathcal{U}_\pm := S^2 \setminus \{\mp e_3\}$ where $\{e_i\}_{i=1}^3$ is the std. basis of $\mathbb{R}^3$.

Define maps $\eta_\pm : \mathcal{U}_\pm \to \mathbb{R}^2$ by stereographic proj:



$\eta_- : p \mapsto x$ defined $\forall p \in S^2 \setminus \{N\}$. given by the formula

$\eta_-(\theta) =$ on the plane $y=0$.

Outside of the plane we rotate by the azimuthal angle φ:

$\eta_-(\theta, \varphi) =$ $e_r +$ e_φ

We can then convert this to Cartesian coordinates:

$\eta_-(x,y,z) =$

Note the relations $\sin(\operatorname{arctg}(\alpha)) = \frac{\alpha}{\sqrt{1+\alpha^2}}$, $\cos(\operatorname{arctg}(\alpha)) = \frac{1}{\sqrt{1+\alpha^2}}$

$\operatorname{ctg}(\tfrac{1}{2}\operatorname{arctg}(\alpha)) = \frac{1+\sqrt{1+\alpha^2}}{\alpha}$, $x^2+y^2+z^2=1$

to get $\eta_-(x,y,z) =$ $\forall (x,y,z) \in S^2 \setminus \{N\}$.

Similarly, $\eta_+(x,y,z) =$

Cl.: $\mathcal{U}_\pm \in \text{Open}(S^2)$ (Complements of closed)

Cl.: $\eta_\pm$ are continuous.

[2] The transition map is defined on .
It is given by
.

Note

η_-^{-1}: is given by

$x \longmapsto ($

Cl.: $\eta_+ \circ \eta_-^{-1}$: is smooth.

$f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) := \|x\|^2 - 1$ $\boxed{3}$

Def.: $x \in \mathbb{R}^3$ is a regular point of f iff $(df)_x : T_x\mathbb{R}^3 \to T_{f(x)}\mathbb{R}$ is surjective.

Def.: $x \in \mathbb{R}$ is a regular value of f iff $\forall\, y \in f^{-1}(\{x\})$, y is a regular point of f.

Cl.: 0 is a regular value of f.

Pf.: $f^{-1}(\{0\}) = S^2$

$(df)_i(x) = (\partial_i f)(x) = 2x_i$

Of course $2x_i : \underbrace{T_p\mathbb{R}^3}_{\cong \mathbb{R}^3} \to \underbrace{T_p\mathbb{R}}_{\cong \mathbb{R}}$ is surjective.

Cl.: If $f: M \to N$ is smooth and $x \in N$ is a regular value of f, then $f^{-1}(\{x\})$ is a submanifold of M of dimension $m-n$.

$\Rightarrow S^2$ is a submanifold of $\mathbb{R}^3$ with dimension 2.

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Q2 Tensors 6

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a) Cl.: $\{v_1 \otimes v_2 \in V \otimes V\} \neq V \otimes V$

Pf.: Let $T \in V \otimes V$ be given of rank 2 or higher.
Let $\{e_i\}_i$ be an ONB for V.
Then $T =$
Assume $\exists\, (v_1, v_2) \in V^2$: $T = v_1 \otimes v_2$.
Then

$\boxed{1}$

b) Cl.: $V \otimes W^* \cong \underbrace{\mathcal{L}(W, V)}_{\text{lin. maps } W \to V}$ as v/sp. isomorphism.

Pf.:

$$\Rightarrow \boxed{\sum_j w_j^*(\cdot)\, v_j}$$