Q1 Radiation Dominated Universe

We want to model a homogeneous and isotropic universe (like the Friedmann model) but where matter is modeled not as dust (as before) but rather as a radiation field, that is,
$$T = \frac{\varepsilon}{3}(4u^{b}\otimes u^{b} - g) \qquad \text{(HW6Q1)}$$
where ε is the energy density and u is the velocity field w.r.t. which the isotropy applies: $u^{\mu} = \delta^{\mu}{}_{0}$ in a co-moving chart. By homogeneity, ε depends only on time (in a co-moving chart).

Since g is of the form $g = dt^2 - a(t)^2 g_0$ $\exists\, g_0$ a metric on the spacelike 3-mnfld at a given instance of time, solving the E.F.E $G = 8\pi T$ $\Leftrightarrow$ solving an equation for a and ε (the friedmann eq-ns):

LHS of EFE RHS of EFE

$$\begin{cases} a(\dot a^2 + k) - \frac{1}{3}\Lambda a^3 = \frac{1}{3}\varepsilon a^3 & (6.14) \\ 2a\ddot a + \dot a^2 + k - \Lambda a^2 = -pa^2 & (6.15) \end{cases}$$

i) **Cl:** $\frac{1}{3}\varepsilon a^4$ is conserved.

Pf: $\left(\frac{1}{3}\varepsilon a^3\right)^{\cdot} \overset{(6.14)}{=}$

$=$

(6.15) $=$

HW6Q1 $=$ $=$

$p = w\varepsilon$ with $w = \frac{1}{3}$ $=$

$\Rightarrow \left(\frac{1}{3}\varepsilon a^3\right)^{\cdot} +$ $= 0$

But $\left(\frac{1}{3}\varepsilon a^3\right)^{\cdot} + w\varepsilon\frac{1}{3}(a^3)^{\cdot} =$

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Indeed,

$=$

$=$

$\Rightarrow \quad \varepsilon a^{3(w+1)} = q \quad \exists\, q \in \mathbb{R}$, if $a \neq 0$.

For us $w = \frac{1}{3}$, so $\varepsilon a^4 = q$

For photons, there is a grav. redshift (HW6Q3) which says that, e.g. $\frac{\nu_2}{\nu_1} = \frac{a(t_1)}{a(t_2)}$ (6.10)

ν is $\propto$ to the photon energy.

ii) Assume $\Lambda = 0$ (radiation dominated universe).

Then we already computed $\varepsilon a^4 = q \quad \exists\, q \in \mathbb{R}$.

Note (6.15) says $\frac{1}{\dot a}\left(\frac{1}{3}\varepsilon a^3\right)^{\cdot} = -w\varepsilon a^2$, and by (i) this is equivalent to $\varepsilon a^4 = q$. So we only have to solve (6.14): $a(\dot a^2 + k) = \frac{1}{3}\varepsilon a^3$

$$\Leftrightarrow \quad \dot a^2 + k - \frac{1}{3}\varepsilon a^2 = 0$$

$$\Leftrightarrow \quad \boxed{\dot a^2 + k - \frac{q^2}{a^2} = 0}$$

Case 1: $k = 0$

s

Case 2: $|k| = 1$

Define $\eta(t) := \int_{[0,t]} a^{-1} \quad \forall\, t \in [0,\infty)$ ("conformal time")

Let $f : [0,\infty) \to [0,\infty)$ be the inverse of η. That is,

$f \circ \eta := \mathbb{1}$ or $f(\eta(t)) \equiv t \quad \forall t.$

(We'll see it really exists ⊛)

Define $\tilde{a} := a \circ f$.

Then $\tilde{a}' =$

chain rule

Differentiate both sides of $f \circ \eta = \mathbb{1}$ to get

$\eta' = a^{-1}$ by Leibniz formula.

$\Rightarrow f' \circ \eta = a \quad \Rightarrow f' = a \circ f$ by $\eta \circ f \equiv \mathbb{1}$.

We find $\tilde{a}' = \qquad \equiv (\dot{a} a) \circ f$

Thus the eq-n $\dot{a}^2 + k - \frac{q^2}{a^2} = 0$ becomes

$(a\dot{a})^2 + a^2 k - q^2 = 0$

Composing it with f we get:

$= 0$

Note also the initial value for $\tilde{a}$:

We've chose $a(0) \equiv 0$, and $\eta(0) = 0 \Rightarrow f(0) = 0$.

$\tilde{a}(0) \equiv$

Note that if $g(x) = \frac{1}{\sqrt{q^2 - kx^2}}$ then the anti-derivative of g is $\int \frac{1}{\sqrt{q^2 - kx^2}}\,dx = \frac{1}{\sqrt{k}} \operatorname{arctg}\left(\frac{\sqrt{k}\,x}{\sqrt{q^2 - kx^2}}\right) =: G(x)$

Indeed, $\operatorname{arctg}'(\alpha) = \frac{1}{1+\alpha^2}$. Then

$G'(x) =$

$= g(x)$

$$\Rightarrow \int_0^\tau \frac{\tilde{a}'}{\sqrt{q^2 - k\tilde{a}^2}} = \int_0^\tau g(\tilde{a})\,\tilde{a}' = \int_0^\tau G'(\tilde{a})\,\tilde{a}'$$

$$= \int_0^\tau (G \circ \tilde{a})' = G \circ \tilde{a}\Big|_0^\tau$$

$\Rightarrow \quad (G \circ \tilde{a})(\tau) = \pm \tau + q' \qquad$ as $G(0) = 0$.

$\Downarrow$

$G^{-1}(0) = 0$

$\Rightarrow \quad \tilde{a}(\tau) = G^{-1}(\pm \tau + q') \Rightarrow q' = 0$

<u>Case 2.1</u>: $k = 1 \Rightarrow G(x) =$

(other choice of sign unphysical).

<u>Case 2.2</u>: $k = -1 \Rightarrow$

(The other choice of sign is unphysical).

(*) We now return to the question of the existence of f:

We saw $\eta' = a^{-1}$, and we assume $a > 0$ and a cont.

and less than ∞ $\Rightarrow$ η is cont. diff. and non-zero

$\Rightarrow$ η is invertible by the inverse function theorem.

We have $\tilde{a}$ but we need a.

$f' = a \circ f \equiv \tilde{a}$

$$\Rightarrow f = \int \tilde{a} + c' = \begin{cases} \xrightarrow{k=1} \\ \xrightarrow{k=-1} \end{cases}$$

The constant c' is found by the constraint $f(0)=0$.

Now that we have f (indep. of a) we can invert it to get η (indep. of a):

$$\eta(t) = \begin{cases} \xrightarrow{k=1} \\ \searrow \end{cases}$$

$\exists\, n \in \mathbb{Z}$

$$\Rightarrow \boxed{a(t) = (\tilde{a} \circ \eta)(t) =}$$

Q2 The Casual Structure of the Friedmann Models

Start with the Ansatz metric $g = dt^2 - a(t)^2 g_0$.

Switch to conformal time coordinates as above:

$$\eta(t) := \int_{[0,t]} a^{-1} \qquad \forall\ t \in [0,\infty)$$

$$d\eta = \eta'(t)\,dt = a^{-1}\,dt$$

$$\Rightarrow dt = a\,d\eta$$

$$\Rightarrow g = \tilde{a}^2\,d\eta^2 - \tilde{a}^2 g_0 \quad \text{with coordinates } (\eta, \chi, \theta, \varphi).$$

With g_0 as in (6.6) ($R_0 \equiv 1$, $k=1$):

$$g_0 = d\chi^2 + \sin(\chi)^2\,(d\theta^2 + \sin(\theta)^2\,d\varphi^2)$$

metric for $S^3 \subseteq \mathbb{R}^4$ with 3 angles (χ, θ, φ).

with the coordinates $(\chi, \theta, \varphi) \in [0,\pi] \times [0, 2\pi)^2$.

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i) For MD (matter dom.) and RD (radiation dom.) we want to compute the range of η s.t. $\tilde{a}$ goes from 0 (the big bang, its initial pt.) to 0 again (the big crunch, in case this happens for finite times).

MD: By (6.21),

RD: By the earlier exercise,

ii) Is it possible to send a light signal from $(\chi, \eta) = (0,0)$ to $(\chi, \eta) = (0,0)$ in either case MD, RD before the end of the universe?

Note that geodesics starting at $\chi = 0$ "see" the metric

$$g_0 = d\chi^2$$

Since the geodesics on the 3-sphere are great-circles (just as is the case for the 2-sphere), the geodesics correspond to fixed θ, φ and varying $\chi \in [0, \pi]$. (this is not a proof).

Thus along geodesics starting at $(\chi, \eta) = (0,0)$, we "see" the metric

$$g = \tilde{a}(\eta)(d\eta^2 - d\chi^2)$$

which is "conformally" equivalent to the Mikowski metric (prove this) so that