Topics in Mathematical Physics:
Mathematical Aspects of Condensed Matter Physics
Princeton University MAT 595 / PHY 508
Lecture Notes

shapiro@math.princeton.edu

Created: Jan 27 2024, Last Typeset: March 7, 2024

Abstract

These lecture notes correspond to a course given in the Spring semester of 2024 in the math and physics departments of Princeton University.

Contents

1 Quantum dynamics 3
  1.1 Locality .................................................. 4
  1.2 Bloch decomposition–the Fourier series ................... 6
  1.3 Consequences of locality .................................. 10
  1.4 Types of quantum motion .................................. 16
    1.4.1 Relation to the diffusion equation .................... 18
  1.5 The relationship between dynamics and spectral type .... 19
  1.6 AC Spectrum–A vague form of delocalization ............. 22
    1.6.1 Stability of AC spectrum .............................. 22
    1.6.2 The limiting absorption principle ..................... 23
    1.6.3 Existence of wave operators .......................... 24
    1.6.4 Mourre theory ........................................... 27
    1.6.5 The non-zero index method ............................ 31
  1.7 Linear response theory: the Kubo formula ................ 31
    1.7.1 Density matrices ....................................... 31
    1.7.2 The many-body Fermionic ground state in single-particle universe 32
    1.7.3 Electric conductivity .................................... 33
    1.7.4 Linear response theory ................................. 33
  1.8 Zero temperature DC conductivity ......................... 36
    1.8.1 time-reversal invariant case ........................... 36
    1.8.2 The general case: IQHE application .................... 41

2 Random operators and Anderson localization 42
  2.1 Why random operators? .................................... 42
  2.2 Basic setup for random operators ........................ 42
    2.2.1 Abstract definitions ................................. 42
    2.2.2 Concrete application: the Anderson model ............ 44
  2.3 The main results known so far and conjectures ............ 46
    2.3.1 Criteria for localization ............................... 46
    2.3.2 Criteria for delocalization ............................. 47
    2.3.3 Established mathematical facts ........................ 48
    2.3.4 Conjectures ............................................. 48
  2.4 The a-priori bound ........................................ 49
  2.5 Sub-harmonicity in space .................................. 55
  2.6 The decoupling lemma ..................................... 58
  2.7 Complete localization at sufficiently strong disorder ...... 61
  2.8 Localization at weak disorder and extreme energies ....... 63
  2.9 What about localization at the edges of the Laplacian’s spectrum? 65
Syllabus

- The main source of material for the lectures: this very document (to be published and weekly updated on the course website—please do not print before the course is finished and the label “final version” appears at the top).

- Official course textbook: No one, main official text will be used but in preparing these notes; I will probably make heavy use of [Sha16], [Rud91] as well as [RS80] and [BB89]. In particular for the part about random operators I will use the textbook by [AW15].

- Other sources one may consult are [Sto11, Kir07].

- Two lectures per week: Tue and Thur, 1:30 pm – 2:50 pm in Jadwin Hall 343.

- People involved:
  - Instructor: Jacob Shapiro shapiro@math.princeton.edu
    Office hours: by appointment.
  - Assistant: ???

- HW will be periodically posted on the course website but is not meant to be submitted.

- Grade: this is an auditing class.

- Anonymous Ed discussion enabled. Use it to ask questions or to raise issues (technical or academic) with the course.

- If you alert me about typos and mistakes in this manuscript (unrelated to the sections marked [todo]) I’ll be most grateful.

  - Thanks goes to: David Shustin, Grace Sommers (×3).

Semester plan

List of (big) theorems and topics aimed at being included:

- Quantum dynamics.
- Quantum transport and linear response theory.
- Effects of disorder on quantum dynamics: Anderson localization and delocalization.
- The quantum Hall effect.
- Topological phases of matter and the classification of insulators.

Semester plan by date:
1. Jan 30th: Quantum dynamics
2. Feb 1st: Quantum dynamics
3. Feb 6th: Connection between spectral types and quantum dynamics: RAGE
4. Feb 8th: Connection between spectral types and quantum dynamics: Mourre
5. Feb 13th: Linear response theory and the Kubo formula
6. Feb 15th: Linear response theory and the Kubo formula: DC conductivity as a function of the Greens function
7. Feb 20th: Anderson localization: introduction to random operators
8. Feb 22nd: Anderson localization: introduction to random operators
10. Feb 29th: Anderson localization: proof of Anderson localization at weak disorder at spectral edges
11. Mar 5th: Anderson localization: proof of complete Anderson localization in 1D
13. Mar 12th: SPRING RECESS
14. Mar 14th: SPRING RECESS
15. Mar 19th: Introduction to the integer quantum Hall effect, Landau Hamiltonian, etc.
16. Mar 21st: Proof that Chern number is well-defined, connection between trace formula, index formula and periodic k-space formula
17. Mar 26th: Some elements of Fredholm theory
18. Mar 28th: Proof that the edge Hall conductivity is an integer and the bulk-edge correspondence for the IQHE
19. Apr 2nd: The $\mathbb{Z}_2$-index and the Fu-Kane-Mele index
20. Apr 4th: The 1D classification of topological insulators
21. Apr 9th: Some elements of K-theory
22. Apr 11th: K-theoretic classification and the Kitaev periodic table
23. Apr 16th: Introduction to many-body quantum mechanics.
25. Apr 23rd: The Fraas et al proof that the Hall conductivity of an interacting system is quantized.
26. Apr 25th: The Kitaev-Fidkowski $\mathbb{Z}_8$ index for interacting systems.

1 Quantum dynamics

Our goal is to understand the dynamics of electrons in solids. To that end, we will (mostly) make the following assumptions:

- Electrons do not interact with each other.
- Real space is a discrete lattice.
- Quantum mechanics is applicable.
As a result of these assumptions, the appropriate setting to explore models of electrons in thus in single-particle Hilbert space
\[ \mathcal{H} = \ell^2 \left( \mathbb{Z}^d \right) \]
where \( d \) is the space dimension. Sometimes we will consider other lattices besides \( \mathbb{Z}^d \), but in principle this generalization is not very important right now. At other times it will also be useful to allow internal degrees of freedom on each lattice site, i.e., that the wave function is a map
\[ \psi : \mathbb{Z}^d \to \mathbb{C} \]
and one way to write that Hilbert space is as
\[ \mathcal{H} = \ell^2 \left( \mathbb{Z}^d \to \mathbb{C} \right) \cong \ell^2 \left( \mathbb{Z}^d \to \mathbb{C} \right) \otimes \mathbb{C}^N. \]
To specify a model for the dynamics of electrons on this Hilbert space, we must pick a Hamiltonian, which for us will be a bounded linear operator \( H \in \mathcal{B}(\mathcal{H}) \) which is furthermore self-adjoint.

1.1 Locality

Beyond being any bounded linear operator which is self-adjoint, a Hamiltonian better be local. To discuss locality we need to single out a basis on Hilbert space, which is the main reason why it is important to stipulate that we are working with \( \ell^2 \left( \mathbb{Z}^d \right) \): because otherwise, all separable Hilbert spaces are isomorphic, so in principle we have the isomorphism
\[ \ell^2 \left( \mathbb{Z}^d \right) \cong \ell^2 \left( \tilde{\mathbb{Z}}^d \right). \]
Hence, let us choose the position basis as
\[ \{ \delta_x \}_{x \in \mathbb{Z}^d} \subseteq \ell^2 \left( \mathbb{Z}^d \right) \]
defined as
\[ (\delta_x)_y \equiv \delta_{xy} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \quad (x, y \in \mathbb{Z}^d). \]
With this basis, we may form matrix elements of the Hamiltonian \( H \) as
\[ H_{xy} \equiv \langle \delta_x, H \delta_y \rangle \quad (x, y \in \mathbb{Z}^d). \]
Note that if \( \mathcal{H} = \ell^2 \left( \mathbb{Z}^d \right) \otimes \mathbb{C}^N \) then \( H_{xy} \) is actually an \( N \times N \) matrix, whose matrix elements are
\[ (H_{xy})_{ij} \equiv \langle \delta_x \otimes e_i, H \delta_y \otimes e_j \rangle \quad (i, j = 1, \ldots , N) \]
where \( \{ e_i \}_{i=1}^N \) is the standard basis of \( \mathbb{C}^N \). With this definition we may finally make the

**Definition 1.1 (Local operator).** The operator \( A \in \mathcal{B}(\ell^2 \left( \mathbb{Z}^d \right) \otimes \mathbb{C}^N) \) is called local (or exponentially-local) iff there exists some \( R < \infty \) and \( \mu > 0 \) such that
\[ -\frac{1}{\| x - y \|} \log (\| A_{x,y} \|) \geq \mu \quad (x, y \in \mathbb{Z}^d : \| x - y \| \geq R). \] (1.1)
Here by \( \| A_{x,y} \| \) we either mean the absolute value (if \( N = 1 \)) or any matrix norm (they are all equivalent) if \( N > 1 \).

**Claim 1.2.** (1.1) is equivalent to: there exist \( C, \mu \in (0, \infty) \) such that
\[ \| A_{x,y} \| \leq Ce^{-\mu \| x - y \|} \quad (x, y \in \mathbb{Z}^d). \] (1.2)

**Proof.** Let us assume (1.1). Taking its exponential we find
\[ \| A_{x,y} \| \leq e^{-\mu \| x - y \|} \quad (x, y \in \mathbb{Z}^d : \| x - y \| \geq R). \]
Since \( A \) is bounded, we of course also have
\[ \| A_{x,y} \| \leq \| A \| \quad (x, y \in \mathbb{Z}^d). \]
At those $x, y \in \mathbb{Z}^d$ for which $\|x - y\| < R$ we have
\[
\| A_{xy} \| \leq \| A \| e^{\mu \| x - y \|} e^{-\mu \| x - y \|}
\]
so if we define
\[
C := \max \{ \| A \| e^{\mu R}, 1 \}
\]
then we find (1.2). Conversely, assuming (1.2), we have
\[
-\frac{1}{\|x - y\|} \log (\| A_{xy} \|) \geq \mu - \log (C) \quad (x, y \in \mathbb{Z}^d).
\]
Take now $R := \frac{\log(C)}{2\mu}$ to get that if $\|x - y\| \geq R$ then
\[
\frac{\log (C)}{\|x - y\|} \leq \frac{1}{2\mu}.
\]

Sometimes it is useful to also have other modes of locality. We shall introduce them as we go along. One can generalize in two possible directions:

- replace exponential decay with various other rates of decay in the off-diagonal direction
- allow the rate of exponential (or any other) decay to depend on the diagonal position. One way to do so will be called weakly-local: there exists some $\mu > 0$ such that for any $\varepsilon > 0$ there exists some $C_\varepsilon < \infty$ with which
\[
\| A_{xy} \| \leq C_\varepsilon e^{\mu \| x - y \| + \varepsilon \| x \|} \quad (x, y \in \mathbb{Z}^d).
\]

Why do we call such operators local? Because we interpret the number (or matrix) $H_{xy}$ as the transition amplitude to go between the state (or space) $\delta_x$ and $\delta_y$. Sometimes these terms are also called hopping terms. We may think of them as the transition amplitude of infinitesimal time, since
\[
\langle \delta_x, e^{-itH_{xy}} \rangle \approx \langle \delta_x, \delta_y \rangle - it \langle \delta_x, H\delta_y \rangle + O(t^2).
\]

**Example 1.3** (Kinetic energy). We present the discrete Laplacian on $\ell^2 (\mathbb{Z}^d) \ni \psi$ as
\[
(-\Delta \psi)_x := \sum_{y \sim x} \psi_x - \psi_y \quad (x \in \mathbb{Z}^d).
\]
Here $y \sim x$ means all vertices $y \in \mathbb{Z}^d$ which share an edge with $x$, i.e., nearest neighbors of $x$. There are different ways to denote the discrete Laplacian, as well as normalize it. First, consider $\{ R_j \}_{j=1}^d$ as the right-shift operators on $\ell^2 (\mathbb{Z}^d)$. In particular, they are defined as
\[
(R_j \psi)_x := \psi_{x - e_j} \quad (x \in \mathbb{Z}^d)
\]
where $\{ e_j \}_{j=1}^d$ is the standard basis for $\mathbb{R}^d$. Then
\[
-\Delta = 2d1 - \sum_{j=1}^d R_j + R^*_j.
\]
We shall shortly see that with this normalization,
\[
\sigma (-\Delta) = \sigma_{ac} (-\Delta) = [0, 4d].
\]
Example 1.4 (A potential). Let $V : \mathbb{Z}^d \to \mathbb{R}$ be any sequence (with fine print on it later). Then we define on $\ell^2 (\mathbb{Z}^d) \ni \psi$, the operator $V (X)$ as

$$(V (X) \psi)_x := V (x) \psi_x \quad (x \in \mathbb{Z}^d).$$

This may be denoted also using the position operator as follows: Define the (unbounded, vector-valued) position operator $X$ on $\ell^2 (\mathbb{Z}^d) \ni \psi$ as:

$$(X \psi)_x := x \psi_x \quad (x \in \mathbb{Z}^d).$$

Then we proceed to interpret $V (X)$ via the measurable functional calculus (using the fact that $[X_i, X_j] = 0$).

Example 1.5 (Non-local operator). It is instructive to consider an example of a non-local operator. To that end, consider the operator $A$ on $\ell^2 (\mathbb{Z})$ given by the matrix elements

$$A_{xy} := \frac{2 \left( -1 + \cos (\pi (x - y)) + \pi (x - y) \sin (\pi (x - y)) \right)}{(x - y)^2} \quad (x, y \in \mathbb{Z}^d).$$

We claim that this definition yields a bounded operator. But from this equation it is clear that its decay is merely like $n \to \frac{1}{n}$ which is very slow, not even summable. We shall never call operators whose integral kernel does not even decay in a summable way local.

1.2 Bloch decomposition–the Fourier series

A basic tool for us to understand and diagonalize certain operators will be the Fourier series. In physics language, this is “going to momentum space” by way of a Fourier transform. Concretely, we define

$$F : \ell^2 (\mathbb{Z}^d) \to L^2 (\mathbb{T}^d)$$

where $\mathbb{T}^d \equiv [-\pi, \pi)^d$ is the $d$-dimensional torus, and we take the $L^2$ space on it with the Lebesgue measure.

$$L^2 (\mathbb{T}^d) \equiv \left\{ \hat{\psi} : \mathbb{T}^d \to \mathbb{C} \left| \int |\hat{\psi} (k)| \, dk < \infty \right. \right\}. $$

The definition is then

$$(F \psi) (k) := \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} \psi_x \quad (k \in \mathbb{T}^d).$$

To make sense initially we would define $F$ on $\ell^1 \cap \ell^2$ and then extend. Its inverse is given by

$$(F^{-1} \hat{\psi} )_x := (2\pi)^{-d} \int_{k \in \mathbb{T}^d} e^{ik \cdot x} \hat{\psi} (k) \, dk.$$ 

With this definition, we have

Claim 1.6 (Parseval). $F$ is a unitary operator (up to a constant).

Proof. We calculate

$$\langle F \psi, F \varphi \rangle_{L^2} = \int_{k \in \mathbb{T}^d} (F \psi) (k) (F \varphi) (k) \, dk$$

$$= \int_{k \in \mathbb{T}^d} \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} \psi_x \sum_{y \in \mathbb{Z}^d} e^{-ik \cdot y} \varphi_y \, dk$$

$$= \sum_{x, y \in \mathbb{Z}^d} \psi_x \varphi_y \int_{k \in \mathbb{T}^d} e^{ik \cdot (x - y)} \, dk$$

$$= (2\pi)^d \sum_{x \in \mathbb{Z}^d} \psi_x \varphi_x$$

$$= (2\pi)^d \langle \psi, \varphi \rangle_{\ell^2}.$$
To complete the proof, in we should use Abel summation (see e.g. my Complex Analysis lecture notes [Sha23], the proof of Theorem 8.5; we avoid these details here).

The big advantage of the Fourier series is in the fact that certain operators are easy to diagonalize using it. These operators are the periodic or translation invariant operators.

**Definition 1.7** (Periodic operator). An operator $A \in \mathcal{B} \left( \ell^2 \left( \mathbb{Z}^d \right) \right)$ is called periodic iff

$$A_{x,y} = A_{x+z,y+z} \quad (x, y, z \in \mathbb{Z}^d). \quad (1.3)$$

**Lemma 1.8.** If $A \in \mathcal{B} \left( \ell^2 \left( \mathbb{Z}^d \right) \right)$ is periodic then there exists some $a : \mathbb{T}^d \to \mathbb{C}$ such that, if $M_a$ is the diagonal multiplication operator on $L^2 \left( \mathbb{T}^d \right)$ by the function $a$, i.e.,

$$(M_a \hat{\psi}) (k) = a (k) \hat{\psi} (k) \quad (k \in \mathbb{T}^d; \psi \in L^2 \left( \mathbb{T}^d \right))$$

then

$$\mathcal{F} A \mathcal{F}^* = M_a.$$

In fact,

$$a (k) := \sum_{x \in \mathbb{Z}^d} e^{i \langle k, x \rangle} A_{0,x} \quad (k \in \mathbb{T}^d).$$

$a$ is called the symbol associated to $A$.

Moreover,

$$\sigma (A) = \sigma_{ac} (A) = \text{im} \left( a \right) = \left\{ a (k) \mid k \in \mathbb{T}^d \right\}.$$ 

**Proof.** We calculate

$$\left( \mathcal{F} A \mathcal{F}^* \hat{\psi} \right) (k) = \sum_{x \in \mathbb{Z}^d} e^{-i \langle k, x \rangle} \sum_{y \in \mathbb{Z}^d} A_{x,y} \int_{p \in \mathbb{T}^d} e^{i \langle p, y \rangle} \hat{\psi} (p) \, dp$$

$$= \sum_{x \in \mathbb{Z}^d} e^{-i \langle k, x \rangle} \sum_{y \in \mathbb{Z}^d} A_{0,y-x} \int_{p \in \mathbb{T}^d} e^{i \langle p, y \rangle} \hat{\psi} (p) \, dp$$

$$= \sum_{x \in \mathbb{Z}^d} e^{-i \langle k, x \rangle} \sum_{z \in \mathbb{Z}^d} A_{0,z} \int_{p \in \mathbb{T}^d} e^{i \langle p, z+x \rangle} \hat{\psi} (p) \, dp$$

$$= \sum_{z \in \mathbb{Z}^d} \int_{p \in \mathbb{T}^d} e^{i \langle p, z \rangle} \delta (k - p) \hat{\psi} (p) \, dp$$

$$= \sum_{z \in \mathbb{Z}^d} e^{i \langle k, z \rangle} A_{0,z} \hat{\psi} (k) .$$

Let us thus define

$$a (k) := \sum_{z \in \mathbb{Z}^d} e^{i \langle k, z \rangle} A_{0,z} \quad (k \in \mathbb{T}^d).$$

The claim about the spectrum follows via the functional calculus of diagonal operators. We leave the part about the spectrum being purely absolutely continuous as an exercise to the reader.

**Example 1.9.** The right-shift in direction $j = 1, \ldots, d$ operator $R_j$ is defined as

$$(R_j \psi)_y = \psi_{y-e_j} \quad (y \in \mathbb{Z}^d, \psi \in \ell^2 \left( \mathbb{Z}^d \right))$$

It is a periodic operator and hence its Fourier representation is a diagonal multiplication operator:

$$\mathcal{F} R_j \mathcal{F}^* = M_{r_j}.$$
with
\[
  r_j(k) = \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} (R_j)_{0,x}
\]
\[
  = \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} \langle \delta_0, R_j \delta_x \rangle
\]
\[
  = \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} \langle \delta_0, \delta_{x-e_j} \rangle
\]
\[
  = \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} \delta_{x-e_j,0}
\]
\[
  = \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} \delta_{x,e_j}
\]
\[
  = e^{-ik j}.
\]
for all \( k \in \mathbb{T}^d \).

**Example 1.10.** The discrete Laplacian \(-\Delta\) defined via
\[
  (-\Delta \psi)_x := \sum_{y \sim x} \psi_y - \psi_x \quad (\psi \in \ell^2(\mathbb{Z}^d), x \in \mathbb{Z}^d)
\]
is periodic. It may be re-written using the right-shift operator as
\[
  -\Delta = 2 \sum_{j=1}^d 1 - \mathbb{R} \{R_j\}
\]
with \( \mathbb{R} \{A\} \equiv \frac{1}{2} (A + A^*) \). In momentum (i.e., Fourier) space, it is given as multiplication by the function
\[
  \delta(k) := 2 \sum_{j=1}^d 1 - \cos(k_j) \quad (k \in \mathbb{T}^d).
\]
We note that
\[
  2 (1 - \cos(k_j)) = 4 \left[ \sin \left( \frac{1}{2} k_j \right) \right]^2
\]
so that for infinitesimal \( k \),
\[
  \delta(k) \approx \|k\|^2
\]
which resembles the dispersion relation of the Laplacian on \( L^2(\mathbb{R}^d) \). Hence, the discrete Laplacian is “accurate” for small momenta and “distorted” for large momenta (small distances). But in condensed matter physics we are mainly interested in large scales, i.e., small momenta, so that this distortion is not something we care about: it is just making our lives easier mathematically speaking.

**Example 1.11.** The position operator in the \( j \)th direction (\( j = 1, \ldots, d \))
\[
  (X_j \psi)_y \equiv y_j \psi_y \quad (y \in \mathbb{Z}^d, \psi \in \ell^2(\mathbb{Z}^d))
\]
gets mapped to derivative with respect to momentum, i.e.,
\[
  \mathcal{F} X_j \mathcal{F}^* = i \partial_{k_j} \quad (j = 1, \ldots, d).
\]
Proof. We calculate

\[
\left( \mathcal{F} X_j \mathcal{F}^* \hat{\psi} \right) (k) = \sum_{x \in \mathbb{Z}^d} e^{-i k \cdot x} \left( X_j \mathcal{F}^* \hat{\psi} \right)_x
\]

\[
= \sum_{x \in \mathbb{Z}^d} e^{-i k \cdot x} x_j \left( \mathcal{F}^* \hat{\psi} \right)_x
\]

\[
= i \partial_{k_j} \left( \mathcal{F} \mathcal{F}^* \hat{\psi} \right) (k).
\]

\[
\Box
\]

Example 1.12. If \( A \) is periodic then the commutator \([X_j, A]\) is mapped to the derivative:

\[
[X_j, A] \mapsto i M \partial_{a_j}.
\]

Proof. We have

\[
\left( \mathcal{F} [X_j, A] \mathcal{F}^* \hat{\psi} \right) (k) = \left( \mathcal{F} [X_j, A] \mathcal{F}^* \hat{\psi} \right) (k)
\]

\[
= \left( \mathcal{F} X_j \mathcal{F}^* A \mathcal{F}^* \hat{\psi} \right) (k) - \left( \mathcal{F} A \mathcal{F}^* X_j \mathcal{F}^* \hat{\psi} \right) (k)
\]

\[
= \left( i \partial_{j} M \hat{\psi} \right) (k) - \left( M \partial_{a_j} \hat{\psi} \right) (k)
\]

\[
\overset{\text{Leibniz}}{=} M i \partial_{k_j} \hat{\psi}.
\]

\[
\Box
\]

Example 1.13. A multiplication operator in real space \( M_v \) by the function \( v : \mathbb{Z}^d \to \mathbb{R} \) is mapped onto the convolution operator \( C \hat{\psi} \) in momentum space.

Proof. Use the convolution theorem for Fourier series:

\[
\left( \mathcal{F} v (X) \mathcal{F}^* \hat{\psi} \right) (k) = \sum_{x \in \mathbb{Z}^d} e^{-i k \cdot x} \left( v (X) \mathcal{F}^* \hat{\psi} \right)_x
\]

\[
= \sum_{x \in \mathbb{Z}^d} e^{-i k \cdot x} v (x) \left( \mathcal{F}^* \hat{\psi} \right)_x.
\]

If we identify

\[
(\mathcal{F} v) (k) \equiv \hat{v} (k) = \sum_{x \in \mathbb{Z}^d} e^{-i k \cdot x} v (x)
\]

then

\[
\left( \mathcal{F} v (X) \mathcal{F}^* \hat{\psi} \right) (k) = \sum_{x \in \mathbb{Z}^d} e^{-i k \cdot x} (2\pi)^{-d} \int_{\mathbb{T}^d} e^{i p \cdot x} \hat{v} (p) \, dp \left( \mathcal{F}^* \hat{\psi} \right)_x
\]

\[
= (2\pi)^{-d} \int_{\mathbb{T}^d} \hat{v} (p) \, dp \sum_{x \in \mathbb{Z}^d} e^{-i (k-p) \cdot x} \left( \mathcal{F}^* \hat{\psi} \right)_x
\]

\[
= \int_{\mathbb{T}^d} \hat{v} (p) \hat{\psi} (k-p) \, dp
\]

\[
= \left( C \hat{\psi} \right) (k).
\]

We thus recognize that

\[
\mathcal{F} M_v \mathcal{F}^* = C \mathcal{F} \psi.
\]
Theorem 1.14. (Riemann-Lebesgue) If $A$ is local as in (1.1) and periodic as in (1.3), so that $\mathcal{F}A\mathcal{F}^* = M_a$, then $a : \mathbb{T}^d \to \mathbb{C}$ is analytic in an annulus.

Proof. We have from Lemma 1.8 that

$$a(z) = \sum_{x \in \mathbb{Z}^d} \prod_{j=1}^d z_j^{x_j} A_{0,x} \quad \left( z \in (S^1)^d \right).$$

Now deforming $z$, we write it instead as of $\tilde{z} = rz$ where $r \in (0, \infty)^d$ so that

$$|a(\tilde{z})| \leq \sum_{x \in \mathbb{Z}^d} \left( \prod_{j=1}^d r_j^{x_j} \right) \|A_{0,x}\| \leq C_A \sum_{x \in \mathbb{Z}^d} \left( \prod_{j=1}^d r_j^{x_j} \right) e^{-\mu\|x\|}.$$  

Using (??)

Now we use

$$\|x\| \geq \sum_{j=1}^d \frac{|x_j|}{\sqrt{d}}$$

to get

$$|a(\tilde{z})| \leq C_A \prod_{j=1}^d \sum_{x_j \in \mathbb{Z}} r_j^{x_j} e^{-\frac{\mu}{\sqrt{d}} |x_j|}$$

Clearly this is finite if $e^{-\frac{\mu}{\sqrt{d}} < \max \left( \left\{ \frac{r_j}{\sqrt{d}} \right\} \right)}$. Hence we get a convergent power series in an annulus about the torus which is equivalent to analyticity on that annulus [TODO: cite this equivalence].

Remark 1.15. We also have the converse statement: if $a : \mathbb{T}^d \to \mathbb{C}$ is analytic in an annulus then $\mathcal{F}M_a\mathcal{F}$ is exponentially local. We leave this as an exercise to the reader (see e.g. [Sha23] Lemma 8.4).

Remark 1.16. More generally, any-rate polynomial decay will be mapped to smooth “symbols”, and $\ell^p$ locality will be mapped to $C^p$ regularity of the symbol.

1.3 Consequences of locality

The significance of locality is clear from the following Lieb-Robinson theorem. It is usually discussed in the context of many-body quantum mechanics [LR72], but here in the single-particle setting, obtains a particularly simple guise, which we take from [AW15, Exercises 2.2 (a)]:

Theorem 1.17. (Single-particle Lieb-Robinson) If $H = H^* \in \mathcal{B} \left( \ell^2 \left( \mathbb{Z}^d \right) \otimes \mathbb{C}^N \right)$ is local as in (1.1), i.e., that there exist $C_H, \mu_H \in (0, \infty)$ such that

$$\|H_{xy}\| \leq C_H e^{-\mu H\|x-y\|} \quad (x, y \in \mathbb{Z}^d).$$

Then there is some velocity $v_H \in (0, \infty)$ and some $D < \infty$ such that for any $v > v_H$,

$$P \left\{ \text{a particle starting at the origin is outside } B_{vt}(0) \text{ after time } t \right\} \leq D e^{-\frac{\mu}{2\sqrt{d}} (v-v_H)t} \quad (t \geq 0). \quad (1.5)$$

Here we mean $B_{vt}(0) \equiv \{ x \in \mathbb{Z}^d \mid \|x\| < vt \}$. 


Proof. First we interpret the LHS probability. We know from quantum mechanics that the state of a particle starting in the origin is $\delta_0$. Since we have internal degrees of freedom we allow for an arbitrary state $\varphi$ in $\mathbb{C}^N$ so we take the initial state of the particle as $\delta_0 \otimes \varphi$. We know that after time $t$, its state, according to quantum mechanics, is

$$e^{-itH} \delta_0 \otimes \varphi$$

and finally, the probability to measure its position at some $y \in \mathbb{Z}^d$ (in some internal state $\psi \in \mathbb{C}^N$) is

$$\left| \langle \delta_y \otimes \psi, e^{-itH} \delta_0 \otimes \varphi \rangle \right|^2.$$

We thus bound the LHS of (1.5) by

$$N \times \sup_{\varphi, \psi \in \mathbb{C}^N : \|\varphi\| = \|\psi\| = 1} \sum_{y \in B_v(0)^c} \left| \langle \delta_y \otimes \psi, e^{-itH} \delta_0 \otimes \varphi \rangle \right|^2 = \sum_{y \in B_v(0)^c} \left\| \langle \delta_y, e^{-itH} \delta_0 \rangle \right\|^2.$$

(1.6)

Now, we begin with a few preliminary estimates: For any $n \in \mathbb{N}$,

$$\left\| (H^n)_{xy} \right\| = \left\| \sum_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} H_{x, z_1} \cdots H_{z_{n-1}, y} \right\|$$

$$\leq \sum_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} \left\| H_{x, z_1} \right\| \cdots \left\| H_{z_{n-1}, y} \right\|$$

$$\leq \sum_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} C_H^n e^{-\mu H (\|x - z_1\| + \cdots + \|z_{n-1} - y\|)} \quad \text{(Locality of } H)$$

$$\leq C_H^n e^{-\frac{\mu H}{d} \|x - y\|} \sum_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} e^{-\frac{\mu H}{d} (\|z_2 - z_1\| + \cdots + \|z_{n-1} - y\|)}$$.

In this last step, we have used the triangle inequality:

$$\|x - z_1\| + \cdots + \|z_{n-1} - y\| \geq \|x - y\|$$

as well as dropping the first term since it is clearly positive. Since $\mathbb{Z}^d$ is invariant under translations, we find

$$\sum_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} e^{-\nu (\|z_2 - z_1\| + \cdots + \|z_{n-1} - y\|)} = \left( \sum_{z \in \mathbb{Z}^d} e^{-\nu \|z\|} \right)^{n-1}$$

but the inner sum is clearly finite. E.g., $\|z\| \geq \frac{1}{\sqrt{d}} \|z\|_1$ with $\|z\|_1 \equiv \sum_{j=1}^d |z_j|$ which then factorizes:

$$D_{\nu, d} := \sum_{z \in \mathbb{Z}^d} e^{-\nu \|z\|}$$

$$\leq \left( \sum_{z \in \mathbb{Z}} e^{-\frac{\nu}{\sqrt{d}} |z|} \right)^d$$

$$= \left[ \coth \left( \frac{\nu}{2 \sqrt{d}} \right) \right]^d$$

$$< \infty.$$

Combining everything together we have the estimate for any $n \in \mathbb{N}_{\geq 1}$,

$$\left\| (H^n)_{xy} \right\| \leq C_H^n e^{-\frac{\mu H}{d} \|x - y\|} \left( D_{\nu, d} \right)^{n-1}$$

$$= \frac{1}{D_{\nu, d}} \left( C_H D_{\nu, d} \right)^{n-1} e^{-\frac{\mu H}{d} \|x - y\|}.$$
Next, we have
\[
\| \langle \delta_y, e^{-itH} \delta_0 \rangle \| = \left\| \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \langle \delta_y, H^n \delta_0 \rangle \right\|
\leq \sum_{n=0}^{\infty} \frac{t^n}{n! \|H^n\|} \|\delta_0\|
\leq \sum_{n=0}^{\infty} \frac{t^n}{n! \sigma_H^\mu_d} \left( C_H D_{\mu_H, d} \right)^n e^{-\frac{\mu_H}{\sigma_H} \|y\|}
= \frac{1}{\sigma_H} e^{t C_H D_{\mu_H, d} - \frac{\mu_H}{\sigma_H} \|y\|}
\]
and hence the RHS of (1.6) is bounded by (using \(y \in B_{vt}(0)\) implies \(\|y\| \geq vt\)):
\[
\sum_{y \in B_{vt}(0)C} \left( D_{\mu_H, d} \right)^{-2} e^{2t C_H D_{\mu_H, d} - \frac{\mu_H}{\sigma_H} \|y\|} \leq \left( D_{\mu_H, d} \right)^{-2} e^{2t C_H D_{\mu_H, d} - \frac{\mu_H}{\sigma_H} \|y\|} \sum_{y \in Z^d} e^{-\frac{\mu_H}{\sigma_H} \|y\|}
\leq \left( D_{\mu_H, d} \right)^{-2} e^{2t C_H D_{\mu_H, d} - \frac{\mu_H}{\sigma_H} \|y\|} \sum_{y \in Z^d} \frac{1}{D_{\mu_H, d}}
= \frac{1}{\sigma_H} e^{t C_H D_{\mu_H, d} - \frac{\mu_H}{\sigma_H} \|y\|}
\]
and so we identify \(v_H := \frac{C_H D_{\mu_H, d}}{\mu_H} \) and \(D := N \frac{1}{\sigma_H} \).

While the Lieb-Robinson bound gives an intuitive sense for what locality implies for quantum dynamics, we will find more for the Combes-Thomas estimate. Again, originally presented in the context of many-body quantum mechanics [CT73], the single-particle version ([AW15, Chapter 10.3]), presented here roughly speaking says that the analytic functional calculus of Hamiltonians preserves locality:

**Theorem 1.18.** (The Combes-Thomas estimate) If \(H = H^* \in \mathcal{B} (\ell^2 (\mathbb{Z}^d) \otimes \mathbb{C}^N)\) is local as in (1.1) with decay estimate \(\mu_H\), and \(z \in \mathbb{C}\) with
\[
\delta := \text{dist} (z, \sigma (H)) > 0
\]
then there is some \(\tilde{\mu}_H > 0\) (which remains finite as \(\delta \to 0\)) such that
\[
\|R (z)_{xy}\| \leq \frac{2}{\delta} e^{-\tilde{\mu}_H \|y\|} \quad (x, y \in \mathbb{Z}^d)
\]
with \(R (z) = (H - z\mathbf{1})^{-1}\) being the resolvent operator and \(\sigma (H) \subseteq \mathbb{R}\) the spectrum of \(H\).

The constant \(\tilde{\mu}_H\) may be expressed in terms of \(\mu_H\) as
\[
\frac{1}{\delta} \min \left\{ \frac{\delta}{4C_H D_{\mu_H, d}}, \frac{\mu_H}{4} \right\}.
\]

**Corollary 1.19.** (The analytic functional calculus of a local self-adjoint operator is local) Assume that \(f : \mathbb{R} \to \mathbb{C}\) is real-analytic, i.e., that,
\[
f (\lambda) = \frac{1}{2\pi i} \oint_G \frac{1}{\lambda - z} f (z) \, dz \quad (\lambda \in \mathbb{R})
\]
for some closed CCW contour \(G\) which encloses \(\sigma (H)\). Then if \(H = H^* \in \mathcal{B} (\ell^2 (\mathbb{Z}^d) \otimes \mathbb{C}^N)\) is local as in (1.1) then \(f (H)\) is local also as in (1.1).
Proof of Corollary 1.19. Write

\[ f(H) = \frac{i}{2\pi} \oint_{\Gamma} R(z) f(z) \, dz \]

where \( \Gamma \) is a closed CCW contour which encloses \( \sigma(H) \). Since \( H \) is bounded, \( \sigma(H) \) has a finite diameter. Let \( \Gamma \) be a contour which always stays distance 1 away from \( \sigma(H) \), so that, say,

\[ \oint_{\Gamma} |dz| \leq 2 (\|H\| + 1) + 2. \]

Then

\[
\| f(H) \|_{xy} \leq \frac{1}{2\pi} \sup_{z \in \Gamma} \| R(z) \|_{xy} \| f(z) \| \oint_{\Gamma} |dz| \\
\leq \frac{1}{2\pi} \left( \frac{2}{1} e^{-\tilde{\mu} \|x-y\|} \|f\|_{L^\infty(\Gamma)} \right) \| f \|_{L^\infty(\Gamma)} (\|H\| + 2) \\
= \frac{2}{\pi} \| f \|_{L^\infty(\Gamma)} (\|H\| + 2) e^{-\tilde{\mu} \|x-y\|}. 
\]

Note we could indeed make the contour bigger so as to make \( \delta \) bigger (and get better exponential decay) but that would worsen the constants outside the exponential.

Proof of Theorem 1.18. Let \( f : \mathbb{Z}^d \to \mathbb{C} \) be a bounded sequence function such that there is some \( \nu \in (0, \infty) \) with

\[ |f(x) - f(y)| \leq \nu \|x - y\| \quad (x, y \in \mathbb{Z}^d). \]

Define then

\[ H_f := e^{f(X)} H e^{-f(X)} \]

which is clearly also bounded. A short calculation yields

\[
\left[ (H_f - z \mathbbm{1})^{-1} \right]_{xy} = \left[ \left( e^{f(X)} H e^{-f(X)} - z \mathbbm{1} \right)^{-1} \right]_{xy} \\
= \left[ \left( e^{f(X)} H e^{-f(X)} - z e^{f(X)} e^{-f(X)} \right)^{-1} \right]_{xy} \\
= e^{-f(X)} R(z) e^{f(X)} \]

Hence,

\[
\| R(z) \|_{xy} = e^{f(x) - f(y)} \| R_f (z) \|_{xy} \\
\leq e^{f(x) - f(y)} \| R_f (z) \|. 
\]

But for any \( \phi \in \mathcal{H} \),

\[
\| (H_f - z \mathbbm{1}) \phi \| = \| (H - z \mathbbm{1}) \phi \| - \| (H_f - H) \phi \| \\
\geq \delta \| \phi \| - \| H_f - H \| \| \phi \|. 
\]
where we have used (1.7) in the last step. Let us remark that

\[
(H_f - H)_{xy} \equiv (H_f)_{xy} - H_{xy} = e^{f(x)}H_{xy}e^{-f(y)} - H_{xy} = \left(e^{f(x)} - e^{-f(y)} - 1\right)H_{xy}.
\]

Hence, using Holmgren's bound (see Lemma 1.20 just below) and the fact that \(|\alpha| \leq \beta \Rightarrow |e^{\alpha} - 1| \leq e^{\beta} - 1\)

we have then

\[
\|H_f - H\| \leq \max_{x \leftrightarrow y} \sup_{x} \sum_{y} \left\| (H_f - H)_{xy} \right\|
\]

\[
= \max_{x \leftrightarrow y} \sum_{y} \left| e^{f(x)} - e^{-f(y)} - 1 \right| \|H_{xy}\|
\]

\[
\leq \max_{x \leftrightarrow y} \sum_{y} \left( e^{\|x - y\|} - 1 \right) C_{H} e^{-\mu H \|x - y\|}
\]

\[
= \sum_{y} \left( e^{\|y\|} - 1 \right) C_{H} e^{-\mu H \|y\|}
\]

\[
\leq 2C_{H} \nu \sum_{y} e^{-(\mu H - 2\nu)\|y\|} (\text{Use } e^{\rho \|y\|} - 1 \leq 2e^{2\nu \|y\|})
\]

where \(D_{\alpha,d} \equiv \sum_{x \in \mathbb{Z}^d} e^{-\alpha \|x\|}\). Assuming that \(\nu \leq \frac{1}{4}\mu_{H}\) we have

\[
D_{\mu H - 2\nu,d} \leq D_{\frac{1}{2}\mu H,d}
\]

so we pick \(\nu\) as

\[
\nu := \min \left( \left\{ \frac{\delta}{4C_{H} D_{\frac{1}{2}\mu H,d}}, \frac{1}{4}\mu_{H} \right\} \right)
\]

we find \(\|R_{f}(z)\| \leq \frac{\delta}{2}\). Now thanks to the freedom \(f \mapsto -f\) we have

\[
\left\| R(z)_{xy} \right\| \leq 2\delta \min \left( \left\{ e^{f(x) - f(y)}, e^{-f(x) + f(y)} \right\} \right)
\]

\[
= \frac{2\delta}{e^{f(x) - f(y)}}
\]

If we now take, for any \(L \geq 0\),

\[
f_{L}(x) := \nu \min \left( \left\{ L, \|y\| \right\} \right)
\]

then clearly \(f_{L}\) is bounded by \(\nu L\) and

\[
f_{L}(x) - f_{L}(y) = 0 - \nu \min \left( \left\{ L, \|x - y\| \right\} \right)
\]

since \(L\) is arbitrary here, we may take the limit \(L \to \infty\) to obtain

\[
\left\| R(z)_{xy} \right\| \leq \frac{2\delta}{e^{\nu \|x - y\|}} \quad (x, y \in \mathbb{Z}^d).
\]

We recognize that

\[
\hat{\mu}_{H} := \frac{1}{\delta} \min \left( \left\{ \frac{\delta}{4C_{H} D_{\frac{1}{2}\mu H,d}}, \frac{1}{4}\mu_{H} \right\} \right)
\]
Above we have used the following basic

**Lemma 1.20. (Holmgren’s bound)** For any operator $A$ on a Hilbert space with an ONB $\{ \psi_j \}_j$ we have

$$\|A\| \leq \sqrt{\sup_j \sum_k |\langle \psi_j, A\psi_k \rangle|} \sqrt{\sup_k \sum_j |\langle \psi_j, A\psi_k \rangle|}$$

**Proof.** Start by the characterization $\|A\| = \sup \{ |\langle \varphi, A\psi \rangle| \mid \|\varphi\| = 1 \}$, and use

$$|\langle \varphi, A\psi \rangle| \leq \sum_{i,j} |\varphi_i| |A_{ij}| |\psi_j|$$

$$= \sum_{i,j} \left( |\varphi_i| \sqrt{|A_{ij}|} \right) \left( \sqrt{|A_{ij}|} |\psi_j| \right)$$

$$\leq \sqrt{\sum_{i,j} |\varphi_i|^2 |A_{ij}|} \sqrt{\sum_{i,j} |A_{ij}| |\psi_j|^2}$$

$$\leq \sqrt{\left( \sup_i \sum_j |A_{ij}| \right) \left( \sum_i |\varphi_i|^2 \right)} \sqrt{\left( \sup_j \sum_i |A_{ij}| \right) \left( \sum_j |\psi_j|^2 \right)}$$

$$= \sqrt{\sup_i \sum_j |A_{ij}|} \sqrt{\sup_j \sum_i |A_{ij}|}.$$

For the sake of concreteness, let us get an estimate on the decay rate for operators which are nearest-neighbors:

**Claim 1.21.** If $H$ is nearest-neighbor, in the sense that

$$H_{xy} = H_{xy} \chi_{\{0,1\}} (\|x-y\|) \quad (x, y \in \mathbb{Z}^d)$$

and $z \in \mathbb{C}$ is such that

$$0 < \delta := \text{dist} (z, \sigma (H)) < e\|H\| \coth \left( \frac{1}{4\sqrt{d}} \right)$$

then

$$\left| \langle \delta_x, (H - z\mathbb{1})^{-1} \delta_y \rangle \right| \leq \frac{2}{\delta} \exp \left( -\frac{C_d}{\|H\|} \delta \|x-y\| \right) \quad (x, y \in \mathbb{Z}^d)$$

with

$$C_d := \left( 4e \coth \left( \frac{1}{4\sqrt{d}} \right) \right)^{-1}.$$

In particular, for the discrete Laplacian on $\ell^2 (\mathbb{Z}^d)$ normalized to have spectrum in $[0, 4d]$ we have $\|H\| = 4d$ and so

$$\left| \langle \delta_x, (-\Delta - z\mathbb{1})^{-1} \delta_y \rangle \right| \leq \frac{2}{\delta} \exp \left( -\frac{1}{16ed \coth \left( \frac{1}{4\sqrt{d}} \right)} \delta \|x-y\| \right) \quad (x, y \in \mathbb{Z}^d).$$

**Proof.** The locality estimate is obeyed with

$$|H_{xy}| \leq \|H\| \chi_{\{0,1\}} (\|x-y\|)$$

$$\leq e\|H\| |e^{-\|x-y\|}|.$$
We thus recognize
\[ C_H := e^{\|H\|} \]
\[ \mu_H := 1 \]
for the locality of \( H \) and from this we conclude
\[ \hat{\mu}_H = \frac{1}{4\delta} \min \left( \left\{ \frac{\delta}{e^{\|H\|} D_{\frac{1}{2},d}}, 1 \right\} \right). \]

Now recall that
\[ D_{\frac{1}{2},d} \equiv \sum_{z \in \mathbb{Z}^d} e^{-\frac{1}{2}\|z\|} \leq \sum_{z \in \mathbb{Z}^d} e^{-\frac{1}{2}\frac{1}{\sqrt{d}}\|z\|_1} = \left( \sum_{z \in \mathbb{Z}} e^{-\frac{1}{2\sqrt{d}}|z|} \right)^d \]
\[ = \coth \left( \frac{1}{4\sqrt{d}} \right). \]

Assuming that \( \delta < e\|H\| \coth \left( \frac{1}{4\sqrt{d}} \right) \) then, we find
\[ \hat{\mu}_H \geq \frac{1}{4e\|H\| \coth \left( \frac{1}{4\sqrt{d}} \right)} \]

1.4 Types of quantum motion

An important quantity in the study of quantum dynamics is the second moment of the position operator:
\[ M_{ij}(t) := \delta_0, e^{itH} X_i X_j e^{-itH} \delta_0 \quad (t > 0, i, j = 1, \ldots, d). \]

It represents the expectation value of \( X_i X_j \) evolved to time \( t \) on a state \( \delta_0 \).

If \( H \) is reflection invariant, i.e., \( H_{xy} = H_{-x,-y} \) then the off-diagonal elements are zero:
\[ M_{ij}(t) = \sum_{x \in \mathbb{Z}^d} x_i x_j \left\langle \delta_0, e^{-itH} \delta_x \right\rangle \left\langle \delta_x, e^{-itH} \delta_0 \right\rangle \]
\[ = \sum_{x \in \mathbb{Z}^d} x_i x_j \left| e^{-itH}(x,0) \right|^2 \]
\[ = 0. \]

For this reason it is mainly the diagonal (and if \( H \) is isotropic, then all of them are the same) that are interesting, so we focus on
\[ M(t) := \sum_{x \in \mathbb{Z}^d} \|x\|^2 \left| e^{-itH}(x,0) \right|^2. \]

This has the probabilistic interpretation of the variance of the position at time \( t \) of a particle starting at the origin at time zero.
Definition 1.22 (Types of motion). We say that the particle exhibits **ballistic motion** iff

\[ M(t) \sim t^2 \quad (t \to \infty) \, . \]

This is because in classical ballistic motion,

\[ x = vt \]

or

\[ x^2 = v^2 t^2 . \]

Conversely, if

\[ M(t) \sim t \quad (t \to \infty) \]

then the motion is called **diffusive**. Finally, if

\[ M(t) \sim O(1) \quad (t \to \infty) \]

then we say the motion is **localized**.

**Proposition 1.23.** If \( H \) is local and periodic, then the motion is ballistic.

**Proof.** Since \( H \) is periodic, it is judicious to write \( M \) in momentum space:

\[
M_{ij}(t) \equiv \langle \delta_0, e^{itH} X_i X_j e^{-itH} \delta_0 \rangle_{L^2} \\
= (2\pi)^{-d} \langle \mathcal{F} \delta_0, \mathcal{F} e^{itH} X_i X_j e^{-itH} \delta_0 \rangle_{L^2} \\
= (2\pi)^{-d} \langle \mathcal{F} \delta_0, \mathcal{F} e^{itH} \mathcal{F} X_i \mathcal{F} X_j \mathcal{F} e^{-itH} \mathcal{F} \delta_0 \rangle_{L^2} .
\]

Now,

\[
(\mathcal{F} \delta_0)(k) = \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} (\delta_0)_x \\
= 1
\]

so we get

\[
M_{ij}(t) = (2\pi)^{-d} \int_{k \in \mathbb{T}^d} e^{ih(k) \cdot \partial_i \partial_j} e^{-it \cdot (\delta_0) (k)} dk \\
= - (2\pi)^{-d} \int_{k \in \mathbb{T}^d} e^{ih(k) \cdot \partial_i} e^{-it \cdot (\delta_0) (k)} (-it (\partial_j h) (k)) dk \\
= - (2\pi)^{-d} \int_{k \in \mathbb{T}^d} (-it (\partial_i \partial_j h) (k) - t^2 (\partial_i h) (k) (\partial_j h) (k)) dk .
\]

We can proceed in various ways. For example, due to reflection symmetry we could say that \( \int \partial_i \partial_j h = 0 \). Another possibility is to say that since in real space \( M_{ij}(t) \) is clearly real, and \( h \) is real-valued as \( H \) is self-adjoint, it must be the case that \( \int \partial_i \partial_j h = 0 \). It any event, we find that

\[
M_{ij}(t) = t^2 (2\pi)^{-d} \int_{\mathbb{T}^d} (\partial_i h) \partial_j h
\]

which is indeed ballistic. We interpret

\[
\sqrt{(2\pi)^{-d} \int_{\mathbb{T}^d} (\partial_i h) \partial_j h}
\]

as the velocity. \( \square \)

Later on we will see non-trivial examples of localized motion, associated with Anderson localization. Here is a trivial example:
Example 1.24 (Localized motion). Assume that $H$ is diagonal in space, i.e., $H_{xy} \sim \delta_{xy}$. Then

$$M_{ij}(t) = \sum_{x \in \mathbb{Z}^d} x_i x_j |e^{-itH}(x,0)|^2$$

$$= \sum_{x \in \mathbb{Z}^d} x_i x_j |e^{-itH}(x,0)|^2 \delta_{x0}$$

$$= 0.$$

1.4.1 Relation to the diffusion equation

The diffusion equation is given by

$$\partial_t n(t, x) = -D\Delta n(t, x) \quad (t > 0, x \in \mathbb{Z}^d, i, j = 1, \ldots, d)$$

where $n$ is the density of particles (as a function of time $t$ and space $x$; perhaps one interprets $-\Delta$ as a discrete Laplacian). Suppose for a moment that this relationship indeed holds. Then, using the $n$-expectation value

$$\langle X_i X_j \rangle_n \equiv \sum_x x_i x_j n(x, t) \sum_x n(x, t)$$

we find

$$\partial_t \sum_x x_i x_j n(x, t) = \sum_x x_i x_j \partial_t n(x, t)$$

Diffusion equation

$$= \sum_x x_i x_j D(-\Delta n)(t, x)$$

$$= D \sum_x [-\Delta (x_i x_j)] n(t, x)$$

$$= 2D\delta_{ij} \sum_x n(t, x).$$

As a result we find the equation

$$\partial_t \langle X_i X_j \rangle_n = 2D\delta_{ij}.$$

In deriving this equation we have assumed that $D$ is isotropic and homogeneous (with obvious generalization otherwise). Integrating this equation we find

$$\langle X_i X_j \rangle_n = 2tD\delta_{ij} + C$$

so that

$$\lim_{t \to \infty} \frac{\langle X_i X_j \rangle_n}{t} = 2D\delta_{ij}.$$

For this reason, we make the following

Definition 1.25 (Diffusion coefficient). Let $\psi \in \mathcal{H}$ with $||\psi|| = 1$. Then if the motion of $H$ and $\psi$ is diffusion, in the sense that

$$\langle \psi, e^{itH} X_i X_j e^{-it^H} \psi \rangle \sim \delta_{ij} t \quad (t \to \infty)$$

then we define the diffusion coefficient associated with $\psi$ as

$$D(\psi) := \frac{1}{2} \lim_{t \to \infty} \frac{\langle \psi, e^{itH} X_i^2 e^{-itH} \psi \rangle}{t}.$$ 

With this we see the relationship between $D$ and $M$, whenever the motion is diffusive.

Remark 1.26. It could very well be that for different initial states $\psi$ the motion has different behavior. The correct thing to imagine is that $\psi$ is a wave packet concentrated in energy in a part of the spectrum that is associated with different kinds of motion (which may well happen).
1.5 The relationship between dynamics and spectral type

We now turn to an interesting relationship between spectral type (i.e., eigenvalues versus continuous spectrum) and dynamics, i.e., bound states versus scattering states.

We first remark that if \( \psi \) is an eigenstate of the Hamiltonian, in the sense that \( \psi \in \ell^2 \) and

\[
H \psi = \lambda \psi
\]

for some \( \lambda \in \mathbb{R} \), then just by being in \( \ell^2 \) we have some form of spatial decay for \( \psi \). However, as we apply time evolution on \( \psi \), we merely get a phase

\[
t \mapsto e^{-itH} \psi = e^{-it\lambda} \psi
\]

so that

\[
\langle \varphi, e^{-itH} \psi \rangle^2 = |\langle \varphi, \psi \rangle|^2
\]

is constant in time, regardless of \( \varphi \). This is in stark difference to any states in the continuous part of the Hilbert space, as the following theorem shows. The material in this section is taken from [Tes09, AW15].

We first start by a measure-theoretic result (see [Tes09] Theorem 5.5):

**Theorem 1.27** (Wiener). Let \( \mu \) be a finite complex Borel measure on \( \mathbb{R} \) and

\[
\hat{\mu} (t) := \int_{E \in \mathbb{R}} e^{-itE} d\mu (E) \quad (t \geq 0)
\]

is its Fourier transform. Then the Cesàro time average of \( \hat{\mu} \) has the following limit

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{\mu} (t)|^2 dt = \sum_{E \in \mathbb{R}} |\mu (\{ E \})|^2
\]

where the sum on the right-hand side is finite.

**Proof.** We write

\[
\frac{1}{T} \int_0^T |\hat{\mu} (t)|^2 dt = \frac{1}{T} \int_0^T \left( \int_{E \in \mathbb{R}} e^{-itE} d\mu (E) \right) \left( \int_{\tilde{E} \in \mathbb{R}} e^{-i\tilde{E}t} d\mu (\tilde{E}) \right) dt
\]

\[
\overset{\text{Fubini}}{=} \int_{E \in \mathbb{R}} \int_{\tilde{E} \in \mathbb{R}} \frac{1}{T} \int_0^T e^{-i(t\tilde{E}-E)} dt \overline{d\mu (E)} d\mu (\tilde{E})
\]

Now, the function in parenthesis is bounded by 1 and converges pointwise to

\[
\chi_{\{0\}} (\tilde{E} - E)
\]

so the dominated convergence theorem of the limit \( T \to \infty \) yields the result

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{\mu} (t)|^2 dt = \int_{E \in \mathbb{R}} \int_{\tilde{E} \in \mathbb{R}} \chi_{\{0\}} (\tilde{E} - E) \overline{d\mu (E)} d\mu (\tilde{E})
\]

\[
= \int_{E \in \mathbb{R}} \mu (\{ E \}) d\mu (E)
\]

\[
= \sum_{E \in \mathbb{R}} |\mu (\{ E \})|^2
\]

\( \square \)

**Remark 1.28.** If \( \mu_{H, \psi, \varphi} \) is the spectral projection associated to the triplet \( H, \psi, \varphi \) as

\[
\mu_{H, \psi, \varphi} (S) \equiv \langle \psi, \chi_S (H) \varphi \rangle
\]

then: if \( \mu_{H, \psi, \varphi} \) is continuous, or absolutely continuous (w.r.t. the Lebesgue measure), then so is \( \mu_{H, \varphi, \psi} \).
Proof. Let $P_\sharp$ be the projection onto the continuous or absolutely continuous part of the Hilbert space, depending on $\sharp$. Then, by definition, $\mu_{H,\psi,\psi}$ is $\sharp$ iff $\psi \in \text{im}(P_\sharp)$. Moreover, $P_\sharp$ commutes with the functional calculus of $H$. Hence

$$\langle \varphi, \chi_S(H)\psi \rangle = \langle \varphi, \chi_S(H)P_\sharp\psi \rangle = \langle \varphi, P_\sharp\chi_S(H)\psi \rangle = \langle P_\sharp\varphi, \chi_S(H)\psi \rangle.$$  

But since the off-diagonal measure is defined via the polarization identity, we find that $\langle \varphi, \chi(H)\psi \rangle$ is also $\sharp$. \qed

Conclusions for us:

1. If $\psi \in \text{im}(P_c)$ then for any $\varphi \in \mathcal{H}$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\mu_{H,\varphi,\psi}(t)|^2 \, dt = 0.$$  

2. If $\psi \in \text{im}(P_{ac})$ then for any $\varphi \in \mathcal{H}$, then already using the Riemann-Lebesgue lemma we know that

$$t \mapsto \mu_{H,\varphi,\psi}(t)$$

is continuous and decays to zero at infinity.

We sharpen this statement with aid of the following intermediate abstract result

**Theorem 1.29.** Let $A$ be a bounded self-adjoint operator and assume that $K$ is bounded and compact, i.e., that $K(A - z 1)^{-1}$ is compact for some (and hence all) $z \in \sigma(A)^c$. Then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|Ke^{-itA}P_c\psi\|^2 \, dt = 0$$

and

$$\lim_{t \to \infty} \|Ke^{-itA}P_{ac}\psi\| = 0$$

for all $\psi \in \mathcal{H}$.

Note: this theorem extends to unbounded operators, see [Tes09] for the details.

Proof. Let $\psi \in \mathcal{H}_\sharp$ with $\sharp \in \{c,ac\}$. Assume that $F$ is finite rank, with $\{\varphi_j\}_{j=1}^n$ an ONB for $\text{im}(F)$. Then $F = F \sum_{j=1}^n \varphi_j \otimes \varphi_j^*$ and hence

$$F\eta = \sum_{j=1}^n \langle \varphi_j, F\eta \rangle \varphi_j = \sum_{j=1}^n \langle F^*\varphi_j, \eta \rangle \varphi_j \quad (\eta \in \mathcal{H})$$

i.e., any finite rank operator $F$ may be written as

$$F = \sum_{j=1}^n \varphi_j \otimes \psi_j^*$$

for some ONB $\{\varphi_j\}_{j=1}^n$ and $\psi_j = F^*\varphi_j$. Then

$$\|Fe^{-itH}\psi\|^2 = \left\| \sum_{j=1}^n \langle \psi_j, \psi_j^* \rangle e^{-itH}\psi \right\|^2$$

$$= \sum_{j=1}^n \|\langle \psi_j, e^{-itH}\psi \rangle\|^2$$  

(ONB property)
We recognize that
\[ \langle \psi_j, e^{-itH} \psi \rangle = \int_{\lambda \in \mathbb{R}} e^{-it\lambda} d\mu_{H,\psi_j,\psi}(\lambda) = \mu_{H,\psi_j,\psi}(t) \]
with \( \mu_{H,\psi_j,\psi} \) the spectral projection associated to the triplet \( H, \psi_j, \psi \). Hence the result follows thanks to the Wiener theorem above.

Now assume that \( K \) is compact and let \( F_n \to K \) be a sequence of finite rank operators such that
\[ \| K - F_n \| \leq \frac{1}{n} \quad (n \in \mathbb{N}) \]
so that
\[ \| Ke^{-itH} \psi \|^2 \leq 2 \left( \| F_n e^{-itH} \psi \| + \frac{1}{n}\| e^{-itH} \psi \| \right)^2 \]
\[ \leq 2 \left[ F_n e^{-itH} \psi \| + \frac{2}{n} \| \psi \| \right]^2 \] (Using \((a + b)^2 \leq 2a^2 + 2b^2\))

Take now the limit \( t \to \infty \) and then \( n \to \infty \) to obtain the result.

One then has the following precise statement due to Ruelle, Amrein, Georgescu and Enss [Rue69, AG74, Ens78]:

**Theorem 1.30 (RAGE).** Let \( H \) be a self-adjoint operator and \( K_n \) a sequence of compact operators such that
\[ \text{s-lim}_{n \to \infty} K_n = 1. \]
Then
\[ \mathcal{H}_c(H) = \left\{ \psi \in \mathcal{H} \mid \lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T \| K_n e^{-itH} \psi \| dt = 0 \right\} \]
and
\[ \mathcal{H}_{pp}(H) = \left\{ \psi \in \mathcal{H} \mid \lim_{n \to \infty} \sup_{t \geq 0} \| (1 - K_n) e^{-itH} \psi \| = 0 \right\} \]
In particular, it is useful to think of
\[ K_n = \chi_{B_n(0)}(X) \]
i.e., the projection onto a ball of size \( n \) about the origin in position space. This is finite rank (and hence compact) and indeed converges strongly to the identity. Then the statement is saying that \( \psi \in \mathcal{H}_{pp}(H) \) iff
\[ \lim_{n \to \infty} \sup_{t \geq 0} \| \chi_{B_n(0)}(X) e^{-itH} \psi \| = 0 \]
i.e., a particle evolved to arbitrary time, will eventually escape a ball of arbitrary size.

**Proof.** Assume first that \( \psi \in \mathcal{H}_c(H) \). Then by Cauchy-Schwarz
\[ \frac{1}{T} \int_0^T \| K_n e^{-itH} \psi \| dt \leq \frac{1}{T} \left( \int_0^T \| K_n e^{-itH} \psi \|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T 1 dt \right)^{\frac{1}{2}} dt \]
\[ = \left( \frac{1}{T} \int_0^T \| K_n e^{-itH} \psi \|^2 dt \right)^{\frac{1}{2}} \]
\[ \to 0 \quad T \to \infty \]
by the previous theorem. Conversely, if \( \psi \notin \mathcal{H}_c(H) \), we may write \( \psi = \psi^c + \psi^{pp} \). By the previous estimate it merely suffices to estimate \( \| K_n e^{-itH} \psi^{pp} \| \) from below. Let us write
\[ \psi^{pp} = \sum_j \alpha_j \psi_j \]
where \( \{ \psi_j \} \) are the eigenfunctions of \( H \) with eigenvalues \( \lambda_j \). Then

\[
e^{-itH}\psi^{pp} = \sum_j e^{-i\lambda_j t} \alpha_j \psi_j.
\]

Truncating this expansion after \( N \) terms, we find that this part converges uniformly by the strong convergences of \( K_n \to 1 \):

\[
\lim_{n \to \infty} \sup_{t \geq 0} \left\| (1 - K_n) \sum_{j=1}^N e^{-i\lambda_j t} \alpha_j \psi_j \right\| \leq \sum_{j=1}^N |\alpha_j| \lim_{n \to \infty} \sup_{t \geq 0} \| (1 - K_n) \psi_j \| \quad (K_n \to 1 \text{ strongly})
\]

By the uniform boundedness principle, we have \( \|K_n\| \leq M \) so that the error can be made arbitrarily small by taking \( N \) sufficiently large.

If \( \psi \in \mathcal{H}^{pp} \), then the claim follows by the estimate we have just proven. Conversely, if \( \psi \notin \mathcal{H}^{pp} \), write again \( \psi = \psi^c + \psi^{pp} \) and it suffices to show that

\[
\left\| (1 - K_n) e^{-itH} \psi^c \right\|
\]

does not tend to zero as \( n \to \infty \). Assume otherwise. Then

\[
0 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\| (1 - K_n) e^{-itH} \psi^c \right\| dt \\
\geq \| \psi^c \| - \lim_{T \to \infty} \frac{1}{T} \int_0^T \| K_n e^{-itH} \psi^c \| dt \\
= \| \psi^c \|
\]

which is a contradiction. \( \square \)

1.6 AC Spectrum—A vague form of delocalization

As we have seen above, there is great interest in establishing that operators actually have purely absolutely continuous spectrum, since this is an indication of either ballistic or diffusive motion (and also has far reaching consequences for scattering theory). In this section we explore various different ways to establish that the spectrum of an operator (on an interval) is absolutely continuous.

1.6.1 Stability of AC spectrum

We begin with a basic observation:

**Theorem 1.31.** The essential spectrum of an operator is stable against compact perturbations.

**Proof.** In a sense this statement is trivial, if we define the essential spectrum appropriately (see [Sha24]). One reasonable definition is

\[
\sigma_{ess}(A) \equiv \{ z \in \mathbb{C} \mid (A - zI) \notin \mathcal{F}(\mathcal{H}) \}
\]

where \( \mathcal{F}(\mathcal{H}) \) is the set of Fredholm operators on a Hilbert space (the space \( \mathcal{F}(\mathcal{H}) \) of those operators \( F \) on \( \mathcal{H} \) such that \( \dim(\ker(F)) \), \( \dim(\ker(F^*)) \) are finite and such that \( \text{im}(F) \) is closed). One basic fact about Fredholm operators is that they are stable under compact perturbations.

To reiterate, we are trying to prove that if \( K \) is compact, then

\[
\sigma_{ess}(A) = \sigma_{ess}(A + K).
\]
Using (1.8) we find that

\[
\begin{align*}
  z \notin \sigma_{\text{ess}}(A + K) &\iff (A + K - z\mathbb{1}) \in \mathcal{F}(\mathcal{H}) \\
  &\iff (A - z\mathbb{1}) \in \mathcal{F}(\mathcal{H}) \\
  &\iff z \notin \sigma_{\text{ess}}(A)
\end{align*}
\]

so we find the result.

We ask whether there is an analogous statement for the absolutely continuous spectrum. It turns out that this is indeed the case, if we replace compactness with the trace class property.

**Theorem 1.32.** Let \( A \) be a normal operator and \( T \) be a trace-class operator so that \( A + T \) is also normal. Then

\[
\sigma_{\text{ac}}(A) = \sigma_{\text{ac}}(A + T).
\]

**Proof.** We postpone the proof of this fact until we can prove the existence of wave operators implies ac spectrum (the proof may be found in [Kat84], pp. 542 Theorem 4.4).

### 1.6.2 The limiting absorption principle

The limiting absorption principle is the statement that in some sense if one goes into the absolutely continuous spectrum, the resolvent still has a bounded limit (though not in \( \ell^2 \)). To warm up, we start with the following characterization of the spectral measure (see more details in [Sha24]):

**Lemma 1.33** (Characterization of measure type via the Borel transform). Let \( \mu \) be a finite Borel measure and \( f \) its Borel transform, given by

\[
f(\lambda) = \int_{\lambda \in \mathbb{R}} \frac{1}{E - \lambda} d\mu(E).
\]

Then

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \text{Im} \{ f(\lambda + i\varepsilon) \}
\]

exists a.e. w.r.t. both \( \mu \) and \( \mathcal{L} \) (the Lebesgue measure). Moreover,

\[
\{ \lambda \in \mathbb{R} \mid \text{Im} \{ f(\lambda + i0^+) \} = \infty \}
\]

and

\[
\{ \lambda \in \mathbb{R} \mid 0 < \text{Im} \{ f(\lambda + i0^+) \} < \infty \}
\]

are the support of the singular and absolutely continuous parts of \( \mu \) respectively. Moreover, the set of point masses of \( \mu \) is given by

\[
\left\{ \lambda \left| \lim_{\varepsilon \to 0^+} \varepsilon \text{Im} \{ f(\lambda + i\varepsilon) \} > 0 \right. \right\}.
\]

**Proof.** See [Jak06].

This we calibrate with the following basic statement about the resolvent implying ac spectrum:

**Proposition 1.34.** Let \( H \) be a self-adjoint operator on a separable Hilbert space \( \mathcal{H} \). Assume that for any \( \varphi \) in a dense subset of \( \mathcal{H} \), either

\[
\sup_{E \in [a, b], \varepsilon \in (0, 1)} \left| \left\langle \varphi, (H - (E + i\varepsilon) \mathbb{1})^{-1} \varphi \right\rangle \right| < \infty. \quad (1.9)
\]

or there exists some \( p > 1 \) such that

\[
\sup_{\varepsilon \in (0, 1)} \int_a^b \frac{1}{\pi} \left| \text{Im} \{ \langle \varphi, R(E + i\varepsilon) \varphi \rangle \} \right|^p dE < \infty. \quad (1.10)
\]

With \( R(z) \equiv (H - z\mathbb{1})^{-1} \). Then \( H \) has purely absolutely continuous spectrum on \([a, b]\).
Proof. We claim first that (1.9) implies (1.10). Actually it is possible to show that Im \{\langle \phi, R(E + i\varepsilon) \phi \rangle \} \geq 0 if \varepsilon > 0 thanks to the Herglotz property, so we can safely take the \( p \) power with no absolute value. To show (1.10), assuming (1.9) holds, we use

\[ \text{Im} \{\langle \phi, R(E + i\varepsilon) \phi \rangle \} \geq 0 \]

as well as

\[ |\langle \phi, R(E + i\varepsilon) \phi \rangle| \leq \frac{1}{\varepsilon} \| \phi \|^2. \]

Hence we assume (1.10) and work towards showing that \langle \phi, \chi_I (H) \phi \rangle\] is ac on \([a, b]\). We have for any interval \( I \), by Stone’s formula

\[ \langle \phi, \chi_I (H) \phi \rangle \leq \lim_{\varepsilon \to 0^+} \int_I \frac{1}{\pi} \text{Im} \{\langle \phi, R(E + i\varepsilon) \phi \rangle \} \, dE. \]

By Hölder’s inequality, the RHS integral is estimated by

\[ \int_I \frac{1}{\pi} \text{Im} \{\langle \phi, R(E + i\varepsilon) \phi \rangle \} \, dE \leq \left( \int_I \left[ \frac{1}{\pi} \text{Im} \{\langle \phi, R(E + i\varepsilon) \phi \rangle \} \right]^p \, dE \right)^{\frac{1}{p}} \left| I \right|^{\frac{1}{q}} \]

with \( q = \left( 1 - \frac{1}{p} \right)^{-1} \). Since we know (1.10) we conclude that

\[ \langle \phi, \chi_I (H) \phi \rangle \leq C |I|^s \]

and hence the measure is absolutely continuous.

The following material is taken from [Tao11]:

**Definition 1.35** (Limiting absorption principle). \( H \) is said to have the limiting absorption principle at \( E \in \mathbb{R} \) iff for any \( \psi \in \ell^2 \) sufficiently “nice” (on the lattice, with finite support is enough), and for any \( \sigma > 0 \) there exists some \( C_\sigma \in (0, \infty) \) (only depending on \( \sigma \)) such that

\[ \sup_{\varepsilon \neq 0} \left\| (H - (E + i\varepsilon) \mathbb{1})^{-1} \psi \right\|_{H^{-\frac{1}{2} - \sigma}} \leq C_\sigma \frac{1}{\sqrt{|E|}} \| \psi \|_{H^{\frac{1}{2} + \sigma}} \]

where

\[ \| \psi \|_{H^s} := \| \langle X \rangle^s \psi \|_{\ell^2} \]

and

\[ \langle x \rangle := \left( 1 + \| x \|^2 \right)^{\frac{1}{2}}. \]

**Claim 1.36.** The limiting absorption principle holds for \( H = -\Delta \) on \( \ell^2 (\mathbb{Z}^d) \).

**Proposition 1.37.** Any operator \( H \) admitting the limiting absorption principle at \( E \) has purely absolutely continuous spectrum in a small interval about \( E \).

**Proof.** Show that

\[ \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \text{Im} \{\langle \phi, R(\cdot + i\varepsilon) \phi \rangle \} \]

exists and is finite.

### 1.6.3 Existence of wave operators

The material in this section is taken from [RS79] Chapter XI.3. We start with a pedestrian criterion for existence of ac spectrum.
Theorem 1.38. Let $A, B$ be two bounded self-adjoint operators on a separable Hilbert space $\mathcal{H}$ and assume that
\[
\lim_{t \to \infty} e^{-itA} e^{itB}
\]
exists. Then $\sigma_{ac}(B) \subseteq \sigma_{ac}(A)$.

Let us motivate this statement a bit with scattering theory. We first note that if $\varphi$ is an eigenvector of $B$ with eigenvalue $\lambda$, then
\[
e^{itA} e^{-itB} \varphi = e^{itA} e^{-it\lambda} \varphi
\]
and we know that that limit exists only if $\varphi$ is also an eigenvector of $A$ (which is generically false). Hence we should really define the wave operator
\[
\Omega^\pm (A, B) := \lim_{t \to \pm \infty} e^{-itA} e^{itB} P_{ac}(B)
\]
if the limit exists. We also define
\[
\mathcal{H}_\pm := \text{im } (\Omega^\pm).
\]

Proposition 1.39. If $\Omega^\pm (A, B)$ both exist, then
1. $\Omega^\pm$ are partial isometries with initial subspaces $P_{ac}(B) \mathcal{H}$ and final subspaces $\mathcal{H}_\pm$.
2. $\mathcal{H}_\pm$ are invariant subspaces for $A$ and
\[
\Omega^\pm (\mathcal{D}(B)) \subseteq \mathcal{D}(A)
\]
\[
A \Omega^\pm = \Omega^\pm B
\]
3. $\mathcal{H}_\pm \subseteq \text{im } (P_{ac}(A))$.

Recall that for a partial isometry $I \in \mathcal{B}(\mathcal{H})$, $\ker (I)^\perp$ is called the initial subspace and $\text{im } (I)$ is called the final subspace. Hence $\Omega^\pm$ define unitary equivalences between
\[
P_{ac}(B) \mathcal{H} \to \mathcal{H}_\pm
\]
and so the first and third point together imply that
\[
\text{im } (P_{ac}(B)) \subseteq \text{im } (P_{ac}(A)).
\]

Proof. For the first item, let us take $\varphi \in (P_{ac}(B) \mathcal{H})^\perp$. Then by definition of the wave operator itself, $\Omega^\pm \varphi = 0$. Conversely, if $\varphi \in P_{ac}(B) \mathcal{H}$, then by unitarity,
\[
\|e^{-itA} e^{itB} P_{ac}(B) \varphi\| = \|\varphi\|
\]
so $\Omega^\pm$ are indeed partial isometries are claimed.

For the second item, note that for any fixed $s$, we have
\[
\Omega^\pm = e^{-iA} \Omega^\pm e^{iB}
\]
\[
\updownarrow
\]
\[
e^{-iA} \Omega^\pm = \Omega^\pm e^{iB}
\]
since this holds for any $s$, Stone’s theorem for unitary groups implies that $A \Omega^\pm = \Omega^\pm B$. Next, if $\varphi \in \mathcal{H}_\pm$, then
\[
\varphi = \Omega^\pm \psi
\]
for some $\psi$, and hence,
\[
A \varphi = A \Omega^\pm \psi = \Omega^\pm B \psi \in \text{im } (\Omega^\pm) \equiv \mathcal{H}_\pm
\]
so that these are indeed invariant subspaces for $A$. This also shows the statement about the domains.

Finally, by the previous arguments, $A|_{\mathcal{H}_\pm}$ is unitarily equivalent via $\Omega^\pm$ to $B|_{P_{ac}(B)\mathcal{H}}$. Hence $A|_{\mathcal{H}_\pm}$ has purely absolutely continuous spectrum. \qed
Proposition 1.40 (Chain rule). If $\Omega^{\pm}(A, B)$ and $\Omega^{\pm}(B, C)$ exist, then $\Omega^{\pm}(A, C)$ exist, and
$$\Omega^{\pm}(A, C) = \Omega^{\pm}(A, B) \Omega^{\pm}(B, C).$$

Proof. By the third item in the proposition above,
$$\text{im} \left( \Omega^{\pm}(B, C) \right) \subseteq \text{im} \left( P_{ac}(B) \right)$$
so
$$\text{s-lim}_{t \to \pm \infty} P_{ac}(B)^\perp e^{-itB} e^{itC} P_{ac}(C) = 0.$$ Hence
$$e^{-itA} e^{itC} P_{ac}(C) \varphi = e^{-itA} e^{itB} P_{ac}(B) e^{-itB} e^{itC} P_{ac}(C) \varphi +
+ e^{-itA} e^{itB} P_{ac}(B)^\perp e^{-itB} e^{itC} P_{ac}(C) \varphi
\to \Omega^{\pm}(A, B) \Omega^{\pm}(B, C) \varphi$$ since the strong limit of a product is the product of the strong limits. \hfill \square

Definition 1.41. We say that we have asymptotic completeness if
$$\mathcal{H}_+ = \mathcal{H}_- = (P_{pp}(A) \mathcal{H})^\perp.$$ Definition 1.42. We say that that the wave operators $\Omega^{\pm}$ are complete iff
$$\mathcal{H}_+ = \mathcal{H}_- = \text{im} \left( P_{ac}(A) \right).$$ Hence the distinction between these two is that asymptotic completeness further requires that $\sigma_{ac}(A) = \emptyset$.

Proposition 1.43. Assume that $\Omega^{\pm}(A, B)$ exist. Then they are complete iff $\Omega^{\pm}(B, A)$ exist.

Proof. Assume that both $\Omega^{\pm}(A, B)$ and $\Omega^{\pm}(B, A)$ exist. By the chain rule,
$$P_{ac}(A) = \Omega^{\pm}(A, A) = \Omega^{\pm}(A, B) \Omega^{\pm}(B, A)$$ so that
$$\text{im} \left( P_{ac}(A) \right) \subseteq \text{im} \left( \Omega^{\pm}(A, B) \right).$$ But we also know that $\text{im} \left( \Omega^{\pm}(A, B) \right) \subseteq \text{im} \left( P_{ac}(A) \right)$, we have completeness.

Conversely, if $\Omega^{\pm}(A, B)$ exist and are complete, let $\varphi \in \text{im} \left( P_{ac}(A) \right)$, Then $\varphi = \Omega^{\pm}(A, B) \psi$ for some $\psi$. This implies that
$$\|e^{-itA} \varphi - e^{-itB} P_{ac}(B) \psi\| \to 0$$ as $t \to \infty$. But $e^{-itB}$ is unitary, so
$$\lim_{t \to \infty} e^{itB} e^{-itA} \varphi$$ exists and equals $P_{ac}(B) \psi$. \hfill \square

Theorem 1.44 (Cook’s method). Assume that $A, B$ are self-adjoint operators and that there exists some set
$$\mathcal{D} \subseteq \mathcal{D}(B) \cap \text{im} \left( P_{ac}(B) \right)$$ which is dense in $\text{im} \left( P_{ac}(B) \right)$ so that for any $\varphi \in \mathcal{D}$, there is some $T_0$ satisfying
1. For any $|t| > T_0$, $e^{-itB} \varphi \in \mathcal{D}(A)$.
2. $\int_{T_0}^{\infty} \left( \|(B - A) e^{-itB} \varphi\| + \|(B - A) e^{itB} \varphi\| \right) \, dt < \infty.$

Then $\Omega^{\pm}(A, B)$ exist.
Proposition 1.46. For quantum dynamics. The material in this section is mostly taken from [CyconKirschFroeseSimon].

In this section we want to establish an additional criterion for pure ac spectrum, which, as we saw above, has consequences for quantum dynamics. The material in this section is mostly taken from [CyconKirschFroeseSimon].

Proof. Define \( \eta(t) := e^{itA}e^{-itB}\varphi \) for some \( \varphi \in D \). For \( t > T_0, e^{-itB}\varphi \in D(A) \cap D(B) \), so \( \eta \) is strongly differentiable on \( (T_0, \infty) \) and

\[
\eta'(t) = -ie^{itA} (B - A) e^{-itB}\varphi.
\]

Hence for \( t > s > T_0 \) we find that

\[
\| \eta(t) - \eta(s) \| \leq \int_s^t \| \eta'(u) \| du \leq \int_s^t \| (B - A) e^{-iuB}\varphi \| du
\]

goes to zero as \( s \to \infty \) by the assumption above. Hence \( \eta \) is Cauchy as \( t \to \infty \), so

\[
\lim_{t \to \infty} e^{itA} e^{-itB} P_{ac}(B) \psi
\]

exists for all \( \psi \in D \). It also exists trivially for all \( \psi \in \text{im} \left( P_{ac}(B) \right) \), so, for all \( \psi \) lying in a dense set. The existence in a dense implies the existence for all \( \psi \) by a \( \frac{\pi}{2} \) argument, which shows that \( \Omega^- \) exists. The proof for \( \Omega^+ \) is identical.

Example 1.45. Imagine that \( B = -\Delta \) (the discrete Laplacian) and \( A = -\Delta + V(X) \) for some \( V \). Then apparently to guarantee that \( A \) has ac spectrum in \([0, \infty)\) we need to verify that

\[
\int_1^\infty \left( \| V(X) e^{-i\Delta} \varphi \| + \| V(X) e^{i\Delta} \varphi \| \right) dt < \infty
\]

for all \( \varphi \) in a dense subset. For instance, the dense subset could be \( \varphi \) with compact support. At this moment it is probably good to mention that the propagator for the continuum Laplacian has time dependence like

\[
e^{-i\Delta} (x, y) \sim t^{-\frac{d}{2}}
\]

so this is going to be integrable if \( d \geq 3 \). In fact, [Krishna 1992] has shown that for the discrete Laplacian it is sufficient to show

\[
\sum_{x \in \mathbb{Z}^d} |V(x)|^2 \left| e^{-i\Delta} (x, 0) \right|^2 \sim |t|^{-2-\varepsilon}
\]

as \( t \to \infty \).

1.6.4 Mourre theory

In this section we want to establish an additional criterion for pure ac spectrum, which, as we saw above, has consequences for quantum dynamics. The material in this section is mostly taken from [CyconKirschFroeseSimon].

Proposition 1.46. Let \( H \) be self-adjoint and assume that for each \( \varphi \) in some dense set there exists some \( C_{\varphi} < \infty \) such that

\[
\limsup_{\varepsilon \to 0^+} \sup_{\mu \in (a, b)} \left\langle \varphi, \| \text{im} \left( (H - (\mu + i\varepsilon) \mathbb{1})^{-1} \right) \varphi \right\rangle \leq C_{\varphi}.
\]

Then \( H \) has purely absolutely continuous spectrum in \((a, b)\).

Proof. Stone’s formula says that

\[
\frac{1}{2} \left\langle \varphi, \left( \chi_{(a', b')} (H) + \chi_{[a', b']} (H) \right) \varphi \right\rangle = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{a'}^{b'} \left\langle \varphi, \| \text{im} \left( (H - (\mu + i\varepsilon) \mathbb{1})^{-1} \right) \varphi \right\rangle d\mu
\]

and using the fact that \( \chi_{(a', b')} \leq \chi_{[a', b']} \), if \((a', b') \subseteq (a, b)\),

\[
\left\langle \varphi, \chi_{(a', b')} (H) \varphi \right\rangle \leq \frac{1}{\pi} \int_{a'}^{b'} C_{\varphi} d\mu = \frac{1}{\pi} C_{\varphi} (b' - a')
\]

27
for a dense subset of $\varphi$'s. But this implies that

$$\langle \varphi, \chi_S(H) \varphi \rangle \leq \frac{1}{\pi} C_\varphi |\Omega|$$

for any measurable $\Omega \subseteq (a, b)$ which implies in turn that the spectral measures $\mu_{H, \varphi, \varphi}$ are absolutely continuous. But since we assume this for a dense subset of $\varphi$'s, the spectrum itself is absolutely continuous.

**Theorem 1.47** (Putnam). Let $H$ and $A$ be bounded and self-adjoint and furthermore assume that

$$i [H, A] = |C|^2 \equiv C^* C$$

for some operator $C$ for which $\ker(C) = \{0\}$. Then $H$ has purely absolutely continuous spectrum.

**Proof.** We have, with $R(z) \equiv (H - z \mathbb{1})^{-1}$, we get via the C-star identity,

$$\| CR(z) \|^2 = \| R(\overline{z}) C^* C R(z) \|$$

$$= \| R(\overline{z}) i [H, A] R(z) \|$$

$$= \| R(\overline{z}) i [H - z \mathbb{1}, A] R(z) \|$$

$$\leq \| R(\overline{z}) (H - z \mathbb{1}) A R(z) \| + \| R(\overline{z}) A (H - z \mathbb{1}) R(z) \|$$

$$= \| R(\overline{z}) (H - (\overline{z} - z) \mathbb{1}) A R(z) \| + \| R(\overline{z}) A \|$$

$$\leq \| A R(z) \| + \| R(\overline{z}) A \| + 2 \| \text{Im} \{z\} \| \| R(\overline{z}) A R(z) \|$$

$$\leq 4 \frac{1}{\| \text{Im} \{z\} \|} \| A \|.$$  

As a result,

$$2 \| \text{Im} \{R(z)\} C^* \| = 2 \| CR(z) 2i \text{Im} \{z\} R(\overline{z}) C^* \|$$

$$\leq 8 \| A \|.$$  

Now, $\text{im}(C^*)$ is dense since we have $\text{im}(C^*)^\perp = \ker(C) = \{0\}$, so using Proposition 1.46 we find the result.

Imagine we could show that

$$i [H, A] \geq \alpha \mathbb{1} \quad (1.11)$$

for some $\alpha > 0$. That is, that not only does $\ker(C) = \{0\}$ but also that it is invertible. Then the previous estimates show us that

$$\| R(z) \| = \| C^{-1} C R(z) \|$$

$$\leq \| C^{-1} \| \| C R(z) \|$$

$$\leq \| C^{-1} \| 2 \frac{1}{\sqrt{\| \text{Im} \{z\} \|}} \sqrt{\| A \|}.$$  

But we also know that for self-adjoint operators,

$$\| R(z) \| = \frac{1}{\text{dist}(z, \sigma(H))}$$

so that

$$\frac{1}{\text{dist}(z, \sigma(H))} \leq \| C^{-1} \| 2 \frac{1}{\sqrt{\| \text{Im} \{z\} \|}} \sqrt{\| A \|} \quad (z \in \mathbb{C}).$$  

If $E \in \sigma(H)$ and we take $z = E + i \varepsilon$ for $\varepsilon > 0$ we get

$$\frac{1}{\varepsilon} \leq 2 \| C^{-1} \| \sqrt{\| A \|} \frac{1}{\sqrt{\varepsilon}} \quad (\varepsilon > 0)$$

which leads to a contradiction, i.e., $\sigma(H) = \emptyset$. Hence (1.11) is impossible for bounded $H, A$.

The Mourre estimate is a weak form of this hypothesis for unbounded operators.
**Definition 1.48** (Mourre estimate). Let $H, A$ be two self-adjoint operators (possibly unbounded) on a separable Hilbert space such that:

1. $\mathcal{D}(A) \cap \mathcal{D}(H)$ is dense in $\mathcal{D}(H)$.
2. The form of $i[H, A]$ defined on $\mathcal{D}(A) \cap \mathcal{D}(H)$ extends to a bounded operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$.

We say that $H$ obeys the Mourre estimate on $\Delta$ (with respect to $A$) iff there exists a positive number $\alpha > 0$ and a compact operator $K$ such that

$$\chi_\Delta(H)i[H, A]\chi_\Delta (H) \geq \alpha \chi_\Delta(H) + K.$$ 

**Example 1.49.** Let 

$$H = -\Delta + V(X)$$

on $L^2(\mathbb{R}^d)$ where $V: \mathbb{R}^d \to \mathbb{R}$ is some function such that

1. $V(X)(-\Delta + 1)^{-1}$ is compact.
2. $(-\Delta + 1)^{-1}X \cdot (\nabla V)(X)(-\Delta + 1)^{-1}$ is compact.

Define $A := \frac{1}{2}(X \cdot P + P \cdot X)$, which is the generator of dilations. Then

$$i[P^2, A] = i\left[\frac{P^2}{2}, \frac{1}{2}(X \cdot P + P \cdot X)\right]$$

$$= i\left[\frac{1}{2}([P_j, X_i]P_i + [P_i, P_j, X_i])\right]$$

$$= i\left[\frac{1}{2}([P_j, P_i, X_i]P_i + P_i[P_j, X_i])\right]$$

$$= i\left[\frac{1}{2}(P_j, X_i)P_i + [P_j, X_i]P_iP_j + P_iP_j, P_i[P_j, X_i] + P_i[P_j, X_i]P_j\right]$$

and now using the fact that $[P_i, X_j] = i\delta_{ij}$ we get that

$$i[P^2, A] = \frac{i}{2}(-P_j[i\delta_{ji}P_i - i\delta_{ji}P_jP_i - P_iP_j,i\delta_{ji} - P_i\delta_{ij}P_j])$$

$$= 2P^2$$

so that $-\Delta \equiv P^2$ itself easily satisfies a Mourre estimate on any interval not containing zero. Using our assumptions on $V$ we find furthermore the same is also true for $H \equiv P^2 + V(X)$:

$$i[H, A] = 2P^2 + i\left[V(X), \frac{1}{2}(X \cdot P + P \cdot X)\right]$$

$$= 2P^2 + \frac{i}{2}(X_i[V(X), P_i] + [V(X), P_i]X_i)$$

$$= 2P^2 + \frac{i}{2}(X_i(i(\partial_i V)(X)) + (i(\partial_i V)(X))X_i)$$

$$= 2P^2 - X \cdot (\nabla V)(X)$$

$$= 2H - 2V(X) - X \cdot (\nabla V)(X)$$

Now by our assumptions: $2V(X) + X \cdot (\nabla V)(X)$ is compact when sandwiched with $\chi_{(a,b)}(H)$. If furthermore, $a > 0$, we find

$$\chi_{(a,b)}(H)2H\chi_{(a,b)}(H) \geq 2a\chi_{(a,b)}(H)$$

so that the Mourre estimate is satisfied.
The Virial Theorem

Let $\psi \in \ell^2$ be an eigenfunction of $H$, in the sense that $H\psi = \lambda \psi$. Then

\[
\langle \psi, i[H, iA] \psi \rangle = i (\langle \psi, HA\psi \rangle - \langle \psi, AH\psi \rangle) \\
= i (\langle H\psi, A\psi \rangle - \lambda \langle \psi, AH\psi \rangle) \\
= i\lambda (\langle \psi, A\psi \rangle - \langle \psi, A\psi \rangle) \\
= 0.
\]

Note that when $H, A$ are unbounded some care must be taken to handle the fact this form is well-defined. We avoid these subtleties here and refer the reader to [CyconKirschFroeseSimon].

Control of embedded eigenvalues

**Theorem 1.50.** Assume that $H$ satisfies the Mourre estimate on the interval $\Delta$ (with respect to $A$). Assume moreover that there exists some self-adjoint operator $H_0$ (in applications this is usually $H_0 = -\Delta$) such that:

1. $(H_0 - z\mathbb{1})^{-1}D(A) \subseteq D(A)$ for some $z \in \rho(H_0)$.
2. $D(H_0) \cap D(H_0A)$ is dense in $D(H)$ and
3. The form $i[H_0, A]$ defined on $D(A) \cap D(H)$ extends to a bounded operator from $D(H)$ to $\mathcal{H}$. Then $H$ has at most finitely many eigenvalues in $\Delta$ and each eigenvalue has finite multiplicity.

Then $H$ has at most finitely many eigenvalues in $\Delta$ and each eigenvalue has finite multiplicity.

**Proof.** Suppose there are infinitely many eigenvalues of $H$ in $\Delta$, or that some eigenvalue has infinite multiplicity. Let $\{\psi_n\}$ be the corresponding orthonormal eigenfunctions of this space. By the Virial theorem and the Mourre estimate we get

\[
0 = \langle \psi_n, i[H, A] \psi_n \rangle \\
= \langle \psi_n, \chi_{\Delta}(H) i[H, A] \chi_{\Delta}(H) \psi_n \rangle \\
\geq \alpha \|\psi_n\|^2 + \langle \psi_n, K \psi_n \rangle.
\]

Since $\|\psi_n\| = 1$ and $\psi_n \to 0$ weakly, and $K$ is compact, we have $\langle \psi_n, K \psi_n \rangle \to 0$ as $n \to \infty$. But this is impossible as $\alpha > 0$. \qed

Absence of singular continuous spectrum

**Lemma 1.51.** Assume that $H$ satisfies the Mourre estimate on the open interval $\Delta$ (with respect to $A$). Then actually the Mourre estimate is obeyed with $K = 0$ away from eigenvalues of $H$.

**Proof.** Let $\Delta' \subseteq \Delta$ be any interval which does not contain an eigenvalue. Then

\[
\chi_{\Delta'}(H) i[H, A] \chi_{\Delta'}(H) \geq \alpha \chi_{\Delta'}(H) + \chi_{\Delta'}(H) K \chi_{\Delta'}(H).
\]

Now since $\Delta'$ does not contain any eigenvalues, $\chi_{\Delta'}(H) K \chi_{\Delta'}(H)$ tends to zero in norm as $\Delta'$ shrinks to zero width about any point. Hence let us pick $\Delta'$ such that

\[
\chi_{\Delta'}(H) i[H, A] \chi_{\Delta'}(H) \geq \alpha \chi_{\Delta'}(H) - \frac{1}{2} \alpha \mathbb{1}.
\]

Now multiply both sides again by $\chi_{\Delta'}(H)$ to get the result. \qed
**Theorem 1.52.** Assume that $H$ satisfies the Mourre estimate on the interval $\Delta$ (with respect to $A$). Assume moreover that there exists some self-adjoint operator $H_0$ (in applications this is usually $H_0 = -\Delta$) such that:

1. $(H_0 - z1)^{-1}D(A) \subseteq D(A)$ for some $z \in \rho(H_0)$.
2. $D(H_0) \cap D(H_0A)$ is dense in $D(H)$ and
3. The form $i[H_0, A]$ defined on $D(A) \cap D(H)$ extends to a bounded operator from $D(H)$ to $H$. Then $H$ has at most finitely many eigenvalues in $\Delta$ and each eigenvalue has finite multiplicity.
4. The form $i[i[H, A], A]$ extends from $D(A) \cap D(H)$ to a bounded map from $D(H)$ to $D(H)^*$.

Then if the Mourre estimate actually holds with $K = 0$ then
\[
\limsup_{\delta \to 0^+} \sup_{\mu \in \Delta} \bigg| (|A| + 1)^{-1} (H - (\mu + i\delta)1)^{-1} (|A| + 1)^{-1} \bigg| \leq C
\]
for some constant $C$, which readily implies pure absolutely continuous spectrum of $H$ within $\Delta$.

We shall not prove this theorem but rather refer the reader to [CyconKirschFroeseSimon].

### 1.6.5 The non-zero index method

Here we describe the fact that if
\[\text{index} \left( \Lambda U \Lambda + \Lambda^\perp \right) \neq 0\]
for some projection $\Lambda$ and some unitary $U$, then $\sigma(U) = \sigma_{ac}(U) = S^1$. In particular if $U = e^{i2\pi A}$ for some self-adjoint $A$ then by the spectral mapping theorem, $\sigma_{ac}(A) = [0, 1]$ or a translate of this interval.

### 1.7 Linear response theory: the Kubo formula

We now want to derive various formulas for the DC conductivity of a system using perturbation theory. In order to do so we first derive a general form of perturbation theory known as the Kubo linear response formula [Kub57]. The main reason for this hurdle is the following

**Example 1.53.** We are interested in calculating the electric conductivity of, say,
\[H = (P - A)^2 - E_0 X_j\]
which is a magnetic system with electric field of strength $E_0$ on the $j$th axis. If we have e.g. constant magnetic field with
\[A = X_2 e_1\]
then there is no dependence on $X_1$ (if $j = 2$) in the Hamiltonian, so the spectrum would not be discrete in this case. Hence we cannot use Rayleigh-Schroedinger perturbation theory.

Hence even though the perturbations we will consider (the electric field) eventually do not depend on time, for regularizing purposes we consider them being ramped up with time very gradually.

Before proceeding we explain why traces with the Fermi projection $P \equiv \chi(-\infty,E_F) (H)$ are of use to us.

### 1.7.1 Density matrices

Usually one talks about states of quantum mechanical systems as vectors $\psi$ in a separable Hilbert space $\mathcal{H}$ with $\|\psi\| = 1$. Equivalently we could speak about rank-1 projections:
\[P := \psi \otimes \psi^* .\]
Then the quantum expectation value of the observable $A$ on the state $\psi$ is given by
\[\langle \psi, A\psi \rangle = \text{tr} (PA) \equiv \text{tr} (\psi \otimes \psi^* A) .\]
Sometimes it is useful however to speak of a classical statistical mixture of states: let \( N \in \mathbb{N} \) and \( \{ p_i \}_{i=1}^N \subseteq [0,1] \) such that \( \sum_{i=1}^N p_i = 1 \). Let also \( \{ \psi_i \}_{i=1}^N \subseteq \mathcal{H} \) be some ONB of some subspace. Then

\[
\rho := \sum_{i=1}^N p_i \psi_i \otimes \psi_i^*
\]

is such a statistical mixture of states. Note that

\[
\text{tr} (\rho) = \sum_{i=1}^N p_i \text{tr} (\psi_i \otimes \psi_i^*) = 1
\]

and actually

\[
\langle \varphi, \rho \varphi \rangle = \sum_{i=1}^N p_i |\langle \psi_i, \varphi \rangle|^2 \geq 0 \quad (\varphi \in \mathcal{H})
\]

so \( \rho \geq 0 \). This leads us to the

**Definition 1.54 (Density matrix).** A density matrix \( \rho \) on a separable Hilbert space is a positive trace-class operator of trace 1.

### 1.7.2 The many-body Fermionic ground state in single-particle universe

In quantum mechanics, the state of a particle is described by a vector in a Hilbert space \( \mathcal{H} \) (or a density matrix, as we have just seen). Conversely, to talk about the state of \( M \) distinguishable particles simultaneously, we need to consider a vector in the \( M \)-fold tensor product Hilbert space \( \bigotimes_{j=1}^M \mathcal{H} \). However, if we have \( M \) indistinguishable particles which are Fermions, which is the situation for electrons in a solid, then the state of these particles is actually a vector in the \( M \)-fold exterior product Hilbert space \( \bigwedge_{j=1}^M \mathcal{H} \), since the state must be anti-symmetric with respect to exchange of any two particles (as a basic axiom of quantum mechanics).

If the single-particle Hamiltonian \( H = H^* \in \mathcal{B} (\mathcal{H}) \) is acting on each particle separately, then the many-body Hamiltonian is given by

\[
d\Gamma (H) := \sum_{j=1}^M 1^{j-1} \wedge H \wedge 1^{M-j},
\]

i.e., the single particle Hamiltonian acts on the \( j \)th particle and doesn’t do anything on all other particles.

Then, if we are interested in the many-body expectation value of a non-interacting single-particle observable, say, \( B \), we would first raise it to the many-body Hilbert space just as above:

\[
B \mapsto \sum_{j=1}^M 1^{j-1} \wedge B \wedge 1^{M-j} =: d\Gamma (B)
\]

and then if our system was in the state \( \Psi \in \bigwedge_{j=1}^M \mathcal{H} \), we would calculate

\[
\langle \Psi, d\Gamma (B) \Psi \rangle.
\]

Now, if \( \Psi \) itself is a product state, i.e., \( \Psi = \psi_1 \wedge \cdots \wedge \psi_M \), where \( \{ \psi_j \}_{j=1}^M \) is an orthonormal collection, then this simplifies to

\[
\langle \Psi, d\Gamma (B) \Psi \rangle = \sum_{j=1}^M \langle \psi_j, B \psi_j \rangle
\]

where we recognize \( \sum_{i=1}^M \psi_i \otimes \psi_i^* \) as the projection operator onto the space spanned by the orthonormal set \( \{ \psi_j \}_{j=1}^M \).

Now, say our Hamiltonian of the solid we wish to describe is \( H \in \mathcal{B} (\mathcal{H}) \), and say its eigenstates are \( \{ \varphi_j \}_{j=1}^N \) (ordered so that \( \varphi_1 \) has the lowest energy, etc). Since no two Fermions can occupy the same quantum mechanical state (this is the
Pauli exclusion principle), if we fill the solid with $M$ electrons, the ground state (i.e., the state of least energy, \textit{at zero temperature}) is the one where the $M$ electrons occupy the $M$ first levels of $H$, i.e., $\varphi_1, \ldots, \varphi_M$. The corresponding state on the many-body Hilbert space $\wedge \mathcal{H}$ (the exterior algebra generated by $\mathcal{H}$) is thus $\varphi_1 \wedge \cdots \wedge \varphi_M$ (which is called the \textit{Slater determinant}).

In conclusion we recognize that the many-body zero-temperature expectation value of a non-interacting observable $B$ in a the ground state corresponding to $M$ filled electrons is $\text{tr}(P_M B)$ where $P_M := \sum_{j=1}^M \varphi_j \otimes \varphi_j$. More generally, if we work in infinite volume we have an infinite number of electrons and it is more judicious to speak of the Fermi energy $E_F$: that energy of the most energetic electron in the system. Then the appropriate expression is the preempted $\text{tr}(P_F B)$ where $P_F \equiv \chi(-\infty, E_F)(H)$.

Indeed, in infinite volume the range of this operator is infinite dimensional. In order to define this operator rigorously one has to apply the measurable functional calculus of bounded self-adjoint operators, see [RS80]. It will turn out that the Fermi projection $P_F$ contains most of the properties we care about in regards to topological insulators. At non-zero temperatures the Fermi-Dirac distribution should be used--we won’t make use of this here.

### 1.7.3 Electric conductivity

We wish to study insulators, for which we would like to calculate their electric conductance, which is phenomenologically defined via Ohms law:

$$\sigma = \frac{I}{V}$$

with $I$ being the current and $V$ the voltage. More generally, the conductivity $\sigma$ is defined as the matrix relating the current density $j$ with the electric field as follows:

$$j = \sigma E.$$

In principle each of these calculations of $\sigma$ depends on the Fermi energy $\mu$ to which we fill the system.

**Definition 1.55.** An electric insulator at Fermi energy $\mu$ is a material filled to $\mu$ whose conductivity matrix at that energy is zero on the diagonal:

$$\sigma_{ii}(\mu) = 0.$$

Why do we only talk about the diagonal conductivity will become clear later when we consider the Hall conductivity.

In the physics literature, for historical and possibly physical reasons, one usually separates the objects of study in an experimental setup where there is a material (a solid) which is described by a Hamiltonian $H$ and the external driving electric field. Hence, if we calculate the conductivity associated with $H$ alone, it should be zero (since it would typically have no spontaneous currents) and only once we \textit{perturb} with an external electric field it does it actually make to calculate $\sigma$. Thus, we are at the task of perturbation theory, by, say a constant electric field. As we know from undergraduate quantum mechanics, this means adding a term of the form

$$E_0 X_i$$

if the field is of strength $E_0$ in direction $i = 1, \ldots, d$.

Typically, however, the type of perturbation theory taught in undergraduate quantum mechanics (Rayleigh-Schrödinger perturbation theory) is inappropriate for most systems we want to deal with, since it only deals with systems with discrete spectrum (finitely degenerate isolated eigenvalues). Also, generally one likes to do perturbation theory of the more general density matrices. The general theory under which this is done is called \textit{linear response theory} [Kub91].

### 1.7.4 Linear response theory

As we have said the perturbation we are mostly concerned with is something proportional to the position operator and the observable should be the current density, i.e.,

$$j_i = n i [H, X_i] \quad (i = 1, \ldots, d)$$

where $n$ is the density of particles. Indeed, $H$ being the generator of time-translations, $i [H, X_i]$ is associated with $\frac{d}{dt} X_i$, i.e., the velocity.

Furthermore, the perturbations we shall consider are \textit{not} constant in time. Instead, they will be turned on very slowly from being zero at the beginning of time.
Theorem 1.56. (The Kubo formula) Assume a system governed by $H$ is in state described by density matrix $\rho_0$. Assume further that it is perturbed by the time-dependent operator $\varepsilon f (t) A$, i.e.,

$$\tilde{H}(t) := H + \varepsilon f (t) A$$

where $f : \mathbb{R} \to [0, 1]$ is some smooth time-modulation function which obeys $f (-\infty) = 0$ and $f (0) = 1$, $\varepsilon > 0$ is some small order parameter, and $A$ is a time-independent self-adjoint operator. Then the first order (in $\varepsilon$) coefficient of the expectation value of an observable $B = B^*$ for which $\text{tr} (\rho_0 B) = 0$ to the perturbation at time zero is given by

$$\chi_{BA} := -i \int_{-\infty}^{0} \text{tr} \left( e^{-i\tilde{H}t} B e^{iH} [A, \rho_0] \right) f (t) \, dt . \quad (1.12)$$

Proof. The state of the system at time $t$ is governed by the Schrödinger equation for the density matrix, which is

$$i \dot{\rho} (t) = [H + \varepsilon f (t) A, \rho (t)]$$

with initial condition $\rho (-\infty) := \rho_0$. We assume that $\rho_0$ is an equilibrium state for $H$ in the sense that

$$[H, \rho_0] = 0 .$$

We write explicitly the first order term as

$$\rho (t) = \rho_0 + \varepsilon \rho_1 (t)$$

where $\rho_0$ is independent of time since the zero order in $\varepsilon$ has no time dependence in the Hamiltonian. Hence

$$\varepsilon i \dot{\rho}_1 (t) = [H + \varepsilon f (t) A, \rho_0 + \varepsilon \rho_1 (t)] = \varepsilon [H, \rho_1 (t)] + \varepsilon f (t) [A, \rho_0] + O (\varepsilon^2)$$

$$= \varepsilon H^x \rho_1 (t) + \varepsilon f (t) A^x \rho_0 + O (\varepsilon^2)$$

where we used the notation $O^x (\cdot) \equiv [O, \cdot]$ (sometimes also denoted by the adjoint notation $ad_O$ for $O$)

Claim. $e^{a^x} b = e^{a} b e^{-a}$

Proof. One can proceed either in a pedestrian way by computing the explicit expression for $(a^x)^n$ (make guess and proof by induction) or by defining

$$F (t) := e^{ta} b e^{-ta} \quad \forall t \in \mathbb{R}$$

and

$$G (t) := e^{ta^x} b \quad \forall t \in \mathbb{R}$$

Next note that $F$ and $G$ both solve the differential equation

$$\tilde{F}' (t) = a^x F (t)$$

with initial condition $\tilde{F}' (0) = b$. Since the solution to a first order ordinary differential equation is unique, $F = G$ and in particular $F (1) = G (1)$.

Claim. The solution for $\rho_1$ is given by:

$$\rho_1 (t) = i \int_{-\infty}^{t} \exp \left( -i (t - t') H^x \right) \varepsilon A^x \rho_0 f (t') \, dt'$$

$$\equiv i \int_{-\infty}^{t} \exp \left( -i (t - t') H \right) \varepsilon [A, \rho_0] \exp (i (t - t') H) f (t') \, dt'$$
Proof. Using the fact that
\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, y) dy = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} [\partial_x f(x, y)] dy
\]
we have
\[
\partial_t \int_{-\infty}^{t} \exp(-i (t - t') H^x) \varepsilon A^x \rho_0 f(t') dt'
\]
\[
= i \varepsilon A^x \rho_0 f(t) + i \int_{-\infty}^{t} \exp(-i (t - t') H^x) (-i) H^x \varepsilon A^x \rho_0 f(t') dt'
\]
\[
= i \varepsilon A^x \rho_0 f(t) + H^x \int_{-\infty}^{t} \exp(-i (t - t') H^x) \varepsilon A^x \rho_0 f(t') dt'
\]
using the fact that
\[
[\exp(-i (t - t') H^x), H^x] = 0
\]
and also note that the initial value is obeyed: \( \Delta \rho(-\infty) = 0 \).

Then we have
\[
\langle B \rangle_{\rho(0)} = Tr[\rho(0) B]
\]
\[
= Tr[(\rho_0 + \varepsilon \rho_1(0) + \Theta(\varepsilon^2)) B]
\]
\[
= Tr[\rho_0 B] + \varepsilon Tr[\rho_1(0) B] + \Theta(\varepsilon^2)
\]
so that
\[
\chi_{BA} = \frac{1}{\varepsilon} Tr[\rho_1(0) B]
\]
\[
= \frac{1}{\varepsilon} Tr \left[ i \int_{-\infty}^{0} \exp(-i (0 - t') H^x) \varepsilon A^x \rho_0 f(t') dt' B \right]
\]
\[
= i \int_{-\infty}^{0} Tr \left\{ [\exp(iH^x)] (A^x \rho_0) \right\} f(t) B dt
\]
\[
= i \int_{-\infty}^{0} Tr \left[ \exp(itH^x) B \right] f(t) dt
\]
\[
= i \int_{-\infty}^{0} Tr \left[ \exp(-itH^x) B \exp(itH^x) [A, \rho_0] \right] f(t) dt
\]
\[
= i \int_{0}^{\infty} Tr \left[ \exp(itH^x) B \exp(-itH^x) [A, \rho_0] \right] f(-t) dt
\]
We now take care of the limit:
\[
\lim_{\varepsilon \to 0} \chi_{BA} = \lim_{\varepsilon \to 0} i \int_{0}^{\infty} Tr \left[ \exp(itH^x) B \exp(-itH^x) [A, \rho_0] \right] \exp(-\varepsilon t) dt
\]
We now use Lebesgue's dominated convergence theorem ([Rud86] pp. 26) with the dominating function being \( t \to |Tr[\exp(itH) B \exp(-itH) [A, \rho_0]]| \) (need to show it is \( L^1 \)) to take the limit \( \varepsilon \to 0 \) into the integrand and obtain our result. \( \square \)
1.8 Zero temperature DC conductivity

1.8.1 time-reversal invariant case

We now want to apply the formula (1.12) in order to calculate the conductivity of a system. As explained above, the appropriate initial density matrix \( \rho_0 \) to use is the Fermi projection, i.e.,

\[
\rho_0 = P \equiv \chi_{(-\infty, \mu]}(H) .
\]

At non-zero temperature one replaces \( \chi_{(-\infty, \mu]} \) with the Fermi-Dirac distribution:

\[
f_{\text{FD}}(E) \equiv \frac{1}{1 + e^{\beta(E-\mu)}}
\]

where \( \beta \equiv \frac{1}{k_B T} \) with \( k_B \) the Boltzmann constant and \( T \) the temperature. Of course

\[
\lim_{\beta \to \infty} f_{\text{FD}} = \chi_{(-\infty, \mu]} .
\]

The observable \( B \) should be the current density, which is related to the velocity operator in direction \( i \), so we shall take

\[
B = i[H, X_i] .
\]

The perturbation shall be the electric field in direction \( j \), i.e., \( X_j \), so that all together we find that to first order in the electric field,

\[
\sigma_{ij}(\mu) = \lim_{\varepsilon \to 1} \operatorname{tr} \int_{-\infty}^{0} e^{-itH_i} [H, X_i] e^{+itH_i} [X_j, P] f(t) \, dt .
\]

(1.13)

The reason why we take the limit is that eventually we are interested in the static case, where the perturbation is not time dependent (or alternatively in the adiabatic limit where the perturbation is turned on infinitely slowly). We shall make the choice \( f(t) = e^{\varepsilon t} \) and take the limit \( \varepsilon \to 0^+ \).

To proceed further, we shall also make use of the notion of time-reversal in quantum mechanics. Since this hasn’t been introduced yet, let us formally

**Definition 1.57 (Time-reversal).** Time-reversal \( \Theta \) is an anti-unitary operator \( \Theta : \mathcal{H} \to \mathcal{H} \). That means it is anti-\( \mathbb{C} \)-linear:

\[
\Theta(\alpha \psi + \varphi) = \bar{\alpha} \Theta(\psi) + \Theta(\varphi) \quad (\alpha \in \mathbb{C}, \psi, \varphi \in \mathcal{H})
\]

and obeys

\[
(\Theta \psi, \Theta \varphi)_{\mathcal{H}} \equiv (\varphi, \psi)_{\mathcal{H}} \quad (\psi, \varphi \in \mathcal{H}) .
\]

Generally in condensed matter physics, \( \Theta^2 = -1 \) for Fermions and \( \Theta^2 = +1 \) for Bosons by the spin-statistics theorem coming from QFT. A Hamiltonian \( H \) is said to be time-reversal invariant (with respect to the fixed time-reversal operator \( \Theta \)) iff

\[
[H, \Theta] = 0 .
\]

**Theorem 1.58.** If \( H \) is time-reversal invariant as in Definition 1.57 then

\[
\sigma_{ij}(\mu) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \lim_{L \to \infty} \frac{1}{(2L + 1)^3} \sum_{y \in \mathbb{Z}^d, ||y||_1 \leq L} |G(y + x, y; \mu + i\varepsilon)|^2 .
\]

(1.14)

We first note that generically, the operator within the trace appearing in (1.13) is not expected to be trace-class. For that reason, we should rather work with the
Moreover, we also write

\[ \tilde{\text{tr}} (A) \equiv \lim_{L \to \infty} \frac{1}{(2L + 1)^d} \sum_{x \in \mathbb{Z}^d : \|x\| \leq L} \text{tr}_{\mathbb{C}^N} (\langle \delta_x, A \delta_x \rangle) \]

where \( \|x\| \equiv \sum_{j=1}^d |x_j| \) and \( \text{tr}_{\mathbb{C}^N} (\langle \delta_x, A \delta_x \rangle) \) means the trace within \( \mathbb{C}^N \) of the \( N \times N \) matrix \( \langle \delta_x, A \delta_x \rangle \).

**Proof.** In the proof below we assume \( N = 1 \) for simplicity. Given the comment above regarding the trace-class property, our starting point is the following modification of (1.13):

\[
\sigma_{ij} (\mu) = \lim_{\varepsilon \to 0^+} \int_{-\infty}^0 \tilde{\text{tr}} \left( e^{-itH} i [H, X_i] e^{itH} i [X_j, P] \right) e^{\varepsilon t} dt
\]

with \( P \equiv \chi_{(-\infty, \mu]} (H) \). We assume that the limit involved in \( \tilde{\text{tr}} \) exists and start off by re-writing the regulator as

\[
e^{\varepsilon t} = \partial_t \left( \frac{e^{\varepsilon t} - 1}{\varepsilon} \right)
\]

to perform integration by parts and find

\[
\sigma_{ij} (\mu) = - \lim_{\varepsilon \to 0^+} \int_{-\infty}^0 \left( \partial_t \tilde{\text{tr}} \left( e^{-itH} i [H, X_i] e^{itH} i [X_j, P] \right) \right) \frac{e^{\varepsilon t} - 1}{\varepsilon} .
\]

Now

\[
\partial_t \tilde{\text{tr}} \left( e^{-itH} i [H, X_i] e^{itH} i [X_j, P] \right) = \partial_t \tilde{\text{tr}} \left( i [H, X_i] e^{itH} i [X_j, P] e^{-itH} \right) = \tilde{\text{tr}} (i [H, X_i] e^{itH} [i [H, X_j], e^{-itH}]) = \tilde{\text{tr}} (V_i e^{itH} [V_j, P] e^{-itH} ) ,
\]

Let us write

\[
f (H) = \int_{E \in \mathbb{R}} f (E) dQ (E)
\]

where \( Q \) is the projection-valued measure associated to \( H \), and \( f \) is any bounded measurable function. Then

\[
\tilde{\text{tr}} (V_i e^{itH} [V_j, P] e^{-itH} ) = \tilde{\text{tr}} \left( V_i \int_{\lambda_1 \in \mathbb{R}} dQ (\lambda_1) e^{itH} i [V_j, P] e^{-itH} \int_{\lambda_2 \in \mathbb{R}} dQ (\lambda_2) \right) \\
= \int_{\lambda_1 \in \mathbb{R}} \int_{\lambda_2 \in \mathbb{R}} e^{it(\lambda_1 - \lambda_2)} \tilde{\text{tr}} (V_i dQ (\lambda_1) i [V_j, P] dQ (\lambda_2)) \\
= \int_{\lambda_1 \in \mathbb{R}} \int_{\lambda_2 \in \mathbb{R}} e^{it(\lambda_1 - \lambda_2)} (g (\lambda_2) - g (\lambda_1)) \tilde{\text{tr}} (V_i dQ (\lambda_1) V_j dQ (\lambda_2)) .
\]

Here we are using

\[
g (\lambda) \equiv \chi_{(-\infty, \mu]} (\lambda) .
\]

Moreover, we also write

\[
e^{\varepsilon t} - 1 = t \int_0^\varepsilon e^{\eta t} d\eta
\]
to get

$$
\sigma_{ij} (\mu) = -i \lim_{\varepsilon \to 0^+} \int_{t=-\infty}^{0} \int_{\lambda_1 \in \mathbb{R}} \int_{\lambda_2 \in \mathbb{R}} e^{it(\lambda_1 - \lambda_2)} (g(\lambda_2) - g(\lambda_1)) \tilde{\text{tr}}(V_i dQ(\lambda_1)V_j dQ(\lambda_2)) \frac{t}{\varepsilon} \int_{\varepsilon}^{0} e^{\eta t} d\eta .
$$

The time integral we can do explicitly to get

$$
\hat{0}\int_{t=-\infty}^{0} dt e^{it(\lambda_1 - \lambda_2 - i\eta)} = \frac{1}{(\lambda_1 - \lambda_2 - i\eta)^2}
$$

so we get

$$
\sigma_{ij}(\mu) = i \lim_{\varepsilon \to 0^+} \int_{\lambda_1, \lambda_2 \in \mathbb{R}} \int_{\eta=0}^{\varepsilon} d\eta \frac{1}{\varepsilon} (g(\lambda_1) - g(\lambda_2)) dm_{ij}(\lambda_1, \lambda_2)
$$

where we define the velocity measure

$$
dm_{ij}(\lambda_1, \lambda_2) := \tilde{\text{tr}}(V_i dQ(\lambda_1)V_j dQ(\lambda_2))
$$

and by $P(\lambda)$ we mean now the function

$$
\mathbb{R} \ni \lambda \mapsto \chi_{(-\infty, \mu]}(\lambda).
$$

Now for well-behaved functions $f$ we may replace

$$
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\eta=0}^{\varepsilon} f(\eta) d\eta = \partial_{\varepsilon \mid \varepsilon = 0} \int_{\eta=0}^{\varepsilon} f(\eta) d\eta = \lim_{\eta \to 0^+} f(\eta).
$$

Moreover, we have the so-called Kramers-Kronig relation [Sha23] (Corollary 7.61)

$$
\lim_{\varepsilon \to 0^+} \frac{1}{x \pm i\varepsilon} \overset{\mathcal{P}}{=} \mp i\pi \delta(x) + \mathcal{P}\left(\frac{1}{x}\right)
$$

where $\mathcal{P}$ is the Cauchy principal value of an integral. If we take the derivative of this relation w.r.t. $x$ we get

$$
\lim_{\varepsilon \to 0^+} -\frac{1}{(x \pm i\varepsilon)^2} \overset{\mathcal{P}}{=} \mp i\pi \delta'(x) + \mathcal{P}'\left(\frac{1}{x}\right).
$$

Finally, if our Hamiltonian is time-reversal invariant, then using Lemma 1.60 right below we get

$$
dm_{ij}(\lambda_1, \lambda_2) = dm_{ij}(\lambda_2, \lambda_1).
$$

Then the function $\mathcal{P}'\left(\frac{1}{x}\right)$ is even (seen from the derivative of the Kramers-Kronig relation) so that we integrate the odd function of $\lambda_1, \lambda_2$

$$
(g(\lambda_1) - g(\lambda_2)) dm_{ij}(\lambda_1, \lambda_2)
$$

against $\mathcal{P}'$ we get zero. we are thus left only with the $\delta'$ term to get

$$
\sigma_{ij}(\mu) = \pi \int_{\lambda_1, \lambda_2 \in \mathbb{R}} \delta'(\lambda_1 - \lambda_2)(g(\lambda_1) - g(\lambda_2)) dm_{ij}(\lambda_1, \lambda_2).
$$

Next we write

$$
\delta'(\lambda_1 - \lambda_2) \overset{\mathcal{P}}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\delta(\lambda_1 - \lambda_2 + \varepsilon) - \delta(\lambda_1 - \lambda_2))
$$
and we get

$$
\sigma_{ij} (\mu) = \lim_{\varepsilon \to 0^+} \frac{\pi}{\varepsilon} \int_{\lambda_1} \left[ (g (\lambda_1) - g (\lambda_1 + \varepsilon)) \, dm_{ij} (\lambda_1, \lambda_1 + \varepsilon) \right] - \left[ (g (\lambda_1) - g (\lambda_1)) \, dm_{ij} (\lambda_1, \lambda_1) \right]
$$

$$
= \pi \int_{\lambda_1} \frac{\partial X (\lambda_1)}{\lambda_1} \, dm_{ij} (\lambda_1, \lambda_1)
$$

$$
= \pi \int_{\lambda_1} \frac{\partial (\lambda_1 - \mu)}{\lambda_1} \, dm_{ij} (\lambda_1, \lambda_1)
$$

$$
= \pi \int_{\lambda_1} \int_{\lambda_2} \frac{\partial (\lambda_1 - \mu \lambda_2 - \mu \delta m_{ij} (\lambda_1, \lambda_2))}{\lambda_2}
$$

$$
= \pi \lim_{\varepsilon \to 0^+} \int_{\lambda_1} \int_{\lambda_2} \frac{\partial (\lambda_1 - \mu \lambda_2 - \mu \delta m_{ij} (\lambda_1, \lambda_2))}{\lambda_2}
$$

$$
= \pi \lim_{\varepsilon \to 0^+} \int_{\lambda_1} \int_{\lambda_2} \frac{\partial (\lambda_1 - \mu \lambda_2 - \mu \delta m_{ij} (\lambda_1, \lambda_2))}{\lambda_2}
$$

$$
= \pi \lim_{\varepsilon \to 0^+} \int_{\lambda_1} \int_{\lambda_2} \frac{\partial (\lambda_1 - \mu \lambda_2 - \mu \delta m_{ij} (\lambda_1, \lambda_2))}{\lambda_2}
$$

$$
= \pi \lim_{\varepsilon \to 0^+} \left[ \delta \left( H - \mu \right) [H, X_i] \delta \left( H - \mu \right) [X_j, H] \right].
$$

Here we have used the identity of distributions for the approximate delta function

$$
\delta \left( H - \mu \right) = \frac{1}{\pi} \text{Im} \left\{ R (\mu + i \varepsilon) \right\}.
$$

We now have

$$
\pi \text{Tr} \left( \frac{1}{\pi} \text{Im} \left\{ R (\mu + i \varepsilon) \right\} [H, X_i] \frac{1}{\pi} \text{Im} \left\{ R (\mu + i \varepsilon) \right\} [X_j, H] \right)
$$

$$
= \frac{\varepsilon^2}{\pi} \text{Tr} \left( R (\mu + i \varepsilon) R (\mu - i \varepsilon) [H, X_i] R (\mu - i \varepsilon) R (\mu + i \varepsilon) [X_j, H] \right)
$$

$$
= \frac{\varepsilon^2}{\pi} \text{Tr} \left( R (\mu + i \varepsilon) R (\mu - i \varepsilon) [H, X_i] R (\mu - i \varepsilon) R (\mu + i \varepsilon) [X_j, H] \right)
$$

$$
= \frac{\varepsilon^2}{\pi} \text{Tr} \left( R (\mu - i \varepsilon) [H, X_i] R (\mu - i \varepsilon) R (\mu + i \varepsilon) [X_j, H] R (\mu + i \varepsilon) \right)
$$

$$
= \frac{\varepsilon^2}{\pi} \text{Tr} \left( [R (\mu - i \varepsilon), X_i] [X_j, R (\mu + i \varepsilon)] \right).
$$

We proceed by plugging in

$$
1 = \sum_{x \in \mathbb{Z}^d} \delta_x \otimes \delta_x^*
$$

and the definition of the trace-per-unit-volume to get

$$
\sigma_{ij} (\mu) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \text{Tr} \left( [R (\mu - i \varepsilon), X_i] [X_j, R (\mu + i \varepsilon)] \right)
$$

$$
= \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \lim_{L \to \infty} \frac{1}{(2L + 1)^d} \sum_{x \in \mathbb{Z}^d} \sum_{y \leq L} \langle \delta_y, [R (\mu - i \varepsilon), X_i] \delta_x \rangle \langle \delta_x, [X_j, R (\mu + i \varepsilon)] \delta_y \rangle
$$

$$
= \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \lim_{L \to \infty} \frac{1}{(2L + 1)^d} \sum_{x \in \mathbb{Z}^d} \sum_{y \leq L} G (x, y; \mu - i \varepsilon) (x - y) (x - y_j) G (x, y; \mu + i \varepsilon)
$$

$$
= \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \lim_{L \to \infty} \frac{1}{(2L + 1)^d} \sum_{x \in \mathbb{Z}^d} \sum_{y \leq L} (x - y_i) (x - y_j) G (x, y; \mu + i \varepsilon)^2
$$

$$
= \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \lim_{L \to \infty} \frac{1}{(2L + 1)^d} \sum_{y \leq L} |G (x + y, \mu + i \varepsilon)|^2
$$
Lemma 1.60. Assume $H$ is time-reversal invariant as in Definition 1.57 and let $dm_{ij}(\lambda_1, \lambda_2)$ be the associated velocity measure

$$dm_{ij}(\lambda_1, \lambda_2) \equiv \text{tr} (V_i dQ(\lambda_1)V_j dQ(\lambda_2))$$

where

$$V_i := i[H, X_i]$$

is the velocity operator in the $i$th direction and $Q$ is the projection-valued measure associated to $H$.

Then

$$dm_{ij}(\lambda_1, \lambda_2) = dm_{ij}(\lambda_2, \lambda_1).$$

Proof. TODO \hfill \Box

Corollary 1.61 (Random ergodic operators). We will have a thorough discussion later about random ergodic operators, but let us just remark here that if $H$ is actually a random ergodic operator, so that Birkhoff’s theorem applies on it (in the sense that space averages may be exchanged for disorder averages) then we get

$$\sigma_{ij}(\mu) = \lim_{\varepsilon \to 0^{+}} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \mathbb{E} \left[ |G(x, 0; \mu + i\varepsilon)|^2 \right].$$

(1.15)

Proof. We employ Birkhoff’s theorem

$$\lim_{L \to \infty} \frac{1}{(2L + 1)^d} \sum_{x \in \mathbb{Z}^d : \|x\|_1 \leq L} \langle \delta_x, f(H) \delta_x \rangle = \mathbb{E} [\langle \delta_0, f(H) \delta_0 \rangle]$$

where $f$ is any measurable function of the Hamiltonian. We thus start from the last displayed equation in the above proof to get

$$\text{tr} ([R(\mu - i\varepsilon), X_i] [X_j, R(\mu + i\varepsilon)]) = \mathbb{E} [\langle \delta_0, [R(\mu - i\varepsilon), X_i] [X_j, R(\mu + i\varepsilon)] \delta_0 \rangle].$$

With that we have

$$\sigma_{ij}(\mu) = \lim_{\varepsilon \to 0^{+}} \frac{\varepsilon^2}{\pi} \mathbb{E} [\langle \delta_0, [R(\mu - i\varepsilon), X_i] [X_j, R(\mu + i\varepsilon)] \delta_0 \rangle]$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{\varepsilon^2}{\pi} \mathbb{E} \left[ \langle \delta_0, [R(\mu - i\varepsilon), X_i] \sum_{x \in \mathbb{Z}^d} \delta_x \otimes \delta_x^* [X_j, R(\mu + i\varepsilon)] \delta_0 \rangle \right]$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} \mathbb{E} [\langle \delta_0, [R(\mu - i\varepsilon), X_i] \delta_x \rangle \langle \delta_x, [X_j, R(\mu + i\varepsilon)] \delta_0 \rangle]$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \mathbb{E} \left[ G(0, x; \mu - i\varepsilon) G(x, 0; \mu + i\varepsilon) \right].$$

But $H = H^*$ so $G(x, y; z) = \overline{G(y, x; \overline{z})}$ and hence

$$\sigma_{ij}(\mu) = \lim_{\varepsilon \to 0^{+}} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \mathbb{E} \left[ |G(x, 0; \mu + i\varepsilon)|^2 \right]$$

which is what we were trying to show. \hfill \Box
Remark 1.62 (Gapped systems are insulators). It is clear that if \( \mu \notin \sigma(H) \), then \( \sigma_{ij}(\mu) = 0 \). Indeed, in that case we may invoke the Combes-Thomas estimate Theorem 1.18 to obtain

\[
\sup_{\varepsilon > 0} |G(x, y; \mu + i \varepsilon)|^2 \leq \frac{4}{\delta^4} \exp(-2\delta \mu \|x - y\|) \quad (x, y \in \mathbb{Z}^d)
\]

where \( \delta := \text{dist} (\mu, \sigma(H)) > 0 \) is the gap size. Since this is an estimate uniform in \( \varepsilon > 0 \), we get summability in the \( x \) variable before even taking the limit \( \varepsilon \to 0^+ \). Then end result is then

\[
\sigma_{ij}(\mu) = \lim_{\varepsilon \to 0^+} \varepsilon^2 \times \text{(something uniformly bounded as } \varepsilon \to 0^+) = 0.
\]

Remark 1.63 (DC conductivity for ballistic motion is infinite). What might happen if we have ballistic motion? For instance, can we show that \( \sigma_{ij}(\mu) = 0 \) if \( \mu \in \sigma_{ac}(H) \)? As a case study, take the discrete Laplacian in 1D, whence we have

\[
\sigma_{11}(\mu) = \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}} x^2 \left| \left( (-\Delta - \mu \mathbb{1} - i \varepsilon \mathbb{1})^{-1} \right)_{x,0} \right|^2.
\]

To proceed we have two options, we could either first calculate

\[
(-\Delta - z\mathbb{1})_{x,0}^{-1} = \frac{1}{2\pi} \int_{k=0}^{2\pi} e^{ikx} \frac{1}{2 - 2\cos(k) - z} dk
\]

where \( z = \mu + i \varepsilon \) and say, \( \mu \) lies in the middle of the spectrum, say, at 2, and doing a residue calculation. Instead, we may bring this integral to momentum space as

\[
\sum_{x \in \mathbb{Z}} x^2 \left| \left( (-\Delta - \mu \mathbb{1} - i \varepsilon \mathbb{1})^{-1} \right)_{x,0} \right|^2 = \langle \delta_0, \mathcal{F} R(\pi) X^2 R(z) \delta_0 \rangle_{L^2} = \langle \mathcal{F} \delta_0, \mathcal{F} R(\pi) \mathcal{F} X^2 \mathcal{F} R(z) \mathcal{F} \delta_0 \rangle_{L^2}.
\]

We now re-call from the proof of Proposition 1.23 that

\[
\mathcal{F} X^2 \mathcal{F}^* = -\partial_k^2 \\
\mathcal{F} \delta_0 = k \mapsto 1
\]

so that (with \( z = 2 + i \varepsilon \), say)

\[
\sum_{x \in \mathbb{Z}} x^2 \left| \left( (-\Delta - \mu \mathbb{1} - i \varepsilon \mathbb{1})^{-1} \right)_{x,0} \right|^2 = -\frac{1}{2\pi} \int_{k=0}^{2\pi} \frac{1}{2 - 2\cos(k) - \varepsilon} \partial_k^2 \frac{1}{2 - 2\cos(k) - \varepsilon} dk
\]

\[
\text{Mathematica} = \frac{2}{\varepsilon^3 \sqrt{4 + \varepsilon^2}}.
\]

We see clearly that as \( \varepsilon \to 0^+ \), the expression

\[
\lim_{\varepsilon \to 0^+} \varepsilon^2 \pi \times \frac{2}{\varepsilon^3 \sqrt{4 + \varepsilon^2}} = \infty.
\]

This is not an accident: periodic operators in general will exhibit infinite DC conductivity, i.e., zero resistivity.

1.8.2 The general case: IQHE application

Before proceeding we make an important modification:

Definition 1.64 (Switch function). A switch function on the \( j \)th axis \( (j = 1, \ldots, d) \) is a projection

\[
\Lambda_j \equiv \chi_{\mathbb{N}}(X_j)
\]

to the \( j \)th positive half-space.
We want to replace $X_j$ with $\Lambda_j$ so that will turn out to yield trace class operators. For the perturbation (i.e., the application of the electric field) the justification is easy. It replaces the constant field with a delta field. For the observable, it means calculating the amount of charge accumulated on the half-space rather than velocity.

**Theorem 1.65.** For two-dimensional systems that do not have time-reversal-invariance, such as integer quantum Hall systems, if $\mu$ is within a spectral gap of $H$, one can bring (1.13) to the form

$$
\sigma_{ij}(\mu) = \text{itr} (P[[\Lambda_1, P], [\Lambda_2, P]]) .
$$

Here, $\Lambda_j$ is a projection operator onto the positive half-space defined by the $j$th axis:

$$(\Lambda_j \psi)_x \equiv \begin{cases} 
\psi_x & x_j \geq 1 \\
0 & x_j \leq 0
\end{cases} .$$

Part of the statement of the theorem is that the above expression is indeed trace-class in two-dimensions (in higher dimensions it is not and one should rather use the trace per unit volume).

We delay the proof of this statement until we go on to talk about the Chern number of integer quantum Hall systems.

2 Random operators and Anderson localization

In this chapter we set up the necessary machinery for discussing the phenomenon of Anderson localization: this is the set up of random ergodic operators. We shall then present different proofs of this fact in various regimes and dimensions and conclude by presenting the big open problem of delocalization.

The theory of localization started with the ground breaking work of Anderson [And58]. Roughly speaking it says that if electrons are placed in a sufficiently disordered medium–neglecting electron-electron interactions–they will get “stuck” in confined regions rather than flow throughout space (compare this with translation-invariant media where Bloch theorem says that electrons are blind to the crystal structure and flow through it freely). One important consequence is that the DC electrical conductivity at the corresponding Fermi energy is zero (1.15), which means we should associate such materials with insulators. Mathematically the first proof of localization appeared in [FS83]; a simpler, different proof appeared in [AM93] which was further developed in [AG98], allowing for the understanding of the role of localization in the plateaus of the IQHE.

2.1 Why random operators?

Anderson’s strategy to understand a disordered material was to toss coins in order to generate a random potential, and make statements which hold almost surely with respect to the probability distribution of the coins or alternatively statements about expectations (w.r.t. disorder) of physical quantities. While an actual experiment is performed on one single material (and hence corresponding to a deterministic Hamiltonian), the theory should describe the outcome of an average over many experiments so that such theoretical statements about inherently random objects could actually describe (an ensemble of) experiments. The individual macroscopic sample contains in itself many microscopic subsamples, and hence the averaging. Indeed, the actual process with which disorder is formed in materials is likely described by some probability distribution (ultimately relating to a quantum stochastic process) and our probabilistic model is merely a (gross) simplification of the real one. Another philosophical justification for this approach is via Wigner’s random matrix theory. It says that in the absence of better knowledge about the actual physical laws, we pretend the unknown part of the model is given by a collection of random variables. General physical principles (e.g. locality) will then give constraints on these random variables (e.g., their independence). For an introduction to random operators, see [AW15].

2.2 Basic setup for random operators

2.2.1 Abstract definitions

A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega$ is a set (of possible basic events), $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a sigma-algebra and

$$\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$$

is a probability measure. On such a probability measure we put additional structure as follows

---

1Recall a sigma-algebra is a collection of subsets of $\Omega$ containing $\Omega$ which is closed under complements and countable unions. The smallest sigma-algebra is $\sigma(\varnothing, \Omega)$ and the largest one is $2^\Omega$. 

---

42
Definition 2.1 (Measure-preserving morphism). A map \( T : \Omega \rightarrow \Omega \) is called measure-preserving iff
\[
P(S) = P(T^{-1}(S)) \quad (S \in \mathcal{F})
\]
where \( T^{-1}(S) \) is the pre-image of \( S \) under \( T \). The tuple \((\Omega, \mathcal{F}, P, T)\) is called a measure-preserving dynamical system.

Definition 2.2. Let \( G \) be a group and that \( T : G \rightarrow \text{Aut}(\Omega) \) is a group morphism, where \( \text{Aut}(\Omega) \) is the group of automorphisms of \( \Omega \) (i.e., each element \( T(g) \) is measure-preserving). Then the tuple \((\Omega, \mathcal{F}, P, T)\) is called a measure-preserving \( G \)-dynamical system.

Definition 2.3 (invariant RV, ergodic dynamical systems). A measurable map \( X : \Omega \rightarrow \mathbb{R} \) is called a random variable. A random variable on a measure-preserving \( G \)-dynamical system \((\Omega, \mathcal{F}, P, T)\) is called invariant iff
\[
X \circ T(g) = X \quad (g \in G).
\]
A measure-preserving \( G \)-dynamical system \((\Omega, \mathcal{F}, P, T)\) is called ergodic iff every invariant random variable is constant \( P \)-almost-surely.

Definition 2.4 (Random operator). Let \((\Omega, \mathcal{F}, P)\) be a probability space. That means that \( \Omega \) is a measure space, \( \mathcal{F} \) is a given sigma-algebra on it and \( P : \mathcal{F} \rightarrow [0,1] \) is a probability measure. Let \( \mathcal{H} \) be a fixed separable Hilbert space. A random (self-adjoint) operator \( A \) is a weakly-measurable function
\[
A : \Omega \rightarrow \{ B \in \mathcal{B}(\mathcal{H}) \mid B = B^* \}
\]
i.e., for any \( \varphi, \psi \in \mathcal{H} \), for any measurable \( f : \mathbb{R} \rightarrow \mathbb{C} \)
\[
\Omega \ni \omega \mapsto \langle \varphi, f(A(\omega))\psi \rangle \in \mathbb{C}
\]
is a measurable function.

Definition 2.5 (Ergodic operator). A random operator \( A : \Omega \rightarrow \mathcal{B}(\mathcal{H}) \) (where \( \Omega \) has the structure of a measure-preserving \( G \)-dynamical system \((\Omega, \mathcal{F}, P, T)\)) is called ergodic iff for all \( g \in G \), and for all \( \omega \in \Omega \), \( A(\omega) \) and \( A(T(g)(\omega)) \) are unitary conjugates (the unitary may well depend on both \( \omega \) and \( g \)).

Theorem 2.6 (Birkhoff). Let an ergodic measure-preserving \( \mathbb{Z}^d \)-dynamical system \((\Omega, \mathcal{F}, P, T)\) be given. Let \( X \in L^1(\Omega, P) \) be a random variable. Then the following limit exists \( P \)-almost-surely and equals
\[
\lim_{L \rightarrow \infty} \frac{1}{(2L + 1)^d} \sum_{x \in \mathbb{Z}^d, \|x\|_1 \leq L} X(T_x \omega) = \mathbb{E}[X].
\]

Theorem 2.7 (Pastur). Let an ergodic measure-preserving \( \mathbb{Z}^d \)-dynamical system \((\Omega, \mathcal{F}, P, T)\) be given and \( H = H^* : \Omega \rightarrow \mathcal{B}(\mathcal{H}) \) be an ergodic random self-adjoint operator. Then there are (deterministic) subset \( s, s^\sharp \subseteq \mathbb{R} \) such that \( P \)-almost-surely,
\[
\sigma_{t^\sharp}(H(\omega)) = s^\sharp
\]
where \( \sharp \) is either nothing (in which case we mean the entire spectrum) or \( pp, sc, ac \).

Proof. [TODO: fix this] Consider, for any \( a < b \in \mathbb{R} \) the map
\[
\Omega \ni \omega \mapsto \dim(\text{im} \left( \chi_{(a,b)}(H(\omega)) \right)) \in [0, \infty]
\]
which is measurable. Since $H$ is presumed ergodic, these functions are invariant under translations. Indeed, we have
\[ \dim \left( \text{im} \left( \chi_{(a,b)} \left( H \left( T_x \omega \right) \right) \right) \right) = \text{tr} \left( \chi_{(a,b)} \left( H \left( T_x \omega \right) \right) \right) = \text{tr} \left( \chi_{(a,b)} \left( U_x^* H (\omega) U_x \right) \right) = \text{tr} \left( U_x^* \chi_{(a,b)} \left( H (\omega) \right) U_x \right) = \text{tr} \left( \chi_{(a,b)} \left( H (\omega) \right) \right). \]

So this is an invariant random variable and since our system is ergodic, it implies there are constants $\alpha_{(a,b)} \in [0, \infty]$ such that
\[ P \left[ \left\{ \dim \left( \text{im} \left( \chi_{(a,b)} \left( H (\omega) \right) \right) \right) = \alpha_{(a,b)} \right\} \right] = 1. \]

Since $\sigma (H (\omega))$ is identified as the essential support of the spectral projections of $H (\omega)$, we identify
\[ s := \left\{ E \in \mathbb{R} \mid \forall a, b \in \mathbb{Q} : a < E < b, \alpha_{(a,b)} > 0 \right\}. \]

We choose rational end points to make sure the countable intersection still has probability one:
\[ \bigcap_{a, b \in \mathbb{Q} \mid \alpha_{(a,b)} > 0} \left\{ \dim \left( \text{im} \left( \chi_{(a,b)} \left( H (\omega) \right) \right) \right) = \alpha_{(a,b)} \right\} \subseteq \left\{ \sigma (H (\omega)) = s \right\}. \]

\[ \square \]

### 2.2.2 Concrete application: the Anderson model

We now consider the main setup which will concern us. We are interested in random operators $H_\omega$ on $\ell^2 (\mathbb{Z}^d)$ which are of the form
\[ H_\omega := -\Delta + \lambda V_\omega (X) \quad (2.1) \]
where $-\Delta$ is the discrete Laplacian, $\lambda > 0$ is a coupling constant, and
\[ V_\omega (x) := \omega_x \quad (x \in \mathbb{Z}^d) \]
where $\{ \omega_x \}_{x \in \mathbb{Z}^d}$ is a point in the random configuration space
\[ \Omega := \left\{ \omega : \mathbb{Z}^d \to \mathbb{R} \text{ measurable} \right\} \cong \mathbb{R}^{\mathbb{Z}^d}. \]

Moreover, we are interested in the following product measure
\[ \int_{\omega \in \Omega} f (\omega) \, dP (\omega) := \prod_{x \in \mathbb{Z}^d} \int_{\omega_x \in \mathbb{R}} d\mu (\omega_x) f (\omega) \]
where $\mu$ is a fixed probability measure on $\mathbb{R}$. Formally we write
\[ P = \mu \otimes \mathbb{R}^{\mathbb{Z}^d}. \]

We say that in this case, the stochastic process $\{ \omega_x \}_{x \in \mathbb{Z}^d}$ is iid: it is independent and identically distributed (according to the “single site” probability measure $\mu$). We usually ask that $\mu$ obeys some regularity condition, for example,

\[ \textbf{Definition 2.8 (uniform $\tau$-Hoelder continuity).} \] Let $\tau \in (0, 1]$. The probability measure $\mu : \mathcal{B} (\mathbb{R}) \to [0, 1]$ ($\mathcal{B} (\mathbb{R})$ being Borel measurable subsets of $\mathbb{R}$) is said to be uniformly $\tau$-Hoelder continuous iff there exists some constant $C_\mu > 0$ such that
\[ \mu (J) \leq C_\mu |J|^\tau \quad (J \subseteq \mathbb{R} \text{ interval with } |J| \leq 1) \]
where $|J|$ is the Lebesgue measure of $J$.

In this case, the group we are interested in is
\[ G := \mathbb{Z}^d \]
i.e., the group of lattice translations:
\[ T : \mathbb{Z}^d \to \text{Aut} (\Omega) \]
is defined as

\[(T_x)(\omega) := \omega_{-x} \quad (x \in \mathbb{Z}^d) .\]

These shifts are measure preserving since \(\Omega\) is merely a product space with the product measure. The unitary transformation which relates \(H_\omega\) with \(H_{T_x \omega}\) is of course lattice translations (recall they commute with \(-\Delta\)).

**Remark 2.9.** It is appropriate to look at independent identically distributed random potential values due to the homogeneous (in distribution) nature of materials: we presume that on the whole they obey the same laws of physics throughout space. We can of course generalize this to decaying correlations etc. We avoid doing so here unless otherwise specified.

**Theorem 2.10 (Kunz-Souillard).** If we normalize \(-\Delta\) such that

\[\sigma(-\Delta) = [-2d, 2d]\]

then \(P\)-almost-surely, the spectrum of the Anderson model (2.1)

\[\sigma(-\Delta + \lambda V_\omega(X)) = [-2d, 2d] + \lambda \text{supp} (\mu) .\]

Here we mean the set addition as

\[A + B := \{ E + \tilde{E} \in \mathbb{R} \mid E \in A, \tilde{E} \in B \}\]

and

\[\text{supp} (\mu) \equiv \{ u \in \mathbb{R} \mid \forall \varepsilon > 0, \mu (B_\varepsilon (u)) > 0 \}.\]

Note we have

\[\{ V_\omega (x) \mid x \in \mathbb{Z}^d \} = \text{supp} (\mu) \]

\(P\)-almost-surely.

**Proof.** [TODO: fix this] This statement shall be proven in two steps: \(\subseteq\) and \(\supseteq\). Let us begin with the former. Let

\[E \notin [-2d, 2d] + \lambda \text{supp} (\mu) .\]

That means that

\[\text{dist} (E, \lambda \text{supp} (\mu)) > 2d .\]

But then,

\[-\Delta + \lambda V_\omega(X) - E \mathbb{1} = (\lambda V_\omega(X) - E \mathbb{1}) (\mathbb{1} - (\lambda V_\omega(X) - E \mathbb{1})^{-1} \Delta)\]

and we have

\[\left\| \mathbb{1} - (\lambda V_\omega(X) - E \mathbb{1})^{-1} \Delta \right\| \leq \left\| (\lambda V_\omega(X) - E \mathbb{1})^{-1} \right\| \left\| -\Delta \right\| < 1\]

so the operator

\[\left( \mathbb{1} - (\lambda V_\omega(X) - E \mathbb{1})^{-1} \Delta \right)\]

is invertible and hence

\[E \notin \sigma(-\Delta + \lambda V_\omega(X)) .\]

For the other inclusion, let \(E \in [-2d, 2d] = \sigma(-\Delta)\). We thus build a Weyl sequence [Sha24] for \(-\Delta\): for any \(\varepsilon > 0\) there exists some \(\psi \in \ell^2 (\mathbb{Z}^d)\) with \(\|\psi\| = 1\) such that

\[\|(-\Delta - E \mathbb{1}) \psi\| \leq \varepsilon .\]
Let us assume for a moment that $\psi$ is supported within a large finite box $\Lambda$ (otherwise approximate and use locality of $-\Delta$). Then if $\hat{E} \in \text{supp}(\mu)$, we must have
\[
P\left( \left\{ \omega \in \Omega \left| \sup_{x \in \Lambda} |\lambda \omega_x - \hat{E}| < \varepsilon \right. \right\} \right) = \prod_{x \in \Lambda} \mu \left( B_{\varepsilon} \left( \hat{E} \right) \right) > 0.
\]
For such $\omega$’s, we have
\[
\| \left(-\Delta + \lambda V_\omega \left( x \right) - \left( E + \hat{E} \right) \mathbf{1} \right) \psi \| \leq \| -\Delta - E \mathbf{1} \| \psi \| + \| \left( V_\omega \left( x \right) - \hat{E} \mathbf{1} \right) \psi \| \leq 2\varepsilon
\]
so that actually $\psi$ is a Weyl sequence for $-\Delta + V_\omega \left( x \right)$ and hence
\[
P\left( \left\{ \omega \in \Omega \left| \text{dist} \left( E + \hat{E}, \sigma \left( -\Delta + \lambda V_\omega \left( x \right) \right) \right) < 2\varepsilon \right. \right\} \right) > 0
\]
and so by ergodicity, must equal 1.

2.3 The main results known so far and conjectures

2.3.1 Criteria for localization

Here we survey various criteria for localization. Some imply the others automatically (as we discuss momentarily).

1. Spectral localization: There exists some $\varepsilon > 0$ such that
\[
B_{\varepsilon} \left( E \right) \cap \sigma \left( H \right) = B_{\varepsilon} \left( E \right) \cap \sigma_{pp} \left( H \right)
\]
almost-surely (or an analogous statement about the almost sure spectrum).

2. Decay of eigenfunctions: If $H\psi = E\psi$ then there exists some $C, \mu \in (0, \infty)$ such that
\[
|\psi \left( x \right) | \leq Ce^{-\mu \| x \|} \quad \left( x \in \mathbb{Z}^d \right)
\]
almost-surely. Presumably it is impossible that this happens merely at a single energy and one should rather ask that this holds for every eigenfunction with energy $\hat{E} \in B_{\varepsilon} \left( E \right)$ for some $\varepsilon > 0$.

3. High inverse participation ratio: In finite boxes $\Lambda \subseteq \mathbb{Z}^d$, if $H\psi = E\psi$ with $\| \psi \| = 1$, then for any $x \in \Lambda$, $|\psi \left( x \right) |^2$ could take the values between 0, $\frac{1}{|\Lambda|^2}$, 1. If the state is fully localized in one position, there would be a single $x_0 \in \Lambda$ where $|\psi \left( x_0 \right) | = 1$ and otherwise if it is fully delocalized, it would be completely spread out throughout space so that
\[
|\psi \left( x \right) | \approx \frac{1}{|\Lambda|^2} \quad \left( x \in \Lambda \right)
\]
so that $\sum_{x \in \Lambda} |\psi \left( x \right) |^2 = 1$. Hence, to measure how “flat” the wave-function is, we introduce
\[
\text{IPR} \left( \psi \right) := \sum_{x \in \Lambda} |\psi \left( x \right) |^4. \quad \left( 2.2 \right)
\]
If $\text{IPR} \left( \psi \right) \approx 1$ we say the state is localized. However, if it is fully delocalized we expect
\[
\text{IPR} \left( \psi \right) \approx \sum_{x \in \Lambda} \left( \frac{1}{|\Lambda|^2} \right)^4 = |\Lambda| \frac{1}{|\Lambda|^2} = |\Lambda|^{-1}
\]
which should be tiny if $|\Lambda|$ is large.

4. Localization of transport $[AG98]$: The diagonal elements of the zero-temperature DC conductivity matrix vanish
\[
\sigma_{ii} \left( E \right) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i^2 \mathbb{E} \left[ |G \left( x, 0; E + i\varepsilon \right) |^2 \right] = 0 \quad (i = 1, \ldots, d).
\]
5. **Localization of position**: The second moments of the position operator evolved with time around bounded. That is, there exists some $\varepsilon > 0$ such that
\[
\sup_{t > 0} \mathbb{E} \left[ \left| \langle \chi_{B_\varepsilon (E)} (H) \delta_0, e^{itH} X_i X_j e^{-itH} \chi_{B_\varepsilon (E)} (H) \delta_0 \rangle \right| \right] < \infty .
\]

6. **Dynamical localization** [AM93]: The probability to reach far away places via time evolution decays with distance, uniformly in time. I.e., there exists some $\varepsilon > 0$ such that there exist $C, \mu \in (0, \infty)$ with which
\[
\mathbb{E} \left[ \sup_{t > 0} \left| \langle \delta_x, e^{-itH} \chi_{B_\varepsilon (E)} (H) \delta_y \rangle \right| \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d) .
\]

7. **The fractional moment condition** [AM93]: There exists some $\varepsilon > 0$, $s \in (0, 1)$, $C, \mu \in (0, \infty)$ such that
\[
\sup_{\eta > 0, \bar{E} \in B_\varepsilon (E)} \mathbb{E} \left[ \left| G (x, y; \tilde{E} + i\eta) \right|^s \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d) .
\]

8. **The second moment condition** [Gra94]: There exists some $\varepsilon > 0$ and $C, \mu \in (0, \infty)$ such that
\[
\sup_{\eta > 0, \bar{E} \in B_\varepsilon (E)} \mathbb{E} \left[ \left| f (x, y; \tilde{E} + i\eta) \right|^2 \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d) .
\]

9. **The many-body ground state exhibits decay of correlations** [AG98]: As we have seen above, we should associate
\[
P \equiv \chi_{(-\infty, E)} (H)
\]
with the many-body ground state reduced one-particle density matrix of the system filled to Fermi energy $E$. Then we expect decay of correlations in the many-body ground state to manifest itself as follows: there exists some $C, \mu \in (0, \infty)$ such that
\[
\mathbb{E} \left[ \left\| \langle \delta_x, P \delta_y \rangle \right\| \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d) .
\]

10. **The bounded measurable functional calculus exhibits exponential decay** [AG98]: More generally and abstractly, there exists some $\varepsilon > 0$ such that if $B_1 (B_\varepsilon (E))$ is the space of measurable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ which obey $\|f\|_\infty \leq 1$ as well as being constant above and below $B_\varepsilon (E)$ (with possibly different constants) then there exist constants $C, \mu \in (0, \infty)$ such that
\[
\mathbb{E} \left[ \sup_{f \in B_1 (B_\varepsilon (E))} \left\| \langle \delta_x, f (H) \delta_y \rangle \right\| \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d) .
\]

11. **Poisson statistics** [Min96]: Coming from the direction of random matrices, there appears to be a dichotomy in the stochastic process of gaps between gaps of eigenvalues as follows. Let $H_N$ be a matrix resulting from $H$ restricted to a finite box with $N$ sites (eventually $N \rightarrow \infty$). Then we consider the random measure
\[
\eta_N (B) := \mathrm{tr} \left( \chi_B (N (H_N - E \mathbf{1})) \right) \quad (B \subseteq \mathbb{R}) . \tag{2.3}
\]
Then $\eta_N$ converges, as $N \rightarrow \infty$, to a Poisson point process on $\mathbb{R}$ with intensity equal to the local density of states at $E$ times the Lebesgue measure.

### 2.3.2 Criteria for delocalization

Unfortunately for delocalization we have way less conditions. We merely state

Let $\omega \mapsto H_\omega$ be an ergodic random operator on $\ell^2 (\mathbb{Z}^d)$ and $E \in \mathbb{R}$ be an energy value. We have

1. **Spectral delocalization**: There exists some $\varepsilon > 0$ such that
\[
B_\varepsilon (E) \cap \sigma (H) = B_\varepsilon (E) \cap \sigma_{ac} (H)
\]
almost-surely (or an analogous statement about the almost sure spectrum).
2. Low inverse participation ratio: In finite boxes \( \Lambda \subseteq \mathbb{Z}^d \), if \( H \psi = E \psi \) with \( \| \psi \| = 1 \), then

\[
\text{IPR} (\psi) \approx \frac{1}{|\Lambda|}
\]

where the inverse participation ratio was defined in (2.2).

3. Delocalization of transport: The diagonal elements of the zero-temperature DC conductivity matrix are finite and non-zero:

\[
\sigma_{ii} (E) = \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i^2 E \left[ |G(x, 0; E + i\varepsilon)|^2 \right] > 0 \quad (i = 1, \ldots, d).
\]

4. Delocalization of position: The second moments of the position operator evolved with time around unbounded. That is, there exists some \( \varepsilon > 0 \) such that

\[
\sup_{t > 0} E \left[ |\langle \chi_{B_{\varepsilon}(E)} (H) \delta_0, e^{itH} X_i X_j e^{-itH} \chi_{B_{\varepsilon}(E)} (H) \delta_0 \rangle|\right] = \infty.
\]

We could also boost this to diffusion if we ask that the quantity behaves linearly in \( t \).

5. The many-body ground state exhibits no decay of correlations:

\[
\sum_{x \in \mathbb{Z}^d} E \left[ |\langle \delta_0, P \delta_x \rangle|\right] = \infty.
\]

6. GUE statistics: The measure defined above as (2.3) converges, as \( N \to \infty \), to the GUE statistics point process: The joint probability density of the eigenvalues \( \{ E_j \} \) is given by

\[
\frac{1}{Z} \exp \left( -\frac{1}{2} \sum_{j=1}^N E_j^2 \right) \prod_{i < j} |E_i - E_j|^2
\]

where \( Z \) is a normalization constant. In particular eigenvalues repel as the density is zero for \( E_i = E_j \).

### 2.3.3 Established mathematical facts

Consider the Anderson model

\[
H_\omega = -\Delta + \lambda V_\omega (X)
\]

on \( \ell^2 (\mathbb{Z}^d) \) and \( \lambda > 0 \) the coupling strength. Here \( \{ \omega_x \}_{x \in \mathbb{Z}^d} \) is an IID sequence of real random variables. Then

1. Complete localization in 1D: For \( d = 1 \), for any \( \lambda > 0 \), the system is localized at all energies (see [KLS90] and references therein).

2. Complete localization at high \( \lambda \): For \( d \in \mathbb{N}_{\geq 1} \), there exists some \( \lambda_c > 0 \) such that if \( \lambda \geq \lambda_c \) then the system is localized at all energies (see [FS83, AM93]).

3. Localization at arbitrary \( \lambda \) for extreme energies: For \( d \in \mathbb{N}_{\geq 1} \), given \( \lambda > 0 \), there exists some non-empty subset \( S_\lambda \subseteq \sigma (H) \) such that the system is localized for all energies within \( S_\lambda \). The set \( S_\lambda \) will typically lie near the boundaries of the spectrum (see [FS83, AM93]).

### 2.3.4 Conjectures

For the same Anderson model, one conjectures that

1. Complete localization in 2D: For \( d = 2 \), for any \( \lambda > 0 \), the system is localized at all energies.

2. Delocalization for 3D and higher: For \( d \geq 3 \), there exists some \( \lambda_c > 0 \) such that if \( \lambda \leq \lambda_c \), there exists some energies where the system is delocalized. These energies will typically be in the middle of the spectrum.

Establishing either one of these statements would mean a huge breakthrough in mathematical physics.
2.4 The a-priori bound

A basic tool in the approach to Anderson localization we will consider is the a-priori bound, developed by Aizenman and Molchanov [AM93]. It is built on the following basic observation: the average of the Greens function may not exist, because it is a singularity that behaves like \( \frac{1}{x} \) at \( x = 0 \), which is not integrable. However, as it turns out, a fractional moment of the Greens function is just as good at controlling many dynamical properties, and that object is integrable at the origin. Indeed,

\[
\int_{x=-1}^{1} \frac{1}{|x|} \, dx = \frac{2}{1 - s} \quad (s < 1).
\]

To begin the analysis, we state the basic tool from linear algebra, the Schur complement.

**Lemma 2.11.** Let \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) be a \( \mathbb{Z}_2 \)-grading of a Hilbert space and let

\[
L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

be a block operator on \( \mathcal{H} \). Then if \( D \) is invertible and the Schur operator

\[
S := A - BD^{-1}C : \mathcal{H}_1 \to \mathcal{H}_1
\]

is invertible, we have

\[
L^{-1} = \begin{bmatrix} S^{-1} & -S^{-1}BD^{-1} \\ -D^{-1}CS^{-1} & D^{-1} + D^{-1}CS^{-1}BD^{-1} \end{bmatrix}.
\]

**Proof.** Note the identity

\[
\begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} S - BD^{-1}C & B \\ C & D \end{bmatrix}.
\]

But the matrices \( \begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \), \( \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix} \) are both invertible regardless of \( B, D, C \):

\[
\begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbb{1}_1 & -BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix}, \quad \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbb{1}_1 & 0 \\ -D^{-1}C & \mathbb{1}_2 \end{bmatrix}.
\]

Hence we may take the inverse of the previous identity to get

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \left( \begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} \mathbb{1}_1 & 0 \\ -D^{-1}C & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} = \begin{bmatrix} S^{-1} & 0 \\ -D^{-1}CS^{-1} & D^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} = \begin{bmatrix} S^{-1} & -S^{-1}BD^{-1} \\ -D^{-1}CS^{-1} & D^{-1}CS^{-1}BD^{-1} + D^{-1} \end{bmatrix}.
\]

In analyzing the properties of

\[
G(x, y; z) \equiv (H - z\mathbb{1})^{-1}_{xy}
\]
it turns out it is useful to isolate the dependence of $G(x, y; z)$ on just $\omega_x$ and $\omega_y$, and integrate on them explicitly, before performing all other integrations. This leads to the following theorem, taken from [Gra94] but also appears in [AM93]:

**Theorem 2.12** (Aizenman-Molchanov, Graf). There exists some $s \in (0, 1)$ such that

$$\sup_{x,y \in \mathbb{Z}^d, z \in \mathbb{C}} \mathbb{E} |G(x, y; z)|^s < \infty.$$  

As a warm up, let us first study the case $x = y$. In this case, we define

$$\mathcal{H}_1 := \text{span} \{\delta_x\}$$

and $\mathcal{H}_2 := \mathcal{H}_1^\perp$. Clearly the diagonal part of $H$ restricted to $\mathcal{H}_1$ is precisely

$$\lambda \omega_x$$

whereas the off-diagonal part includes all hopping terms from the Laplacian that may lead into or out of $x$: We write this generally as

$$H = \begin{bmatrix} \lambda \omega_x & P_x (\Delta) P_x^\perp \\ P_x^\perp (\Delta) P_x & H \end{bmatrix}$$

where $P_x := \delta_x \otimes \delta_x^*$ and $\tilde{H} \equiv P_x^\perp H P_x^\perp \in \mathcal{B}(\ell^2(\mathbb{Z}^d \setminus \{x\}))$ is a random operator that, by definition, does not depend on the variable $\omega_x$. Then by Lemma 2.11

$$(H - z\mathbb{1})_{xx}^{-1} = \frac{1}{\lambda \omega_x - z - P_x (\Delta) P_x^\perp \tilde{H} - z \mathbb{1} \mathbb{I}_{\mathcal{H}_2}} P_x^\perp (\Delta) P_x$$

This hinges on verifying the invertibility of the Schur operator as well as $\tilde{H} - z\mathbb{1}$. Let us check those: $\tilde{H} \equiv P_x^\perp H P_x^\perp$ is a self-adjoint operator so if $\text{Im} \{z\} > 0$ it is automatically invertible. As for the Schur operator, since $\text{Im} \{z\} > 0$ and

$$z \mapsto \langle \varphi, (H - z\mathbb{1})^{-1} \psi \rangle$$

is a Herglotz function (see [Sha24]) then necessarily it has a positive imaginary part. This implies that

$$\text{Im} \left\{ \lambda \omega_x - z - P_x (\Delta) P_x^\perp \tilde{H} - z \mathbb{1} \mathbb{I}_{\mathcal{H}_2} \right\}^{-1} P_x^\perp (\Delta) P_x > 0$$

and is hence invertible. Hence we are justified in employing the Schur complement.

The particular form of the operator

$$P_x (\Delta) P_x^\perp \tilde{H} - z \mathbb{1} \mathbb{I}_{\mathcal{H}_2}$$

is unimportant except that it is independent of $\omega_x$, by construction. Just for fun let us study it anyway. We begin with the discrete Laplacian:

$$-\Delta \delta_z = 2d \delta_z - \sum_{y \sim z} \delta_y$$

then

$$\langle \delta_w, -\Delta \delta_z \rangle = 2d \delta_{z,w} - \sum_{y \sim z} \delta_{wy}$$

and so

$$P_x^\perp (\Delta) P_x = P_x^\perp (\Delta) \delta_z \otimes \delta_z^*$$

$$= P_x^\perp \left(2d \delta_x - \sum_{y \sim x} \delta_y \right) \otimes \delta_x^*$$

$$= - \sum_{y \sim x} P_x^\perp \delta_y \otimes \delta_x^*$$

$$= - \sum_{y \sim x} \delta_y \otimes \delta_x^*$$
and similarly

\[
P_x (-\Delta) P_x^\perp = \left( - \sum_{y \sim x} \delta_y \otimes \delta_y^* \right)^* = - \sum_{y \sim x} \delta_x \otimes \delta_y^*.
\]

Hence

\[
P_x (-\Delta) P_x^\perp \left( \tilde{H} - z 1_{\mathbb{R}^2} \right)^{-1} P_x^\perp (-\Delta) P_x = \sum_{y \sim x} \sum_{z \sim x} \delta_x \otimes \delta_y^* \left( \tilde{H} - z 1_{\mathbb{R}^2} \right)^{-1} \delta_z \otimes \delta_x^* = P_x \sum_{y \sim x} \sum_{z \sim x} \left( \tilde{H} - z 1_{\mathbb{R}^2} \right)_y^z.
\]

We thus find

\[
(H - z 1)_{xx}^{-1} = \frac{1}{\lambda\omega_x - z - \sum_{y \sim x} \sum_{z \sim x} (P_x^\perp H P_x^\perp - z 1_{\mathbb{R}^2})_y^z}.
\]

We emphasize again, we will not make use of any information about

\[
\sum_{y \sim x} \sum_{z \sim x} (P_x^\perp H P_x^\perp - z 1_{\mathbb{R}^2})_y^z
\]

except that it is independent of \(\omega_x\).

**Theorem 2.13** (a-priori bound, diagonal version (Aizenman-Molchanov)). For any \(s < \tau\), we have

\[
\sup_{x \in \mathbb{Z}^d} \sup_{z \in \mathbb{C} : \operatorname{Im}(z) > 0} \mathbb{E} \left[ |G(x, x; z)|^s \right] < \frac{\tau}{\tau - s} C_\mu^s \left( \frac{2}{\lambda} \right)^s
\]

where \(C_\mu < \infty\) is the constant of regularity of the single-site probability measure \(\mu\) which is assumed to be \(\tau\)-Hoelder regular as in Definition 2.8.

**Proof.** Thanks to (2.4) we find that

\[
G(x, x; z) = \frac{1}{\lambda\omega_x - w}
\]

for some \(w \in \mathbb{C}\) which is independent of \(\omega_x\). Hence in taking the expectation \(\mathbb{E}\) which is essentially an integral over all variables \(\{\omega_z\}_{z \in \mathbb{Z}^d}\), we may first integrate over \(\omega_x\) before all other variables. Hence we must bound

\[
\sup_{w \in \mathbb{C}} \int_{\omega_x \in \mathbb{R}} \frac{1}{|\lambda\omega_x - w|^s} d\mu(\omega_x)
\]

where \(\mu\) obeys some \(\tau\)-Hoelder regularity as in Definition 2.8. To that end, let us estimate

\[
\int_{\omega_x \in \mathbb{R}} \frac{1}{|\lambda\omega_x - w|^s} d\mu(\omega_x) \leq D + \int_{\omega_x \in \mathbb{R} : |\lambda\omega_x - w|^s \geq b} \frac{1}{|\lambda\omega_x - w|^s} d\mu(\omega_x)
\]

which holds for any \(D > 0\). The reason for separating into above and below \(D\) is in order to regularize the second term, as will become apparent momentarily. The second term then may be rewritten using the so-called layer-cake representation

\[
\int_{x : f(x) \geq t} f(x) d\mu(x) = \int_{t' = t}^\infty \mu(\{x \in \mathbb{R} : f(x) \geq t'\}) dt'.
\]
Indeed, the two expressions are equal by re-writing
\[
\int_{t=0}^{\infty} \chi_{[0,f(x)]}(t) \, dt = \int_{t=0}^{\infty} \chi_{\{ y \in \mathbb{R} \mid f(y) > t \}}(x) \, dt
\]
and so
\[
\int_{x : f(x) \geq t} f(x) \, d\mu(x) = \int_{x : f(x) \geq t} \int_{t'=0}^{\infty} \chi_{\{ y \in \mathbb{R} \mid f(y) > t' \}}(x) \, dt' \, d\mu(x)
\]
\[
= \int_{x} \int_{t'=t}^{\infty} \chi_{\{ y \in \mathbb{R} \mid f(y) > t' \}}(x) \, dt' \, d\mu(x)
\]
\[
= \int_{t'=t}^{\infty} \mu(\{ y \in \mathbb{R} \mid f(y) > t' \}) \, dt'.
\]

But now,
\[
|\lambda \omega - w|^{-s} > t \iff |\lambda \omega - w|^{-1} > t^{\frac{1}{s}}
\]
\[
\iff |\lambda \omega - w| < t^{-s}
\]
\[
\iff |\omega - \frac{1}{\lambda} w| < \frac{1}{\lambda} t^{-s}
\]
\[
\iff \omega \in B_{\frac{1}{\lambda} t^{-s}} \left( \frac{1}{\lambda} \Re \{w\} \right).
\]

The \(\tau\)-Hoelder regularity then implies
\[
\mu(\{ |\lambda \omega - w|^{-s} > t \}) \leq C \left( 2^{1-\frac{s}{s}} \right)^{\tau}
\]
so that
\[
\int_{\omega \in \mathbb{R} : \frac{1}{|\lambda \omega - w|^{s}} \geq D} \frac{1}{|\lambda \omega - w|^{s}} \, d\mu(\omega) = \int_{t'=D}^{\infty} \mu(\{ \frac{1}{|\lambda \omega - w|^{s}} > t' \}) \, dt'
\]
\[
\leq \int_{t'=D}^{\infty} C \left( \frac{2^{1/\lambda} t'^{-\frac{1}{s}}}{} \right)^{\tau} \, dt'
\]
\[
= C \left( \frac{2}{\lambda} \right)^{\tau} \int_{t'=D}^{\infty} t'^{-\frac{1}{s}} \, dt'
\]
\[
= C \left( \frac{2}{\lambda} \right)^{\tau} D^{1-\frac{s}{s}} s \quad \text{(if } s < \tau\).
\]

Together we find
\[
\int_{\omega \in \mathbb{R}} \frac{1}{|\lambda \omega - w|^{s}} \, d\mu(\omega) \leq D + \frac{C}{\frac{2^{1/\lambda}}{\lambda} - 1} \left( \frac{2}{\lambda} \right)^{\tau} D^{1-\frac{s}{s}}.
\]

Note that \(s < \tau\) so \(-\alpha := 1 - \frac{s}{s} < 0\) and hence, even though \(D > 0\) was arbitrary, we actually have
\[
\inf_{D > 0} \left( D + \frac{C}{\alpha - 1} D^{-\alpha} \right) = \left( 1 + \frac{1}{\alpha} \right) C^{1+\alpha}.
\]

Hence
\[
\int_{\omega \in \mathbb{R}} \frac{1}{|\lambda \omega - w|^{s}} \, d\mu(\omega) \leq \frac{\tau}{\tau - s} C_{\mu} \left( \frac{2}{\lambda} \right)^{s}.
\]  
(2.5)
Since this upper bound is independent of \( w \), integrating over all other variables and taking \( \sup_{z \in \mathbb{C}} \) does not change the bound. We also see that it is the regularity of \( \mu \) which dictates the allowed values of \( s \): any \( s \in (0, \tau) \) would do. 

**Remark 2.14.** Of course once we now that \( \mathbb{E}[X^s] < \infty \) for some \( s \in (0, 1) \) then the same holds for all \( s' \in (0, s) \). Indeed this is merely a consequence of the Hölder inequality: Let \( p := \frac{s}{s'} > 1 \) and \( q \) so that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then

\[
\mathbb{E}[X^{s'}] = \mathbb{E}[X^s]^{\frac{1}{p}} \mathbb{E}[|Y|^\frac{1}{q}].
\]

**Proof of Theorem 2.12.** We now turn to the proof that

\[
\sup_{\varepsilon > 0} \sup_{x, y \in \mathbb{Z}^d} \mathbb{E}[|G(x, y; E + i\varepsilon)|^s] < \infty \quad (E \in \mathbb{R}).
\]

We shall follow [Gra94]. We begin by extracting the dependence of \( G(x, y; z) \) on both \( \omega_x \) and \( \omega_y \). Assuming that \( x \neq y \), we need to study the rank-2 perturbation theory. Let us rewrite

\[
\mathcal{H} = \mathcal{H}_{xy} \oplus \mathcal{H}_{\perp xy}
\]

where

\[
\mathcal{H}_{xy} := \text{im} (P_{xy})
\]

with \( P_x \equiv \delta_x \otimes \delta_y^* \) and \( P_{xy} \equiv P_x + P_y \). With this notation, we may write

\[
H = \begin{bmatrix}
\lambda \omega_x P_x + \lambda \omega_y P_y & P_{xy} (-\Delta) P_{\perp xy} \\
P_{\perp xy} (-\Delta) P_{xy} & P_{\perp xy} H P_{\perp xy}
\end{bmatrix}.
\]

If \( \text{Im} z > 0 \) then again thanks to the Herglotz property, \( P_{\perp xy} (H - z \mathbb{1}) P_{\perp xy} \) will be invertible and so will the Schur operator

\[
\lambda \omega_x P_x + \lambda \omega_y P_y - z P_{xy} - P_{xy} (-\Delta) (P_{\perp xy} (H - z \mathbb{1}) P_{\perp xy})^{-1} (-\Delta) P_{xy}.
\]

Hence we find thanks to the Schur complement **Lemma 2.11** that

\[
P_{xy} (H - z \mathbb{1})^{-1} P_{xy} = \frac{1}{\lambda} \begin{bmatrix}
\omega_x & 0 \\
0 & \omega_y
\end{bmatrix} + M \right)^{-1}
\]

for some \( 2 \times 2 \) matrix \( \lambda M = -z P_{xy} - P_{xy} (-\Delta) (P_{\perp xy} (H - z \mathbb{1}) P_{\perp xy})^{-1} (-\Delta) P_{xy} \) with the following properties:

1. It has a positive imaginary part \( \text{Im} \{ M \} > 0 \) thanks to the Herglotz property.
2. It does depend on \( z \in \mathbb{C} \) and on \( \omega_{\hat{x}} \) for all \( \hat{x} \neq x, y \).

Hence if we manage to come up with an upper bound by integrating over only \( \omega_x, \omega_y \) and uniformly in \( M, z \) we’d be finished. We have

\[
M := \begin{bmatrix}
m_{xx} & m_{xy} \\
m_{yx} & m_{yy}
\end{bmatrix}
\]

and

\[
\text{Im} \{ M \} = \frac{1}{2i} \left( (M - M^*) \right) = \frac{1}{2i} \left( \begin{bmatrix}
m_{xx} & m_{xy} \\
m_{yx} & m_{yy}
\end{bmatrix} - \begin{bmatrix}
m_{xx} & m_{yx} \\
m_{xy} & m_{yy}
\end{bmatrix} \right) = \begin{bmatrix}
\text{Im} \{ m_{xx} \} & \frac{1}{2i} (m_{xy} - m_{yx}) \\
\frac{1}{2i} (m_{yx} - m_{xy}) & \text{Im} \{ m_{yy} \}
\end{bmatrix}.
\]
Hence
\[
\lambda G(x, y; z) = \left( \begin{array}{cc}
\omega_x & 0 \\
0 & \omega_y
\end{array} + \begin{bmatrix}
m_{xx} & m_{xy} \\
m_{yx} & m_{yy}
\end{bmatrix} \right)^{-1}
\] top right corner
\[
= \frac{-m_{xy}}{(m_{xx} + \omega_x) (m_{yy} + \omega_y) - m_{xy} m_{yz}}.
\]

Using the trivial \(|w| \geq |\Re \{w\}|\) or \(|w| \geq |\Im \{w\}|\) and the notation \(\tilde{\omega}_x := \omega_x + \Re \{m_{xx}\}\) and \(\tilde{\omega}_y := \omega_y + \Re \{m_y\}\) we get the two possible estimates
\[
\lambda |G(x, y; z)| \leq \frac{|m_{xy}|}{|\tilde{\omega}_x \Im \{m_{yy}\} + \tilde{\omega}_y \Im \{m_{xx}\} - \Im \{m_{xy} m_{yz}\}|}.
\]
as well as
\[
\lambda |G(x, y; z)| \leq \frac{|m_{xy}|}{|\tilde{\omega}_x \Re \{m_{yy}\} \Im \{m_{xx}\} - \Im \{m_{xy} m_{yz}\}|}.
\]

Moreover, since \(\Im \{M\} > 0\), we have
\[
0 < \det(\Im \{M\}) \leq \Im \{m_{xx}\} \Im \{m_{yy}\} + \frac{1}{4} (m_{xy} - m_{yz}) (m_{yx} - m_{xy})
\]
\[
= \Im \{m_{xx}\} \Im \{m_{yy}\} + \frac{1}{4} \left(m_{xy} m_{yx} + m_{xy} m_{yx} - |m_{xy}|^2 - |m_{yx}|^2\right)
\]
\[
= \Im \{m_{xx}\} \Im \{m_{yy}\} + \frac{1}{2} \Re \{m_{xy} m_{yx}\} - \frac{1}{4} \left(|m_{xy}|^2 + |m_{yx}|^2\right)
\]

Case 1: Assume that
\[
\max \{|\Im \{m_{xx}\}|, |\Im \{m_{yy}\}|\} < \frac{1}{2} |m_{xy}|.
\]
Then
\[
c^2 := \frac{\Im \{m_{xx}\} \Im \{m_{yy}\} + \Re \{m_{xy} m_{yx}\}}{\det(\Im \{M\}) > 0}
\]
\[
y \geq \frac{1}{2} \left(|m_{xy}|^2 + |m_{yx}|^2\right) - \Im \{m_{xx}\} \Im \{m_{yy}\}
\]
\[
y \geq \frac{1}{2} |m_{xy}|^2.
\]

Hence
\[
\lambda |G(x, y; z)| \leq \frac{2c}{|\tilde{\omega}_x \tilde{\omega}_y - c^2|} = \frac{2c}{|c^2 \tilde{\omega}_x \tilde{\omega}_y - 1|}.
\]

Next, define \(f(w) := \frac{1}{c} \min \{1, w^2\}\). Then we claim
\[
|ab - 1| \geq \min \{|a - f(b)|, |b - f(a)|\} \quad (a, b \in \mathbb{R}).
\]

Indeed, if \(a^2 \geq 1\) then
\[
|b - f(a)| = \left|b - \frac{1}{a}\right| \leq |ab - 1|.
\]
Similarly if \(b^2 \geq 1\). If, however, both \(a^2, b^2 < 1\) then
\[
(a - f(b))^2 = (a - b)^2.
\]
and 
\[(b - f(a)) = (a - b)^2\]
which equals 
\[(a - b)^2 = (ab - 1)^2 - (1 - a^2)(1 - b^2) < (ab - 1)^2\].

We conclude that Section 2.4 holds. We use it as 
\[|ab - 1|^{-1} \leq \frac{1}{\min(|a - f(b)|, |b - f(a)|)}\]
\[= \max\left\{\left|\frac{a - f(b)}{b - f(a)}\right|^{-1}, |b - f(a)|^{-1}\right\}\]
\[\leq \left|a - f(b)\right|^{-1} + |b - f(a)|^{-1} .\]

We conclude that
\[\lambda |G(x, y; z)| \leq \frac{2c^{-1}}{|c^{-2}\tilde{\omega}_x \tilde{\omega}_y - 1|}\]
\[\leq 2c^{-1}\left(c^{-1}\tilde{\omega}_x - f(c^{-1}\tilde{\omega}_y)\right)^{-1} + |c^{-1}\tilde{\omega}_y - f(c^{-1}\tilde{\omega}_x)|^{-1}\]
\[= 2\left(|\tilde{\omega}_x - cf(c^{-1}\tilde{\omega}_y)|^{-1} + |\tilde{\omega}_y - cf(c^{-1}\tilde{\omega}_x)|^{-1}\right) .\]

But we have just seen above in (2.5) that
\[
\int |\omega_x - z|^{-s} d\mu(\omega_x) < \frac{\tau}{\tau - s} C^{\frac{\tau}{\tau - s}} .
\]

Case 2: Conversely, if (2.7) we must have
\[|\text{Im} \{m_{\alpha}\}| \geq \frac{1}{2}|m_{xy}| \quad (\alpha = x \vee \alpha = y) .
\]

Assume that \(\alpha = y\). Then using Section 2.4 we get
\[\lambda |G(x, y; z)| \leq \frac{|m_{xy}|}{\tilde{\omega}_x \text{Im} \{m_{yy}\} + \tilde{\omega}_y \text{Im} \{m_{xx}\} - \text{Im} \{m_{xy}m_{yx}\}}\]
\[\leq \frac{1}{2} \frac{\text{Im} \{m_{yy}\}}{\text{Im} \{m_{yy}\} + \text{Im} \{m_{xx}\} - \text{Im} \{m_{xy}m_{yx}\}}\]
\[= \frac{1}{\text{Im} \{m_{yy}\}} \text{Im} \{m_{xx}\} - \text{Im} \{m_{xy}m_{yx}\}}\]
and again we know how to estimate the \(s\) moment of this.

### 2.5 Sub-harmonicity in space

The next ingredient we will need is a basic statement about integral kernels of operators, called sub-harmonicity. The basic statement is essentially that if a kernel decays faster than the massive Laplacian would then it exhibits exponential decay (because the massive Laplacian does).
Lemma 2.15 (Subharmonicity implies exponential decay). Assume that an integral kernel $B : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ obeys the sub-harmonicity bound

$$B_{xy} \leq \gamma \sum_{u \sim x} B_{u,y} \quad (x, y \in \mathbb{Z}^d) \quad (2.9)$$

for some $\gamma < \frac{1}{2d}$; $u \sim x$ means $u, x$ share an edge on $\mathbb{Z}^d$. Then

$$B_{xy} \leq \frac{2}{m} e^{-\frac{1}{2}m\|x-y\|} \quad (x, y \in \mathbb{Z}^d)$$

with $m := \frac{1}{\gamma} - 2d$.

Proof. With the Laplacian $-\Delta$ defined so that

$$\sigma (-\Delta) = [0, 4d]$$

we have

$$-\Delta = 2d1 - A$$

where $A$ is the adjacency matrix with $\sigma (A) = [-2d, 2d]$ and

$$(A\psi)_x = \sum_{y \sim x} \psi_y \quad (x \in \mathbb{Z}^d).$$

With this notation,

$$\sum_{u \sim x} B_{u,y} \equiv (AB)_{xy}$$

and so we find that (2.9) is equivalent to

$$B_{xy} \leq \gamma (AB)_{xy}$$

$$\Delta \quad \uparrow$$

$$[ (1 - \gamma A) B ]_{xy} \leq 0 \quad \downarrow$$

$$\left[ \left( \frac{1}{\gamma} - A \right) B \right]_{xy} \leq 0 \quad \downarrow$$

$$\left[ \left( \frac{1}{\gamma} - A \right) B \right]_{xy} \leq \delta_{xy}. \downarrow$$

But we may re-write the operator as

$$\frac{1}{\gamma} - A = \left( 2d + \frac{1}{\gamma} - 2d \right) 1 - A$$

$$= -\Delta + \left( \frac{1}{\gamma} - 2d \right) 1.$$}

The condition $\gamma < \frac{1}{2d}$ implies that the mass term $m := \frac{1}{\gamma} - 2d$ is positive, so we find by Theorem 1.18 that

$$(-\Delta + m1)^{-1} \leq \frac{2}{m} \exp \left( -\tilde{\mu}m\|x-y\| \right) \quad (x, y \in \mathbb{Z}^d).$$
Since the integral kernel obeys \((-\Delta + m\mathbb{1})^{-1}_{xy} \geq 0\), we learn that

\[
B_{xy} \leq \frac{2}{m} \exp\left(-\tilde{\mu} m \|x - y\|\right) \quad (x, y \in \mathbb{Z}^d).
\]

To see that \((-\Delta + m\mathbb{1})^{-1}_{xy} \geq 0\), note that \(-\Delta \geq 0\) so \(-\Delta + m\mathbb{1} > m\mathbb{1}\) and hence \((-\Delta + m\mathbb{1})^{-1} \geq \frac{1}{4d+m}\mathbb{1}\). Indeed, the integral kernel of the Laplacian is positive for all positions. To see this, we write

\[
(-\Delta + m\mathbb{1})^{-1}_{xy} = \int_0^\infty e^{-t(-\Delta + m)} \, dt
\]

\[
= \int_0^\infty e^{tm} \frac{1}{(2\pi)^\frac{d}{2}} \int_{k \in \mathbb{T}^d} e^{ilk \cdot (x-y) - tk(k)} \, dk \, dt
\]

where \(k \equiv 2d - \sum_{j=1}^d 2 \cos(k_j)\). Hence

\[
(-\Delta + m\mathbb{1})^{-1}_{x0} = \int_0^\infty e^{tm} e^{-2dt} \prod_{j=1}^d \left( \frac{1}{\sqrt{2\pi}} \int_{k_j = 0}^{2\pi} e^{ik_j x_j - 2t \cos(k_j)} \, dk_j \right) \, dt
\]

\[
= \int_0^\infty e^{tm} e^{-2dt} \prod_{j=1}^d \left( \sqrt{2\pi} I_{x_j}(2t) \right) \, dt
\]

where \(I_{x_j}(2t)\) is the modified Bessel function of order \(x_j\), which is known to be positive, for instance using the representation

\[
I_{a}(2t) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + a + 1)} t^{2m+a}.
\]

Another related result is

**Lemma 2.16.** Assume that \(f : \mathbb{Z}^d \to [0, \infty)\) with \(\|f\|_\infty < \infty\) obeys

\[
f(x) \leq g(x) + \sum_{y \in \mathbb{Z}^d} K(x, y) f(y) \quad (x \in \mathbb{Z}^d)
\]

for some kernel \(K : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)\) which obeys

\[
\sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} K(x, y) < 1
\]

as well as

\[
r := \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \frac{W(x)}{W(y)} K(x, y) < 1
\]

and

\[
b := \sum_{x \in \mathbb{Z}^d} W(x) g(x) < \infty
\]

for some \(W : \mathbb{Z}^d \to [0, \infty)\).

Then

\[
\sum_{x \in \mathbb{Z}^d} W(x) f(x) \leq \frac{b}{1-r}.
\]
Proof. We apply the estimate (2.10) repeatedly many times to get
\[ f \leq g + Kf \]
\[ \leq g + K(g + Kf) \]
\[ \leq \ldots \]
\[ \leq \sum_{j=0}^{n-1} K^j g + K^n f \quad (n \in \mathbb{N}_1). \]

Now,
\[ |K^n f(x)| = \left| \sum_{x_1, \ldots, x_n} K(x, x_1) K(x_1, x_2) \cdots K(x_{n-1}, x_n) f(x_n) \right| \]
\[ \leq \sum_{x_1, \ldots, x_n} |K(x, x_1)| |K(x_1, x_2)| \cdots |K(x_{n-1}, x_n)| |f(x_n)| \]
\[ \leq \|f\|_{\infty} \left( \sup_{x, y} |K(x, y)| \right)^n \]
\[ \to 0 \quad (n \to \infty). \]

Hence
\[ f(x) \leq \sum_{j=0}^{\infty} (K^j g)(x). \]

Thus
\[ \sum_x W(x) f(x) \leq \sum_x W(x) \sum_{j=0}^{\infty} (K^j g)(x) \]
\[ = \sum_{j=0}^{\infty} \sum_x W(x) \sum_{x_1, \ldots, x_j} K(x, x_1) \cdots K(x_{j-1}, x_j) g(x_j) \]
\[ = \sum_{j=0}^{\infty} \sum_x \sum_{x_1, \ldots, x_j} \frac{W(x)}{W(x_1)} K(x, x_1) \cdots \frac{W(x_{j-1})}{W(x_j)} K(x_{j-1}, x_j) W(x_j) g(x_j) \]
\[ \leq \sum_{j=0}^{\infty} r^j b \]
\[ = \frac{b}{1-r}. \]

We learn in particular that for any \( x \in \mathbb{Z}^d \)
\[ W(x) f(x) \leq \sum_{\tilde{x}} W(\tilde{x}) f(\tilde{x}) \leq \frac{b}{1-r} \]
i.e.,
\[ f(x) \leq \frac{b}{1-r} \frac{1}{W(x)} \quad (x \in \mathbb{Z}^d). \]

### 2.6 The decoupling lemma

In the sequel we will use the so-called decoupling lemma which goes as follows:
Lemma 2.17 (Decoupling lemma). Let $s \in (0, \tau)$ and $\alpha, \beta \in \mathbb{C}$. Then
\[
\int_{\omega \in \mathbb{R}} \frac{\omega - \alpha}{|\omega - \beta|^s} \, d\mu(\omega) \geq \frac{\tau}{\tau - s} C^{5/2} 2^{s+1} \int_{\omega \in \mathbb{R}} \frac{1}{|\omega - \beta|^s} \, d\mu(\omega)
\]
where $C$ is a constant independent of $\alpha, \beta$ coming from the a-prior bound.

Proof. We first claim that
\[
|v - \beta|^{-s} + |u - \beta|^{-s} \leq \frac{|v|^s}{|v - \beta|^s} \left(|u|^{-s} + |u - \beta|^{-s}\right) + \frac{|u|^s}{|u - \beta|^s} \left(|v|^{-s} + |v - \beta|^{-s}\right)
\]
for all $u, v, \beta \in \mathbb{C}$ unless the denominators vanish. To see this, multiply by $|v - \beta|^s |u - \beta|^s$ and re-arrange to get the equivalent claim
\[
0 \leq \left(|v|^s |u|^{-s} - 1\right) |u - \beta|^s + \left(|u|^s |v|^{-s} - 1\right) |v - \beta|^s + |u|^s + |v|^s .
\]
Since this expression is symmetric in $u \leftrightarrow v$, suffice to show it for $|u - \beta| \geq |v - \beta|$. By the triangle inequality, we have
\[
|u - \beta|^s = (|v + u - \beta|)^s \leq (|v - \beta| + |u + \beta|)^s \leq |v - \beta|^s + |u + \beta|^s
\]
or $|u|^s + |v|^s \geq |u - \beta|^s - |v - \beta|^s$. Applying this we find that our equivalent claim reduces to
\[
\left(|v|^s |u|^{-s} - 1\right) |u - \beta|^s + \left(|u|^s |v|^{-s} - 1\right) |v - \beta|^s + |u|^s + |v|^s \\
\geq \left(|v|^s |u|^{-s} - 1\right) |u - \beta|^s + \left(|u|^s |v|^{-s} - 1\right) |v - \beta|^s + |u - \beta|^s - |v - \beta|^s \\
= |v|^s |u|^{-s} |u - \beta|^s + \left(|u|^s |v|^{-s} - 2\right) |v - \beta|^s \\
\geq |v|^s |u|^{-s} |v - \beta|^s + \left(|u|^s |v|^{-s} - 2\right) |v - \beta|^s \\
= \left(|v|^s |u|^{-s} + |u|^s |v|^{-s} - 2\right) |v - \beta|^s \\
\geq 0
\]
where in the last line we have used $t + \frac{1}{t} \geq 2$ for all $t > 0$. Hence (2.11) is proven. We use it by replacing $v$ with $v - \alpha$, $u$ with $u - \alpha$ and finally replace $\beta$ with $\beta - \alpha$ to get
\[
|v - \beta|^{-s} + |u - \beta|^{-s} \leq \frac{|v - \alpha|^s}{|v - \beta|^s} \left(|u - \alpha|^{-s} + |u - \beta|^{-s}\right) + \frac{|u - \alpha|^s}{|u - \beta|^s} \left(|v - \alpha|^{-s} + |v - \beta|^{-s}\right).
\]
We then integrate on both $u, v$ with respect to $\mu$ (using $\mu(\mathbb{R}) = 1$) (after renaming some variables and dividing by 2)
\[
\int_u |v - \beta|^{-s} \, d\mu(v) \leq \left(\int_u \frac{|v - \alpha|^s}{|v - \beta|^s} \, d\mu(v)\right) \int_u \left(|u - \alpha|^{-s} + |u - \beta|^{-s}\right) \, d\mu(u).
\]
On the latter integral on the RHS we use the same proof as in Section 2.4, in particular (2.5), to get
\[
\int_v |v - \beta|^{-s} \, d\mu(v) \leq \frac{\tau}{\tau - s} C^{5/2} 2^{s+1} \int_v \frac{|v - \alpha|^s}{|v - \beta|^s} \, d\mu(v).
\]

There is a converse type of decoupling lemma that we shall also need.
Lemma 2.18. For the regular probability measure $\mu$, assume further that it has some fractional moment in the sense that for some $s \in (0, 1)$,

$$B_s := \int_{v \in \mathbb{R}} |v|^s \, d\mu(v) < \infty.$$ 

Then there exists some $s \in (0, 1)$ and constant $D_s < \infty$ (independent of $\alpha$ below) such that

$$\int_{v \in \mathbb{R}} \frac{|v|^s}{|\lambda v - \alpha|^s} \, d\mu(v) \leq D_s \int_{v \in \mathbb{R}} \frac{1}{|\lambda v - \alpha|} \, d\mu(v) \quad (\alpha \in \mathbb{C}, \lambda > 0).$$ (2.12)

**Proof.** Using the Cauchy-Schwarz inequality, we have

$$\int_{v \in \mathbb{R}} \frac{|v|^s}{|\lambda v - \alpha|^s} \, d\mu(v) \leq \sqrt{B_{2s}} \sqrt{\int_{v \in \mathbb{R}} |\lambda v - \alpha|^{-2s} \, d\mu(v)}.$$

Now by assumption the first factor is bounded whereas the second one was shown to be bounded above in Theorem 2.13. We get an upper bound of the form

$$\int_{v \in \mathbb{R}} \frac{|v|^s}{|\lambda v - \alpha|^s} \, d\mu(v) \leq \sqrt{\frac{s}{s - 2}} C^{\frac{2s}{s}} \left(\frac{s}{s - 2}\right)^{\frac{2s}{s}}.$$

Conversely, we seek a lower bound on

$$\int_{v \in \mathbb{R}} |\lambda v - \alpha|^{-s} \, d\mu(v) \geq \int_{v:|\lambda v| \leq Q} |\lambda v - \alpha|^{-s} \, d\mu(v)$$

$$\geq \int_{v:|\lambda v| \leq Q} \frac{1}{(|\lambda v| + |\alpha|)^s} \, d\mu(v)$$

$$\geq \frac{1}{(Q + |\alpha|)^s} \int_{v:|\lambda v| \leq Q} \, d\mu(v)$$

$$= \frac{1}{(Q + |\alpha|)^s} \left( \left\{ v \in \mathbb{R} \mid |v| \leq \frac{Q}{\lambda} \right\} \right)$$

$$= \frac{1}{(Q + |\alpha|)^s} \left( 1 - \mu \left( \left\{ v \in \mathbb{R} \mid |v| \geq \frac{Q}{\lambda} \right\} \right) \right) .$$

Now, by Markov’s inequality,

$$\mu \left( \left\{ v \in \mathbb{R} \mid |v| \geq \frac{Q}{\lambda} \right\} \right) \leq \frac{B_{2s}}{\left(\frac{Q}{\lambda}\right)^{2s}}.$$

Hence if we pick $Q$ such that

$$\frac{B_{2s}}{\left(\frac{Q}{\lambda}\right)^{2s}} := \frac{1}{2}$$

$$Q = (2\lambda^{-2s} B_{2s})^{\frac{1}{s}} .$$

We find

$$\int_{v \in \mathbb{R}} |\lambda v - \alpha|^{-s} \, d\mu(v) \geq \frac{1}{2} \frac{1}{\left(2\lambda^{-2s} B_{2s}\right)^{\frac{1}{s}} + |\alpha|} .$$

If $|\alpha| \leq (2\lambda^{-2s} B_{2s})^{\frac{1}{s}}$ we get

$$(2\lambda^{-2s} B_{2s})^{\frac{1}{s}} + |\alpha| \leq 2 (2\lambda^{-2s} B_{2s})^{\frac{1}{s}}.$$
and hence
\[
\int_{v \in \mathbb{R}} |\lambda v - \alpha|^{-s} d\mu(v) \geq \frac{1}{2} \left(2 \left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2}}\right)^{-s}
\]
and we need to define \(D_s\) so that
\[
D_s \frac{1}{2} \left(2 \left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2}}\right)^{-s} \geq \sqrt{B_{2s}} \sqrt{\frac{\tau}{\tau - 2s} C_{\mu}^s \left(\frac{2}{\lambda}\right)^{2s}}.
\]
If, on the other hand, \(|\alpha| \geq (2\lambda^{-2s}B_{2s})^{\frac{1}{2s}}\), we can estimate
\[
\int_{v \in \mathbb{R}} \frac{|v|^s}{|\lambda v - \alpha|^s} d\mu(v) \leq \left(\frac{2}{|\alpha|}\right)^s \int_{|v| \leq \frac{|\alpha|}{2}} |v|^s d\mu(v) + \left(\frac{2}{|\alpha|}\right)^s \int_{|v| \geq \frac{|\alpha|}{2}} \frac{|v|^s}{|\lambda v - \alpha|^s} d\mu(v)
\]
\[
\leq \left(\frac{2}{|\alpha|}\right)^s B_s + \left(\frac{2}{|\alpha|}\right)^s (B_{2s} + \cdots)
\]
\[
= : \left(\frac{2}{|\alpha|}\right)^s M_s.
\]
Then we ask that
\[
D_s \frac{1}{2} \left(\frac{1}{(2\lambda^{-2s}B_{2s})^{\frac{1}{2s}} + |\alpha|}\right)^s \geq \left(\frac{2}{|\alpha|}\right)^s M_s
\]
which can clearly be fulfilled for large \(|\alpha|\).

\[\square\]

**Example 2.19 (Gaussian distribution).** Consider the case where
\[
\frac{d\mu(v)}{dv} = \frac{1}{\sqrt{\pi}} \exp(-v^2) .
\]

### 2.7 Complete localization at sufficiently strong disorder

We are now ready to prove complete localization (i.e., at all energies) for sufficiently strong disorder using all the ingredients at our disposal.

**Theorem 2.20 (Aizenman-Molchanov 1993).** Let \(H = -\Delta + \lambda V_\omega(X)\) be the Anderson model on \(\ell^2(\mathbb{Z}^d)\) with \(\lambda > 0\) and \(\{\omega_x\}_{x \in \mathbb{Z}^d}\) an IID sequence with common single site measure \(\mu\) which obeys Definition 2.8. If
\[
\lambda > \left(\frac{2d}{\tau - 2s} C_{\mu}^s 2^{s+1}\right)^{\frac{1}{2}}
\]
then for any \(E \in \mathbb{R}\) there exist \(C, \mu \in (0, \infty)\) and \(s \in (0, 1)\) such that
\[
\sup_{\eta > 0} \mathbb{E} \left[|G(x, y; E + i\eta)|^s\right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d) .
\]

In particular, we have exponential decay of the fractional moments of the Greens function for all energies.

**Proof.** We begin by writing
\[
(-\Delta + \lambda V - zI) R(z) \equiv 1
\]
for \( R(z) \equiv (\Delta + \lambda V - z I)^{-1} \). Separating this equation to its diagonal and non-diagonal parts we find

\[
(2d - z + \lambda \omega_x) G(x, y; z) = \delta_{xy} + \sum_{x' \sim x} G(x', y; z)
\]

Raise this to some power \( s \in (0, 1) \) after taking absolute values to find

\[
|2d - z + \lambda \omega_x|^s |G(x, y; z)|^s = \left| \delta_{xy} + \sum_{x' \sim x} G(x', y; z) \right|^s \leq \left( \delta_{xy} + \sum_{x' \sim x} |G(x', y; z)| \right)^s.
\]

Next, note that if \( a, b \geq 0 \) then \( (a + b)^s \leq a^s + b^s \). Indeed, we have

\[
a + b \geq a \Updownarrow (a + b)^{s-1} \leq a^{s-1}
\]

and so

\[
a^s = aa^{s-1} \geq a (a + b)^{s-1}.
\]

Similarly \( b^s \geq b (a + b)^{s-1} \) and so adding those two inequalities we find

\[
a^s + b^s \geq (a + b)^s.
\]

Hence

\[
|2d - z + \lambda \omega_x|^s |G(x, y; z)|^s \leq \delta_{xy} + \sum_{x' \sim x} |G(x', y; z)|^s.
\]

Now take \( \mathbb{E} [\cdot] \) of both sides of the equation to get

\[
\mathbb{E} \left[ |2d - z + \lambda \omega_x|^s |G(x, y; z)|^s \right] \leq \delta_{xy} + \sum_{x' \sim x} \mathbb{E} \left[ |G(x', y; z)|^s \right].
\]

Now we want to integrate the LHS only over \( \omega_x \) to get a lower bound. For that we need the explicit dependence of \( G(x, y; z) \) on \( \omega_x \). Similarly to how we handled \( G(x, x; z) \) in Section 2.4, we find

\[
(H - z I)^{-1} = \begin{bmatrix} \lambda \omega_x & P_x (-\Delta) \\ (-\Delta) P_x & H \end{bmatrix}^{-1}
\]

and hence

\[
G(x, y; z) = \begin{cases} 
- \left( \lambda \omega_x - z - \sum_{y' \sim x \sim x'} \left( P_x^\dagger H P_x^\dagger - z I P_x^\dagger \right)^{-1} \right)^{-1} \left( P_x (-\Delta) P_x^\dagger \left( P_x^\dagger H P_x^\dagger - z I P_x^\dagger \right)^{-1} P_x^\dagger \right)_{xy} & \\
- \left( \lambda \omega_x - z - \sum_{y' \sim x \sim x'} \left( P_x^\dagger H P_x^\dagger - z I P_x^\dagger \right)^{-1} \right)^{-1} \left[ (-\Delta) P_x^\dagger \left( P_x^\dagger H P_x^\dagger - z I P_x^\dagger \right)^{-1} \right]_{xy} & \\
\end{cases}
\]

\[
G(x, y; z) = \left( \lambda \omega_x - z - \sum_{y' \sim x \sim x'} \left( P_x^\dagger H P_x^\dagger - z I P_x^\dagger \right)^{-1} \right)^{-1} \sum_{y' \sim x} \left( P_x^\dagger H P_x^\dagger - z I P_x^\dagger \right)^{-1}.
\]

Again the particular form of this expression is unimportant, since we only care where \( \omega_x \) dependence appears. In that sense, we shall write

\[
G(x, y; z) = \frac{1}{\lambda \omega_x - \alpha} \beta.
\]
Using now Lemma 2.17 we find then that, with $M := \frac{x}{\tau} C^2 2^{s+1}$
\[
\mathbb{E} \left[ |2d - z + \lambda \omega_x|^s |G(x, y; z)|^s \right] = \mathbb{E} \left[ \int_{\omega_x \in \mathbb{R}} d\mu(\omega_x) |2d - z + \lambda \omega_x|^s |\lambda \omega_x - \alpha|^{-s} \beta^s |\omega_x| \right] \\
\geq \lambda^s M \mathbb{E} \left[ \int_{\omega_x \in \mathbb{R}} d\mu(\omega_x) |\lambda \omega_x - \alpha|^{-s} \beta^s |\omega_x| \right] \\
= \lambda^s M \mathbb{E} \left[ |G(x, y; z)|^s \right] .
\]
Collecting everything together we have
\[
\lambda^s M \mathbb{E} \left[ |G(x, y; z)|^s \right] \leq \delta_{xy} + \sum_{x' \sim x} \mathbb{E} \left[ |G(x', y; z)|^s \right] .
\]
Let us define $g(x, y) := \mathbb{E} \left[ |G(x, y; z)|^s \right] \geq 0$. Then we have
\[
(\lambda^s M g - Ag)_{xy} \leq \delta_{xy} \\
(-\Delta g + (\lambda^s M - 2d) g)_{xy} \leq \delta_{xy} .
\]
Now from Lemma 2.15 we learn that if $\lambda^s M > 2d$ we have exponential decay, as desired.

2.8 Localization at weak disorder and extreme energies
Another regime in which localization may be established is at arbitrarily small $\lambda$ but at extreme energies. We first start with a technical lemma

Using Lemma 2.18 and the Combes-Thomas estimate Theorem 1.18, we find:

**Theorem 2.21.** If $\lambda > 0$ there exists some $E_c(\lambda) \in \mathbb{R}$ such that if $E \geq E_c(\lambda)$ then there exists some $s \in (0, 1)$ and $C, \mu \in (0, \infty)$ such that
\[
\sup_{x > 0} \mathbb{E} \left[ |G(x, y; E + i\varepsilon)|^s \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d) .
\]

**Proof.** We start by writing the resolvent identity between the operators
\[
H_\omega = -\Delta + \lambda V_\omega(X)
\]
and
\[
H_0 := -\Delta .
\]
It yields, at some $z \in \mathbb{C} \setminus \mathbb{R}$,
\[
R_\omega(z) = R_0(z) + R_0(z) (H_0 - H_\omega) R_\omega(z) \\
= R_0(z) - R_0(z) \lambda V_\omega(X) R_\omega(z) .
\]
Taking the $x, y$ matrix elements, expectation w.r.t some $s \in (0, 1)$ moment and using the triangle inequality, we find
\[
\mathbb{E} \left[ |G_\omega(x, y; z)|^s \right] \leq |G_0(x, y; z)|^s + \lambda^s \sum_{\tilde{x} \in \mathbb{Z}^d} |G_0(x, \tilde{x}; z)|^s \mathbb{E} \left[ |\omega_{\tilde{x}}|^s \right] |G_\omega(\tilde{x}, y; z)|^s .
\]
We already know the dependence of $G_\omega(\tilde{x}, y; z)$ on $\omega_{\tilde{x}}$ from our study of finite rank perturbation theory above in (2.13):
\[
G_\omega(\tilde{x}, y; z) = \frac{1}{\lambda \omega_{\tilde{x}} - \alpha} \beta
\]
for some \( \alpha, \beta \in \mathbb{C} \) which are independent of \( \omega_x \). Using the regularity condition (2.12) we then find

\[
\mathbb{E} \left[ |G_\omega (x, y; z)|^s \right] \leq |G_0 (x, y; z)|^s + \lambda^s D_s \sum_{\tilde{x} \in \mathbb{Z}^d} |G_0 (x, \tilde{x}; z)|^s \mathbb{E} \left[ |G_\omega (\tilde{x}, y; z)|^s \right].
\]

Using now Theorem 1.18, assuming \( \text{Re} \{ z \} \notin \sigma (-\Delta) \), we find that there exists some \( \delta (z) > 0 \) such that

\[
|G_0 (x, y; z)| \leq \frac{2}{\delta (z)} \exp \left( -\tilde{\mu} \delta (z) \| x - y \| \right) \quad (x, y \in \mathbb{Z}^d).
\]

Hence

\[
\mathbb{E} \left[ |G_\omega (x, y; z)|^s \right] \leq \frac{2^s}{\delta (z)} \exp \left( -s \tilde{\mu} \delta (z) \| x - y \| \right) + \frac{2^s}{\delta (z)^s} D_s \lambda^s \sum_{\tilde{x} \in \mathbb{Z}^d} \exp \left( -s \tilde{\mu} \delta (z) \| x - \tilde{x} \| \right) \mathbb{E} \left[ |G_\omega (\tilde{x}, y; z)|^s \right].
\]

Such a condition implies in itself exponential decay of \( \mathbb{E} \left[ |G_\omega (x, y; z)|^s \right] \) if \( \lambda \) is sufficiently small. Indeed, let us re-write it as

\[
f (x) := \mathbb{E} \left[ |G_\omega (x, y; z)|^s \right]
\]

to get

\[
f (x) \leq Q \exp (-\nu \| x - y \| ) + Q \lambda^s \sum_{\tilde{x}} e^{-\nu \| \tilde{x} - x \|} f (\tilde{x})
\]

for some \( Q < \infty \) and \( \nu > 0 \). Let us define

\[
W (x) := \exp (+\xi \| x - y \| )
\]

\[
K (x, y) := Q \lambda^s e^{-\nu \| x - y \|}
\]

\[
g (x, y) := Q e^{-\nu \| x - y \|}
\]

with which we verify:

\[
\sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} K (x, y) = \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} Q \lambda^s e^{-\nu \| x - y \|}
\]

\[
= Q \lambda^s \left( \sum_{x \in \mathbb{Z}^d} e^{-\nu \| x \|} \right)
\]

\[
< 1
\]

which implies

\[
\lambda < \left[ \frac{Q \left( \sum_{x \in \mathbb{Z}^d} e^{-\nu \| x \|} \right)}{1} \right]^{\frac{1}{s}}.
\]

Moreover,

\[
r = \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \exp (+\xi \| x - y \| ) Q \lambda^s e^{-\nu \| x - y \|}
\]

is indeed smaller than 1 iff \( \xi < \nu \) and \( \lambda \) is even smaller. Finally,

\[
b := \sum_{x \in \mathbb{Z}^d} \exp (+\xi \| x - y \| ) Q e^{-\nu \| x - y \|} < \infty.
\]
Hence Lemma 2.16 applies and we find
\[ f(x) \leq \frac{b}{1 - r} \exp(-\xi \|x - y\|). \]

2.9 What about localization at the edges of the Laplacian’s spectrum?

The argument we presented just above manages to establish localization when
\[ E \notin \sigma(-\Delta) \]
essentially using the Combes-Thomas estimate for \(-\Delta\). It turns out that localization also holds for \( E \in \sigma(-\Delta) \) but at its fringes, through a somewhat more intricate mechanism: very low density of states due to the so-called Lifschitz tails. To better study this phenomenon, we need to study finite volume restrictions of \( H \) onto boxes
\[ \Lambda_L := [-L, L]^d \cap \mathbb{Z}^d. \]
The boundary conditions don’t matter a lot, it turns out, so pick Dirichlet for simplicity. Denote by \( H_L \) the operator \( H \) restricted to \( L^2(\Lambda_L) \) with Dirichlet boundary conditions. Clearly \( H_L \) is just a finite matrix which has \((2L + 1)^d\) eigenvalues.

We do not show the full argument but just explain the mechanism which enables this other form of localization. The full argument will be found in [AW15], Corollary 11.6:

1. First of all, through a finite rank perturbation theory argument,
\[ |G(x, y; z)| \leq C|G_L(x, y; z)| \]
and
\[ |G_L(x, y; z)| \leq C|G_L(0, L; z)| \]
for some constant \( C \) independent of disorder and independent of \( L \).

2. Sufficiently fast polynomial decay of \( |G_L(0, L; z)| \) implies that actually it has exponential decay.

3. \( H_L \) has very low density of eigenvalues at the fringes of \( \sigma(-\Delta) \). This is called Lifschitz tails.

2.9.1 Low density of states implies polynomial decay of Greens function

Fix some energy \( E \in \mathbb{R} \), size of box \( L \in \mathbb{N} \) and \( \delta := CL^{-\beta} \) for some \( C < \infty \) and \( \beta \in (0, 1) \). Using the deterministic Combes-Thomas estimate, we know that on the set of realizations \( \omega \) such that
\[ \Omega(E, \delta) := \{ \omega \in \Omega \mid \text{dist}(\sigma(H_L), E) > \delta \} \]
we have exponential decay of the Greens function:
\[ |G_{L, \omega}(0, x; E)| \leq \frac{2}{\delta} \exp(-c\delta|x|) \quad (x \in \Lambda_L, \omega \in \Omega(E, \delta)). \]

Hence we get
\[ \mathbb{E}[|G_{L, \omega}(0, x; E)|^p] \leq \mathbb{E}[|G_{L, \omega}(0, x; E)|^p \chi_{\Omega(E, \delta)}] + \mathbb{E}[|G_{L, \omega}(0, x; E)|^p \chi_{\Omega(E, \delta)^c}] \leq \frac{2^s}{\delta^s} \exp(-c\delta|x|) + \mathbb{E}[|G_{L, \omega}(0, x; E)|^p]^{\frac{1}{p}} \mathbb{P}[\Omega(E, \delta)^c]^{\frac{1}{p}} \]
for some \( p \in \left(1, \frac{s}{\beta}\right) \) to make Hölder work. To deal with \( \mathbb{E}[|G_{L, \omega}(0, x; E)|^p]^{\frac{1}{p}} \) we use the a-priori bound Theorem 2.12. Hence if we managed to show that
\[ \mathbb{P}[\{ \omega \in \Omega \mid \text{dist}(\sigma(H_L), E) \leq CL^{-\beta} \}] \leq CL^{-\alpha} \tag{2.14} \]
then we would have the estimate
\[ \mathbb{E}[|G_{L, \omega}(0, L; E)|^s] \leq \frac{2^s}{C^s}L^{\beta s} \exp(-csCL^{1-\beta}) + C\left(CL^{-\alpha}\right)^{1-\frac{1}{\beta}} \]
which yields polynomial decay in \( L \).
2.9.2 Fast enough polynomial decay implies exponential decay

[TODO]: This proceeds by a finite rank perturbation argument.

2.9.3 The Lifschitz tails argument

[TODO]: Establish (2.14) for energies in the fringes of \( \sigma (-\Delta) \).

2.10 Complete localization in one dimension for arbitrary strength of disorder

In this chapter we show yet another mechanism of localization: complete localization in one-dimension. We largely follow [CPSS22] in its mechanism but restrict our attention to the one-dimensional Anderson model instead of the random band matrices treated in that reference. Another standard technique (which we do not present here) is the transfer matrix approach. It is harder and gives less quantitative information, but it came beforehand, see [KLS90]. It is also more robust.

Consider then

\[ H := -\Delta + \lambda V_\omega (X) \]

on \( \ell^2 (\mathbb{Z}) \). To facilitate the discussion, it will useful to restrict this to \([1, L] \cap \mathbb{Z}\) and thus discuss the \( L \times L \) matrix

\[
H_L := \begin{pmatrix}
2 + \lambda \omega_1 & -1 & & & \\
-1 & 2 + \lambda \omega_2 & -1 & & \\
& -1 & \ddots & \ddots & \\
& & 2 + \lambda \omega_{L-1} & -1 & \\
& & & -1 & 2 + \lambda \omega_L
\end{pmatrix}.
\]

Furthermore assume also that \( \omega_j \) has a standard Gaussian distribution (for simplicity):

\[
\exp (-\pi \omega_x^2) d\omega_x.
\]

We have seen above that it is only necessary to study the corner element

\[
G_L (1, L; E) = (H_L - E \mathbb{1})^{-1}
\]

and establish that its fractional moment has exponential decay with \( L \). Note that since we have a finite matrix we do not need to go off the real axis to get an invertible operator: almost surely any given \( E \) does not hit an eigenvalue of \( H_L \) so we may contend ourselves to real \( E \). Hence our goal is to establish

\[
\mathbb{E}[|G_L (1, L; E)|^s] \leq e^{-cL}
\]

for some \( c > 0 \).

2.10.1 Lower bound on fluctuations implies exponential decay

Let us begin with a remark about Hölder’s inequality: For \( 0 < r < s < 1 \) and \( Y \geq 0 \) we have

\[
\mathbb{E}[Y^r] \leq (\mathbb{E}[Y^s])^{r/s}.
\]

We shall require a strengthening of this into

**Lemma 2.22.** Let \( 0 < r < s < 1 \) and \( Y \geq 0 \). Then

\[
\mathbb{E}[Y^r] = (\mathbb{E}[Y^s])^{r/s} \exp \left( - \int_0^s f_{r,s}(q) \text{Var}_q [\log (Y)] dq \right)
\]

where

\[
f_{r,s}(q) := \frac{1}{s} \min \{ r, q \} (s - \max \{ r, q \}) \quad (q \in (0, s))
\]

and

\[
\text{Var}_q [X] := \mathbb{E}_q [\mathbb{E}_q [X] - X]^2, \quad \mathbb{E}_q [Z] \equiv \mathbb{E} \left[ \frac{Z e^{qX}}{e^{qX}} \right].
\]
If we apply this lemma on \( Y := |G_L(1, L; E)| \) we find, using Theorem 2.12, that thanks to \( f_{rs}(q) > 0 \), it suffices to prove
\[
\forall q \in (0, s) \quad \log (|G_L(1, L; E)|) \geq C_q L
\]
for some constant \( C_q > 0 \) where \( q \in (0, s) \).

To get a lower bound on fluctuations, we shall use the basic

**Lemma 2.23.** Let \( X \) be a real-valued RV distributed according to a probability measure \( P \) and such that there are some
\[
0 < \alpha < a
\]
and \( \varepsilon \in (0, 1), \beta \in (0, \infty) \) with which
\[
P\{ |X| \leq \alpha \} \leq \beta \sqrt{P\{ X \geq a \} P\{ X \leq -a \}} + \varepsilon.
\]
Then
\[
E[X^2] \geq \frac{1 - \varepsilon}{1 + \frac{1}{2} \beta} \alpha^2.
\]

**Proof.** TODO

### 2.10.2 Factorizing the Greens function

**Symplectic transfer matrices** For the sake of completeness, let us first study the usual factorization, which is using the transfer matrix. We have the eigenequation
\[
H \psi = E \psi
\]
\[
-\psi_{x+1} - \psi_{x-1} + (2d + \lambda \omega_x) \psi_x = E \psi_x
\]
\[
\psi_{x+1} = -(E - 2d - \lambda \omega_x) \psi_x - \psi_{x-1}.
\]
If we define
\[
\Psi_x := \begin{bmatrix} \psi_{x+1} \\ \psi_x \end{bmatrix}
\]
then we find the Schroedinger equation is equivalent to
\[
\Psi_x = \begin{bmatrix} -(E - 2d - \lambda \omega_x) & -1 \\ 1 & 0 \end{bmatrix} \Psi_{x-1} \quad (x \in \mathbb{Z}).
\]
We call the matrix
\[
A_x(E) := \begin{bmatrix} -(E - 2d - \lambda \omega_x) & -1 \\ 1 & 0 \end{bmatrix}
\]
a transfer matrix. We note a few properties of it:

1. It has real entries for the usual Anderson model at real energies.

2. It obeys the symplectic condition \( A_x(E)^T \Omega A_x(E) = \Omega \) for \( \Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) which ultimately is a consequence of probability preservation in quantum mechanics. The symplectic condition implies that its eigenvalues are symmetric about the unit circle.

From the Schroedinger equation it is apparent that the product of many transfer matrices controls the eigenfunctions as
\[
\Psi_x = A_x(E) \cdots A_2(E) \Psi_1.
\]
To understand better the decay and growth properties of these matrices, we study the *Lyapunov exponents*
\[
\gamma(E) := \lim_{x \to \infty} \frac{1}{x} \log (\|A_x(E) \cdots A_2(E)\|)
\]
where we note that due to the symplectic condition, it is only the top singular value that is necessary to extract since the bottom one is symmetric about the unit circle. By (abstract arguments we do not want to get into, see manuscript by Lacroix) we also have almost-surely

$$\gamma(E) = \lim_{x \to \infty} \frac{1}{x} \mathbb{E} [\log (||A_x (E) \cdots A_2 (E)||)] .$$

It turns out that the Greens function can similarly be enlarged as

$$G(x, y; E) = \begin{bmatrix} G(x + 1, y; E) \\ G(x, y; E) \end{bmatrix}$$

in order to yield

$$G(x, y; E) = A_x (E) \cdots A_{y-2} (E) G(y - 1, y; E) \quad (x \leq y - 1) .$$

Thanks to this identity and the a-priori bound Theorem 2.12, after some manipulations it is sufficient to prove that

$$\gamma(E) > 0 . \quad (2.15)$$

Indeed, [TODO: explain how]. Establishing (2.15) is covered by Furstenberg’s theory [GMP77] which shows that the Lyapunov spectrum of sufficiently rich sequences of random matrices is simple and hence avoids zero by the symplectic condition. We shall not take that route here.

The self-adjoint factorization  Instead of the symplectic factorization of the Greens function, we instead factorize the Greens function using Gaussian elimination.

**Claim 2.24.** We have

$$G_L(1, L; E) = \Gamma_1^{-1} \cdots \Gamma_L^{-1}$$

where

$$\Gamma_1 := \lambda \omega_1 - E \quad \Gamma_j := \lambda \omega_j - E - \Gamma_{j-1}^{-1} \quad (j = 2, \ldots, L) .$$

**Proof.** Consider $H_{L-1}$ as an $L \times L$ matrix with 0 added:

$$H_{L-1} \oplus 0 .$$

Then the resolvent identity yields

$$(H_L - z \mathbb{1})^{-1} = (H_{L-1} \oplus 0 - z \mathbb{1})^{-1} + (H_{L-1} \oplus 0 - z \mathbb{1})^{-1} (H_{L-1} \oplus 0 - H_L) (H_L - z \mathbb{1})^{-1} .$$

But now, take the 1, $L$ matrix elements. Since $H_{L-1} \oplus 0$ does not couple the site $L$ with the rest of the matrix, that matrix element will be zero. Hence

$$G_L(1, L; z) = \sum_{j, k=1}^L \left[ (H_{L-1} \oplus 0 - z \mathbb{1})^{-1} \right]_{1,j} (H_{L-1} \oplus 0 - H_L)_{j,k} G_L(k, L; z) .$$

Moreover,

$$H_{L-1} \oplus 0 - H_L = \begin{bmatrix} 0 & 0 \\ 0 & 0 & 0 \\ & 0 & \cdots \\ & & 0 & 1 \\ & & & 1 - (2 + \lambda \omega_L) \end{bmatrix}$$

and $\left[ (H_{L-1} \oplus 0 - z \mathbb{1})^{-1} \right]_{1,j} = 0$ if $j = L$, whereas by the above, $(H_{L-1} \oplus 0 - H_L)_{j,k} = 0$ if $j \leq L - 1$ but then
\[ k = L, \text{ so we get} \]
\[ G_L(1, L; z) = \left[ (H_{L-1} \oplus 0 - z1)^{-1} \right]_{1,L-1} G_L(L, L; z) \]
\[ = G_{L-1}(1, L - 1; z) G_L(L, L; z). \]

Iterating this identity \( L - 1 \) more times we find
\[ G_L(1, L; z) = G_1(1, 1; z) \cdots G_{L-1}(L - 1, L - 1; z) G_L(L, L; z). \]

Hence, let us define
\[ \Gamma_j := G_j(j, j; z)^{-1}. \]

Let us now use the Schur complement formula Lemma 2.11 on \( H_j \) (decomposing \( C_j = C_j^{-1} \oplus C \)) to find
\[ G_j(j, j; z) = \left( \lambda \omega_j - z - G_{j-1}(j - 1, j - 1; z)^{-1} \right)^{-1}. \]

\[ \square \]

2.10.3 The change of variable argument

Since the numbers \( \Gamma_j \) are also real, we make a change of variable
\[ \omega_j \mapsto \Gamma_j. \]

Since the dependence of the \( \Gamma_j \) is only on the past, the determinant of the Jacobian is identity and we find now the distribution of the random variables
\[ \exp (-E(\Gamma)) \, d\Gamma_1 \cdots d\Gamma_L \]

where
\[ E(\Gamma) := \pi \lambda^{-2} (\Gamma_1 + E)^2 + \pi \lambda^{-2} \sum_{j=2}^{L} (\Gamma_j + E + \Gamma_{j-1})^2 \]

and
\[ X := \log \left( |\Gamma_1^{-1} \cdots \Gamma_L^{-1}| \right). \]

Let us define a collective change of variables on \( \{ \Gamma_j \} \) as follows
\[ \Gamma_j^\pm := \exp (\pm \delta F_j) \Gamma_j \quad (j = 1, \ldots, L) \]
where \( \delta, F_j \) are to be determined. Then
\[ X^\pm = \log \left( |\Gamma_1^{\pm-1} \cdots \Gamma_L^{\pm-1}| \right) \]
\[ = X \pm \delta F \]

with \( F := \sum_j F_j \). Moreover, we also have
\[ X = \frac{1}{2} X^+ + \frac{1}{2} X^- . \]

We are interested, thanks to Lemma 2.23, in
\[
\begin{align*}
|X| & \leq \alpha \\
|X - \mathbb{E}_q[X]| & \leq \alpha \\
\mathbb{E}_q[X] - \alpha & \leq X \leq \mathbb{E}_q[X] + \alpha \\
\mathbb{E}_q[X] - \alpha & \leq X^\pm = \delta F \leq \mathbb{E}_q[X] + \alpha \\
\mathbb{E}_q[X] \pm \delta F - \alpha & \leq X^\pm \leq \mathbb{E}_q[X] \pm \delta F + \alpha \\
\pm \delta F - \alpha & \leq X^\pm \leq \pm \delta F + \alpha.
\end{align*}
\]
Finally, we estimate

$$\mathbb{P}_q \left\{ \{ |X| \leq \alpha \} \cap M \right\} = \frac{\mathbb{E} \left[ e^{qX} \right]}{Z} \int_{\Gamma \in \mathbb{R}^E \cap M} e^{qX - E(\Gamma)} \chi\{ |X| \leq \alpha \} (\Gamma) \, d\Gamma.$$  

We now make the following replacements in this integral

1. $X = \frac{1}{2} X^+ + \frac{1}{2} X^-.$
2. $\mathcal{R}(\Gamma) := \frac{1}{2} E(\Gamma^+) + \frac{1}{2} E(\Gamma^-) - E(\Gamma).$
3. $\chi\{ |X| \leq \alpha \}(\Gamma) = \sqrt{\chi\{ \delta F - \alpha \leq X^- \leq \delta F + \alpha \}(\Gamma)} \chi\{ -\delta F - \alpha \leq X^- \leq -\delta F + \alpha \}(\Gamma).$
4. If $\eta_\pm : \Gamma \mapsto \Gamma \pm$ then let $J_\pm := |\det (D\eta_\pm)|$ be the Jacobian.

With all of these, we get, with $R_M := \left\| e^{\mathcal{R}(\Gamma)} \right\|_{L^\infty(M)},$

$$\mathbb{P}_q \left\{ \{ |X| \leq \alpha \} \cap M \right\} \frac{Z}{\mathbb{E} \left[ e^{qX} \right]} \leq R_M \int_{\Gamma \in M} \prod_{\sigma \in \{ \pm \}} e^{qX^\sigma - E(\Gamma^\sigma)} \chi\{ \sigma \delta F - \alpha \leq X^\sigma \leq \sigma \delta F + \alpha \}(\Gamma) \frac{1}{J_\sigma} \, d\Gamma$$

$$\leq C_S R_M \left( \prod_{\sigma \in \{ \pm \}} \int_{\Gamma \in M} e^{qX^\sigma - E(\Gamma^\sigma)} \chi\{ \sigma \delta F - \alpha \leq X^\sigma \leq \sigma \delta F + \alpha \}(\Gamma) \, d\Gamma \right)^{1/2}$$

Now we would like to apply the change of variables formula, but the set

$$\left\{ \delta F - \alpha \leq X^+ \leq \delta F + \alpha \right\}$$

depends in a complicated way on $F,$ and through that $\Gamma.$ Instead of using that, we note that this set is a subset of

$$\left\{ \delta \inf_M F - \alpha \leq X^+ \right\}$$

and similarly for the other sign. Hence this set is defined non-randomly, and we may apply the change of variables formula to get

$$\mathbb{P}_q \left\{ \{ |X| \leq \alpha \} \cap M \right\} \frac{Z}{\mathbb{E} \left[ e^{qX} \right]} \leq R_M \sqrt{\prod_{\sigma \in \{ \pm \}} \int_{\Gamma \in M} e^{qX^\sigma - E(\Gamma^\sigma)} \chi\{ \sigma \delta \inf_M F - \alpha \leq X^\sigma \}(\Gamma) \, d\Gamma}$$

where we have applied

$$\int_{\mathbb{R}^E} f \circ \eta_\sigma J_\sigma \, d\Gamma = \int f \, d\Gamma.$$  

Finally, we estimate

$$\mathbb{P}_q \left\{ \{ |X| \leq \alpha \} \right\} = \mathbb{P}_q \left\{ \{ |X| \leq \alpha \} \cap M \right\} + \mathbb{P}_q \left\{ \{ |X| \leq \alpha \} \cap M^c \right\}$$

and

$$\mathbb{P}_q \left\{ \{ |X| \leq \alpha \} \cap M^c \right\} = \mathbb{E} \left[ \chi\{ \{ |X| \leq \alpha \} \cap M^c \} e^{qX} \right].$$

Now,

$$\mathbb{E} \left[ e^{qX} \right] \geq \mathbb{E} \left[ e^{q\overline{Z}X} \right]^{2s}.$$  

Let us further assume, by contradiction, that

$$\mathbb{E} \left[ e^{q\overline{Z}X} \right] \geq e^{-cL}$$

since if that is false then we are anyway finished with localization. Hence, as $q \in \left( \frac{s}{2}, s \right)$ we have

$$\mathbb{E} \left[ e^{qX} \right] \geq e^{-c \overline{Z}L}.$$  

70
For the numerator we have
\[
E \left[ \chi_{\{ |X| \leq \alpha \} \cap M^c} \right] \leq \sqrt{E [e^{2qX}] P [\{ |X| \leq \alpha \} \cap M^c]}
\leq \sqrt{E [e^{2qX}] P [M^c]}
\leq C \sqrt{P [M^c]}
\]
where in the last step we have invoked the a-priori bound Theorem 2.12. Combining everything together we find
\[
P_q \left[ \{ |X| \leq \alpha \} \right] \leq e^{\frac{2qL}{2}} C \sqrt{P [M^c]} + R_M \prod_{\sigma \epsilon \{+, -\}} \int_{\Gamma \in M} e^{qX - E(\Gamma)} \chi_{\{ \sigma \inf_{M} F - \alpha \leq X \}}(\Gamma) d\Gamma.
\]
It is readily seen that this estimate is of the form Lemma 2.23 if we can arrange that:
1. \( R_M \) is bounded in \( L \).
2. \( e^{\frac{2qL}{2}} C \sqrt{P [M^c]} < \frac{1}{2} \).
3. \( \inf_{M} F \geq \varphi L \) for some fraction \( \varphi \in (0, 1) \) independent of \( L \).

To fulfill these, we need to ask that
\[
\delta \varphi L - \alpha = 2\alpha
\]
which fixes \( \delta \) as
\[
\delta = \frac{3}{\varphi L} \alpha = \frac{3}{\varphi \sqrt{L}}.
\]

2.11 Consequences of the fractional moment condition

2.11.1 Decay of the Fermi projection

**Theorem 2.25.** For a given random operator \( \omega \mapsto H_\omega \) for which
\[
\sup_{\varepsilon > 0} E \left[ |G(x, y; E + i\varepsilon)|^s \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d)
\]
we have
\[
E \left[ \left( \chi_{(-\infty, E)} (H) \right)_{xy} \right] \leq C e^{-\mu \|x-y\|} \quad (x, y \in \mathbb{Z}^d).
\]

**Proof.** We now that almost surely \( E \) is not an eigenvalue of \( H \) (TODO, explain why). Then we may use the contour formula
\[
\chi_{(-\infty, E)} (H) = \frac{1}{2\pi i} \oint R(z) \, dz
\]
where the contour is a rectangle that passes \( E \) vertically and otherwise passes through another vertical line below \(-\|H\|\). Thanks to the Combes-Thomas estimate Theorem 1.18 we only need to obtain exponential decay of the vertical line that passes through \( E \):
\[
\int_{\varepsilon = -1}^{1} R(E + i\varepsilon) \, d\varepsilon.
\]
Taking the \( x, y \) matrix elements we get
\[
\int_{\varepsilon = -1}^{1} |G(x, y; E + i\varepsilon)| \, d\varepsilon = \int_{\varepsilon = -1}^{1} |G(x, y; E + i\varepsilon)|^s |G(x, y; E + i\varepsilon)|^{1-s} \, d\varepsilon
\leq \int_{\varepsilon = -1}^{1} |G(x, y; E + i\varepsilon)|^s \frac{1}{\varepsilon^{1-s}} \, d\varepsilon
Now we take expectation and we find
\[
\mathbb{E}\left[\left(\chi_{(-\infty,E)}(H)\right)_{xy}\right] \leq \int_{\varepsilon=-1}^{1} \mathbb{E}\left[|G(x,y;E+i\varepsilon)|^s\right] \frac{1}{\varepsilon^{1-s}} d\varepsilon
\]

2.12 The physics argument for delocalization

3 Topology in condensed matter physics

3.1 The main idea

3.2 The chiral one-dimensional case

3.3 The integer quantum Hall effect and time-reversal invariant systems

References


[CPSS22] Giorgio Cipolloni, Ron Peled, Jeffrey Schenker, and Jacob Shapiro. Dynamical localization for random band matrices up to \( w \ll n^{1/4} \), 2022.


