

Functional Analysis
Princeton University MAT520
HW9, Assigned on Nov 15 2025

November 15, 2025

1 Compact operators

1. In an infinite dimensional Hilbert space, show that if A is invertible then it cannot be compact.
2. Show that if A_n is a sequence of operators such that for any bounded $B \subseteq \mathcal{H}$, $\overline{A_n(B)}$ is a *compact set* in $\mathcal{H}$, and $A_n \rightarrow A$ in norm, then also $\overline{A(B)}$ is a compact subset.
3. Let $A \in \mathcal{B}(\mathcal{H})$ be compact and $\{\varphi_n\}_n \subseteq \mathcal{H}$ converge *weakly* (in the sense of the Banach space weak topology on $\mathcal{H}$) to some $\varphi \in \mathcal{H}$. Show that $A\varphi_n \rightarrow A\varphi$ in norm.
4. Figure out if the following operators are compact or not (and prove what you think):
 - (a) $u \otimes v^*$ for some $u, v \in \mathcal{H}$.
 - (b) On the Banach space $X := C([0, 1] \rightarrow \mathbb{C})$ with $\|\cdot\|_\infty$, let $A : X \rightarrow X$ be given

$$(A\varphi)(x) := \int_{y=0}^1 K(x, y) \varphi(y) dy$$

where $K : [0, 1]^2 \rightarrow \mathbb{C}$ is some *continuous* function.

- (c) $A := \frac{1}{1+X^2}$ on $\ell^2(\mathbb{Z})$ where X is the position operator given by

$$(X\psi)(n) \equiv n\psi(n) \quad (n \in \mathbb{Z}; \psi \in \ell^2(\mathbb{Z}))$$

and we employ the holomorphic functional calculus to define A .

5. Let $\mathcal{H}$ be some separable Hilbert space. On $\mathcal{H} \oplus \mathcal{H}$ (external direct sum), let

$$H := \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$$

for some $S \in \mathcal{B}(\mathcal{H})$. Find the polar decomposition of H .

6. Show that an idempotent is compact if and only if it is of finite rank.
7. Show that no nonzero multiplication operator on $L^2([0, 1])$ is compact.
8. Show that if $A \in \mathcal{B}(\mathcal{H})$ is compact and $\{e_n\}_n$ is an ONB then $\|Ae_n\| \rightarrow 0$. Find a counter-example of the converse.
9. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded region with a smooth boundary surface $\partial\Omega$. Let $f : \partial\Omega \rightarrow \mathbb{C}$ be continuous. Fix some parameter $m > 0$. It is a theorem that there exists a unique function $\varphi_f : \overline{\Omega} \rightarrow \mathbb{C}$ which is twice differentiable in Ω and continuous on $\overline{\Omega}$ such that

$$\begin{aligned} (-\Delta + m^2 \mathbf{1}) \varphi_f &= 0 \\ \varphi_f|_{\partial\Omega} &= f. \end{aligned}$$

Indeed, there is a continuous function $K : \overline{\Omega} \times \partial\Omega \rightarrow \mathbb{C}$ (called the Poisson kernel of $-\Delta + m^2 \mathbb{1}$ in the interior of Ω) which allows the solution of the above Dirichlet problem be written as

$$\varphi_f(x) = \int_{y \in \partial\Omega} K(x, y) f(y) d\sigma(y)$$

where σ is the volume measure on $\partial\Omega$.

Show that the operator

$$T : C(\partial\Omega) \rightarrow C(\overline{\Omega}) : f \mapsto \varphi_f$$

is a bounded linear operator which is *compact*.

[For more background on, see Gilbarg–Trudinger, *Elliptic Partial Differential Equations of Second Order*].

2 Fredholm operators

11. Provide an example of a norm continuous path of operators $[0, 1] \ni t \mapsto A_t$ for which $\dim \ker A_t$ has jump discontinuities.
12. Show that, for $A \in \mathcal{B}(\mathcal{H})$, $\dim \operatorname{coker}(A) < \infty$ iff both: (1) $\dim \ker A^* < \infty$ and (2) $\operatorname{im}(A) \in \operatorname{Closed}(\mathcal{H})$.
13. Show that if X is the position operator on $\ell^2(\mathbb{N})$ then $\frac{1}{X}$ does *not* have a closed image.
14. Show that if $A : V_1 \rightarrow V_2$ is a linear map between two finite dimensional Hilbert spaces then A is Fredholm and

$$\operatorname{index}(A) = \dim V_1 - \dim V_2.$$

Conclude that square matrices always have zero Fredholm index.

15. Prove that a compact operator cannot be Fredholm if $\dim \mathcal{H} = \infty$.
16. Show that if K is compact then $\mathbb{1} - K$ is Fredholm.