Functional Analysis
Princeton University MAT520
HW8, assigned on Nov 8 2025

November 8, 2025

Here $\mathcal{H}$ is a separable Hilbert space.

1. Let $\{ e_i \}_i$ be an ONB in $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$. Show that the sequence

$$\left\{ \sum_{i,j=1}^{N} \langle e_i, A e_j \rangle e_i \otimes e_j^* \right\}_N$$

converges in SOT to A. Find an example of an A for which this convergence does *not* hold in operator norm.

2. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$:

$$R e_j \quad := \quad e_{j+1} \qquad (j \in \mathbb{N})$$

where $\{ e_j \}_{j\in\mathbb{N}}$ is the standard basis of $\ell^2(\mathbb{N})$ and extend linearly.

   (a) Calculate R^*.

   (b) Calculate $|R|^2$ and $|R^*|^2$.

   (c) Show that R is a partial isometry.

   (d) Calculate $\sigma(R), \sigma(R^*), \sigma\left(|R|^2\right)$ and $\sigma\left(|R^*|^2\right)$.

3. Let $\hat{R}$ be the bilateral right shift operator on $\ell^2(\mathbb{Z})$:

$$\hat{R} e_j := e_{j+1} \qquad (j \in \mathbb{Z})$$

where $\{ e_j \}_{j\in\mathbb{Z}}$ is the standard basis of $\ell^2(\mathbb{Z})$ and extend linearly.

   (a) Calculate $\hat{R}^*$.

   (b) Calculate $\left|\hat{R}\right|^2$ and $\left|\hat{R}^*\right|$.

   (c) Show that $\hat{R}$ is a unitary.

   (d) Calculate $\sigma\left(\hat{R}\right), \sigma\left(\hat{R}^*\right), \sigma\left(\left|\hat{R}\right|^2\right)$ and $\sigma\left(\left|\hat{R}^*\right|^2\right)$.

4. Let $\frac{1}{X} \in \mathcal{B}\left(\ell^2(\mathbb{N})\right)$ be given by

$$\frac{1}{X} e_j := \frac{1}{j} e_j \qquad (j \in \mathbb{N})$$

and extend linearly.

   (a) Calculate $\left(\frac{1}{X}\right)^*$.

   (b) Calculate $\sigma\left(\frac{1}{X}\right)$.

   (c) Show that $\frac{1}{X}$ does *not* have closed range (but show it directly, not via the indirect route we saw in class).

5. Show that if M is a closed linear subspace and $P_M : \mathcal{H} \to \mathcal{H}$ is given by

$$P_M \psi \quad := \quad a$$

where $\psi = a + b$ in the unique decomposition $\mathcal{H} = M \oplus M^\perp$, then P_M is a *self-adjoint projection*, i.e., show that $P_M = P_M^* = P_M^2$. Conversely, given any self-adjoint projection $P \in \mathcal{B}(\mathcal{H})$, find a closed linear subspace M such that $P = P_M$.

6. Let $\{ A_n \}_n \subseteq \mathcal{B}(\mathcal{H})$ such that for any $\varphi, \psi \in \mathcal{H}$,

$$\exists \lim_n \langle \varphi, A_n \psi \rangle \ .$$

Show there exists $A \in \mathcal{B}(\mathcal{H})$ such that $A_n \to A$ weakly.

7. For any $t > 0$, let $T_t \in \mathcal{B}\left(L^2(\mathbb{R})\right)$ be given by

$$T_t \varphi \quad := \quad \varphi(\cdot + t) \qquad \left(\varphi \in L^2\right) \ .$$

   (a) Calculate $\|T_t\|$.
   (b) Find a limit to which T_t converges as $t \to \infty$ (in which operator topology?).

8. Show that multiplication is not jointly continuous as a map

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

if $\mathcal{B}(\mathcal{H})$ is given the strong operator topology.

9. Let $A_n \to A, B_n \to B$ in the strong operator topology. Show that $A_n B_n \to AB$ in the strong operator topology.

10. Let $A_n \to A, B_n \to B$ in the weak operator topology. Find a counter example for $A_n B_n \to AB$ in the weak operator topology.

11. Show that for $A \in \mathcal{B}(\mathcal{H})$,

$$\|A\|_{\text{op}} = \sup\left(\{ |\langle \varphi, A\psi \rangle| \mid \|\varphi\| = \|\psi\| = 1 \}\right)$$

and if $A = A^*$ then

$$\|A\|_{\text{op}} = \sup\left(\{ |\langle \varphi, A\varphi \rangle| \mid \|\varphi\| = 1 \}\right) \ .$$

12. Show that if $A_n \geq 0$, $A_n \to A$ in norm (resp. strongly) then $\sqrt{A_n} \to \sqrt{A}$ in norm (resp. strongly).

13. Show that if $A_n \to A$ in norm then $|A_n| \to |A|$ in norm.

14. Show that if $A_n \to A$ and $A_n^* \to A^*$ strongly then $|A_n| \to |A|$ strongly.

15. Find a counter example to

$$\| |A| - |B| \| \leq \|A - B\| \, .$$

16. Let P, Q be two orthogonal projections onto subspaces M, N in a Hilbert space $\mathcal{H}$ such that $[P, Q] = 0$.
   (a) Show $P^\perp \equiv \mathbb{1} - P$, $Q^\perp$, PQ, $P + Q - PQ$ and $P + Q - 2PQ$ are orthogonal projections.
   (b) What is the relation between the projections in the previous item and M, N?

17. Let P, Q be two orthogonal projections onto subspaces M, N in a Hilbert space $\mathcal{H}$. Show that

$$\operatorname*{s-lim}_{n\to\infty} (PQ)^n$$

exists and is the orthogonal projection onto $M \cap N$.

18. Let $A \in \mathcal{B}(\mathcal{H})$. Show that the set of $\lambda \in \sigma(A)$ such that λ is not an eigenvalue of A and $\operatorname{im}(A - \lambda\mathbb{1})$ is closed but not the whole of $\mathcal{H}$ is an open subset of $\mathbb{C}$.

19. Define the numerical range $N(A)$ of $A \in \mathcal{B}(\mathcal{H})$ via

$$N(A) := \{ \langle \psi, A\psi \rangle \mid \psi \in \mathcal{H} \wedge \|\psi\| = 1 \} \ .$$

(a) Show that

$$\sigma(A) \subseteq \overline{N(A)}\,.$$

(b) Find an example where $N(A)$ is not closed and

$$\sigma(A) \not\subseteq N(A)\ .$$

(c) Find an example where

$$\sigma(A) \neq N(A) = \overline{N(A)}\,.$$

20. Show that if $A \in \mathcal{B}(\mathcal{H})$ has $A = A^*$ then

$$\left\|(A - z\mathbb{1})^{-1}\right\| \leq \frac{1}{|\mathbb{Im}\{z\}|} \qquad (z \in \mathbb{C} : |\mathbb{Im}\{z\}| > 0)\ .$$

21. Show that if $A \in \mathcal{B}(\mathcal{H})$ is an isometry then $\operatorname{im}(A)$ is closed in $\mathcal{H}$.

22. Let $V \in \mathcal{B}\left(L^2([0,1] \to \mathbb{C})\right)$ be give by

$$V(\psi) := \int_0^{\cdot} \psi \qquad \left(\psi \in L^2\right)\ .$$

(a) Show that V is well-defined (it is a bounded linear map) with

$$V^*(\psi) = \int_{\cdot}^1 \psi \qquad \left(\psi \in L^2\right)\ .$$

(b) Show that the spectral radius of V, $r(V)$, equals zero and that $\sigma(V) = \{\,0\,\}$.

(c) Show that $\|V\| = \frac{2}{\pi}$.

23. Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2\left(\mathbb{S}^1\right)$ be the Fourier series given by

$$\ell^2(\mathbb{Z}) \ni \psi \mapsto \left([0, 2\pi] \ni k \mapsto \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}kn}\psi_n =: \hat{\psi}(k)\right)\ .$$

Let $A \in \mathcal{B}\left(\ell^2(\mathbb{Z})\right)$ be the discrete Laplacian:

$$A = R + R^*$$

where R is the bilateral right shift operator

$$R\delta_n := \delta_{n+1} \qquad (n \in \mathbb{Z})$$

and $\{\,\delta_n\,\}_{n\in\mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$. Calculate

$$\mathcal{F}A\mathcal{F}^* \in \mathcal{B}\left(L^2\left(\mathbb{S}^1\right)\right)\ .$$

24. Call an operator $A \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right)\right)$ *local* iff

$$\inf_{x,y\in\mathbb{Z}^d} -\frac{1}{\|x-y\|}\log\left(|\langle\delta_x, A\delta_y\rangle|\right) > 0\,.$$

Show that if $A = A^*$ and A is local then $(A - z\mathbb{1})^{-1}$ is local too for any $z \in \mathbb{C} : |\mathbb{Im}\{z\}| > 0$.