Functional Analysis
Princeton University MAT520
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1 C-star algebras

In the following exercises, $\mathscr{A}$ is a C-star algebra with involution $*:\mathscr{A}\to\mathscr{A}$ and norm $\|\cdot\|$. $a,b,\dots\in\mathscr{A}$. Please see the corresponding section in the lecture notes for the definitions of algebraic conditions on elements in a C-star algebra.

When solving these exercises please don't forget that $\mathscr{B}(\mathscr{H})$ is a c-star algebra so everything you prove here will be useful for operators on Hilbert space.

1. Show that if a is a partial isometry (i.e. $|a|^2$ is an idempotent) then $a = aa^*a = aa^*aa^*a$.
2. Show that a is a partial isometry iff a^* is a partial isometry.
3. Show that if p,q are self-adjoint projections then $\|p-q\|\leq 1$.
4. Show that if u,v are unitary then $\|u-v\|\leq 2$.
5. Show that if a is self-adjoint with $\|a\|\leq 1$ then
$$a+\mathrm{i}\sqrt{1-a^2},\qquad a-\mathrm{i}\sqrt{1-a^2}$$
are unitary. Conclude that any $b\in\mathscr{A}$ is the linear combination of four unitaries.
6. Two self-adjoint projections p,q are said to be orthogonal (written $p\perp q$) iff $pq=0$. Show that the following are equivalent:
   (a) $p\perp q$.
   (b) $p+q$ is a self-adjoint projection.
   (c) $p+q\leq \mathbb{1}$.
7. Let $v_1,\dots,v_n$ be partial isometries and suppose that
$$\sum_{j=1}^n |v_j|^2 = \sum_{j=1}^n |v_j^*|^2 = \mathbb{1}\,.$$
Show that $\sum_{j=1}^n v_j$ is unitary.
8. Show that for any $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ such that if a obeys
$$\max\left(\left\{\,\|a-a^*\|,\left\|a^2-a\right\|\,\right\}\right)\leq\delta_\varepsilon$$
then there exists a self-adjoint projection p with $\|a-p\|\leq\varepsilon$.
9. Show that for any $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ such that if a obeys
$$\max\left(\left\{\,\left\||a|^2-\mathbb{1}\right\|,\left\||a^*|^2-\mathbb{1}\right\|\,\right\}\right)\leq\delta_\varepsilon$$
then there exists a unitary u with $\|a-u\|\leq\varepsilon$.

10. Show that $\sigma(p) \subseteq \{0, 1\}$ for an idempotent p.
11. Show that $\|p\| = 1$ for a non-zero self-adjoint projection p.
12. Show that the spectral radius $r(a)$ of a self-adjoint a equals its norm $\|a\|$.
13. Show that $\sigma(u) \subseteq \mathbb{S}^1$ if u is unitary (i.e. $|u|^2 = |u^*|^2 = \mathbb{1}$).
14. Show that $\sigma(a) \subseteq [0, \infty)$ if a is positive (i.e. $a = |b|^2 \, \exists b$).
15. Show that $\sigma(a) \subseteq \mathbb{R}$ if $a = a^*$.
16. Show that a is invertible if $|a|^2 \geq \varepsilon\mathbb{1}$ for some $\varepsilon > 0$ and $|a^*|^2 \geq \delta\mathbb{1}$ for some $\delta > 0$; (recall $a \geq b$ iff $a - b \geq 0$ iff $a - b = |c|^2$ for some c).