Functional Analysis
Princeton University MAT520
HW6 assigned on Oct 24th 2025

October 24, 2025

1 Hilbert spaces

In the following exercises, $\mathcal{H}$ is a Hilbert space and $\varphi, \psi, \dots$ are vectors in it.

1. Show that $\ell^2(\mathbb{N} \to \mathbb{C})$ is a Hilbert space: define an inner product on it and show that the induced metric is complete.
2. Show that $L^2(\mathbb{R})$ (with the Lebesgue measure) is a Hilbert space. Define an inner product and show that the induced metric is complete.
3. Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of bounded linear operators on $\mathcal{H}$. Show (in a concrete example, e.g., $\mathcal{H} = \mathbb{C}^2$) that $\mathcal{B}(\mathcal{H})$ is *not* a Hilbert space by showing the operator norm violates the parallelogram law.
4. Show that when $\dim(\mathcal{H}) = \infty$ then
$$\mathcal{H} \otimes \mathcal{H}^* \subsetneq \mathcal{B}(\mathcal{H}) .$$
Note: this may be hard.
5. Show that if $M \subseteq \mathcal{H}$ is a closed vector subspace of it then $\left(M^\perp\right)^\perp = M$.
6. Show that if $\{ \varphi_n \}_{n\in\mathbb{N}}$ is a sequence of *pairwise orthogonal* vectors in $\mathcal{H}$, then the following are equivalent:
   (a) $\sum_{n\in\mathbb{N}} \varphi_n$ exists in $\|\cdot\|_{\mathcal{H}}$.
   (b) $\sum_{n\in\mathbb{N}} \|\varphi_n\|_{\mathcal{H}}^2 < \infty$.
   (c) For any $\psi \in \mathcal{H}$, $\sum_{n\in\mathbb{N}} \langle \psi, \varphi_n \rangle_{\mathcal{H}}$ exists.
7. Show that if $\{ \varphi_n \}_{n\in\mathbb{N}}$ is a sequence of vectors in $\mathcal{H}$, then item (a) above implies item (c) above. Find an example where item (c) does *not* imply item (a).
8. Let $N \in \mathbb{N}$, $\alpha \in \mathbb{C}$ with $\alpha^N = 1$ and $\alpha^2 \neq 1$. Show that in $\mathcal{H}$, for any $\varphi, \psi \in \mathcal{H}$:
$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \frac{1}{N} \sum_{n=1}^{N} \alpha^n \|\psi + \alpha^n \varphi\|^2 .$$
Show also that
$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\theta} \left\|\psi + \mathrm{e}^{\mathrm{i}\theta} \varphi\right\|^2 \mathrm{d}\theta .$$
9. Let
$$\{ \varphi_n \}_{n\in\mathbb{N}}, \{ \psi_n \}_{n\in\mathbb{N}} \subseteq \{ \xi \in \mathcal{H} \mid \|\xi\| \leq 1 \} .$$
Assume further that $\lim_{n\in\mathbb{N}} \langle \varphi_n, \psi_n \rangle \to 1$. Show that
$$\lim_{n\in\mathbb{N}} \|\varphi_n - \psi_n\| = 0 .$$

10. Let $\{ \varphi_n \}_n \subseteq \mathcal{H}$ converge to some $\varphi \in \mathcal{H}$ weakly (i.e., for any $\xi \in \mathcal{H}$, $\langle \xi, \varphi_n \rangle \to \langle \xi, \varphi \rangle$ in $\mathbb{C}$). Assume further that $\|\varphi_n\| \to \|\varphi\|$ in $\mathbb{R}$. Show that
$$\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0\,.$$

11. Let V be an inner product space and $\{ \varphi_n \}_{n=1}^{N} \subseteq V$ be an orthonormal set. Show that, for fixed ψ, the functional
$$F(\alpha_1, \dots, \alpha_N) := \left\| \psi - \sum_{n=1}^{N} \alpha_n \varphi_n \right\|$$
of the N numbers $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ is minimized with the choice $\alpha_n := \langle \varphi_n, \psi \rangle$.

12. Prove that if A and B are two disjoint measure spaces then
$$L^2(A \sqcup B) \quad \cong \quad L^2(A) \oplus L^2(B)\ .$$

13. Prove that if A and B are two measure spaces then
$$L^2(A \times B) \quad \cong \quad L^2(A) \otimes L^2(B)\ .$$

14. Show that
$$\mathcal{H} := \ell^2(\mathbb{R}) \equiv \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; f^{-1}(\mathbb{C} \setminus \{ 0 \}) \text{ is a countable set and } \sum_{x\in\mathbb{R}} |f(x)|^2 < \infty \right\}$$
is a non-separable Hilbert space.