MAT520 HW5

November 7, 2025

1. See Lemma 6.24 in LN.

3. Since $\sigma(a)$ is disconnected, we can write $\sigma(a) \subset U \cup V$ where U, V are open subsets of $\mathbb{C}$, each of which encloses some components of $\sigma(a)$. Consider the holomorphic function f that is 1 on U and 0 on V. Define $b = f(a)$. Then $b^2 = (f(a))^2 = f(a) = b$ since f squares to itself as a function. In particular b is nontrivial since $\sigma(f(a)) = f(\sigma(a)) = \{0, 1\}$.

4. If $\sigma(a)$ is connected, then the result follows from Theorem 10.20 in Rudin. Otherwise suppose $\sigma(a) \subset \Omega_0 \cup \Omega$ where Ω_0 and Ω are disjoint open subsets and Ω enclosed a component of $\sigma(a)$. For all n large enough, we have $\sigma(a_n) \subset \Omega_0 \cup \Omega$ by Theorem 10.20 in Rudin. Consider the holomorphic function f that is 1 on Ω and 0 on Ω_0. Now $f(a) \neq 0$. Thus
$$\|f(a_n)\| \geq \|f(a)\| - \|f(a_n) - f(a)\| > 0$$
for all n large, since $\|f(a_n) - f(a)\| \to 0$ (to show this, we can use the estimate in Lemma 6.14 and the integral formula (6.7) of Theorem 6.28 in the lecture note, and also the resolvent identity.) Thus $\sigma(a_n) \cap \Omega \neq \varnothing$ for all n large; otherwise $f(a_n) = 0$.

5. That (b) implies (a) is clear. If (a) is true, than $TR(A) = R(B)T$ holds for rational functions R without poles in U. We can approximate a holomorphic function f on U by rational functions $\{R_n\}$ without poles in U, uniformly on compact subsets of U. Thus $R_n(A) \to f(A)$ and $R_n(B) \to f(B)$ in norm (see Theorem 6.28 in the lecture note), and hence $Tf(A) = f(B)T$.

7. Since $\sigma(a)$ is compact, there is $z \in \sigma(a)$ such that $|z| = \operatorname{dist}(0, \sigma(a)) \neq 0$. Now
$$a^{-1} - z^{-1} = a^{-1}(z - a)z^{-1}.$$
Since $z - a$ is not invertible, it follows that $a^{-1} - z^{-1}$ is not. Thus $z^{-1} \in \sigma(a^{-1})$. Then
$$\|a^{-1}\| \geq r(a^{-1}) \geq |z^{-1}| = \frac{1}{\operatorname{dist}(0, \sigma(a))}.$$

8. Using Theorem 6.28 in LN, we can prove the second part. By the second part, we have that for all $\varepsilon > 0$ there is N such that for $m \geq N$, then $r(a_m) \leq r(a) + \varepsilon$. Thus

$$\limsup_n r(a_n) := \inf_N \sup_{m \geq N} r(a_m) \leq \sup_{m \geq N} r(a_m) \leq r(a) + \varepsilon$$

Take $\varepsilon \to 0$.

9. See Theorem 11.23 in Rudin.

10. If $\mathcal{B}$ is a Banach subalgebra of $\mathcal{A}$, then it is clear that $\sigma_{\mathcal{A}}(a) \subset \sigma_{\mathcal{B}}(a)$ for all $b \in \mathcal{B}$. To find an example where the inclusion is strict, consider $\mathcal{A} := C(\mathbb{S}^1)$ the continuous function on circle. The function z has spectrum $\sigma_{\mathcal{A}}(z) = \mathbb{S}^1$. Let $\mathcal{B}$ be the closure of algebra generated by z and 1 (the constant function mapping to 1). Then we have $0 \in \sigma_{\mathcal{B}}(z)$. Indeed, the inverse of z is $|z|$ which cannot be written as polynomial and hence not in $\mathcal{B}$.

11. Since $a^2 = a$, we have $\{z^2 : z \in \sigma(a)\} = \{z : z \in \sigma(a)\}$. It follows that $\sigma(a) \subset \{0, 1\}$.

12. See Lemma 2.1 in Gohberg, Classes of Linear Operators Vol.I. The idea is to use the resolvent identity.