

Functional Analysis

Princeton University MAT520

HW5 Oct 2025

October 11, 2025

1 Banach algebras and the spectra of elements in it

In the following, $\mathcal{A}$ is a $\mathbb{C}$ -Banach algebra.

1. Prove Fekete's lemma: If $\{a_n\}_n \subseteq \mathbb{R}$ is sub-additive then $\lim_{n \rightarrow \infty} \frac{1}{n}a_n$ exists and equals $\inf \frac{1}{n}a_n$.
2. Let $R : \mathbb{C} \rightarrow \mathbb{C}$ be a rational function, i.e.,

$$R(z) = p(z) + \sum_{k=1}^n \sum_{l=1}^q c_{k,l} (z - z_k)^{-l}$$

where p is a polynomial, $n \in \mathbb{N}$, and $\{z_k\}_k, \{c_{k,l}\}_{k,l} \subseteq \mathbb{C}$. Let now $a \in \mathcal{A}$ such that $\{z_k\}_{k=1}^n \subseteq \rho(a)$.

Assume further that we choose some $\sigma(a) \subseteq \Omega \in \text{Open}(\mathbb{C})$ such that R is holomorphic on Ω , and $\gamma_j : [a, b] \rightarrow \Omega$, $j = 1, \dots, m$ a collection of m oriented loops which surround $\sigma(a)$ within Ω , such that

$$\frac{1}{2\pi i} \sum_{j=1}^m \oint_{\gamma_j} \frac{1}{z - \lambda} dz = \begin{cases} 1 & \lambda \in \sigma(a) \\ 0 & \lambda \notin \Omega \end{cases}.$$

Using Lemma 6.26 in the lecture notes (= Lemma 10.24 in Rudin) show that $R(a)$ obeys the Cauchy integral formula, in the sense that

$$p(a) + \sum_{k=1}^n \sum_{l=1}^q c_{k,l} (a - z_k)^{-l} = \frac{1}{2\pi i} \sum_{j=1}^m \oint_{\gamma_j} R(z) (z\mathbb{1} - a)^{-1} dz.$$

3. Let $\mathcal{A}$ be such that there exists some $a \in \mathcal{A}$ with $\sigma(a)$ not connected. Show that then $\mathcal{A}$ contains some non-trivial idempotent (an element $b \in \mathcal{A}$ with $b^2 = b \notin \{0, \mathbb{1}\}$).
4. Assume that $\{a_n\}_n \subseteq \mathcal{A}$ is a sequence such that $\exists \lim_n a_n =: a \in \mathcal{A}$. Let $\Omega \in \text{Open}(\mathbb{C})$ contains a component of $\sigma(a)$. Show that $\sigma(a_n) \cap \Omega \neq \emptyset$ for all sufficiently large n . *Hint:* If $\sigma(a) \subseteq \Omega \sqcup \tilde{\Omega}$ where $\tilde{\Omega} \in \text{Open}(\mathbb{C})$ (in particular this means $\Omega \cap \tilde{\Omega} = \emptyset$), define $f : \mathbb{C} \rightarrow [0, 1]$ such that $f|_{\Omega} = 1$ and $f|_{\tilde{\Omega}} = 0$.
5. Let X, Y be two Banach spaces and A, B be two bounded linear operators on X, Y respectively. Let $T \in \mathcal{B}(X \rightarrow Y)$. Show that the following two assertions are equivalent:
 - (a) $TA = BT$.
 - (b) $Tf(A) = f(B)T$ for any $f : \mathbb{C} \rightarrow \mathbb{C}$ holomorphic in some open set U which contains $\sigma(A) \cup \sigma(B)$.
6. Show that if $a, b \in \mathcal{A}$ and $b \in \mathcal{G}$ then

$$\sigma(bab^{-1}) = \sigma(a).$$

7. Show that if $a \in \mathcal{G}$ then

$$\|a^{-1}\| \geq \frac{1}{\text{dist}(0, \sigma(a))}.$$

8. [Upper semicontinuity of the spectral radius] If $\{a_n\}_n \subseteq \mathcal{A}$ and $a_n \rightarrow a \in \mathcal{A}$ then

$$\sup_n r(a_n) \leq r(a) .$$

Next show that for any $\varepsilon > 0$ there exists some $N_\varepsilon \in \mathbb{N}$ such that if $n \geq N_\varepsilon$ then

$$\sigma(a_n) \subseteq \sigma(a) + B_\varepsilon(0_{\mathbb{C}}) .$$

9. Show that if $a, b \in \mathcal{A}$ with $[a, b] \equiv ab - ba = 0$ then

$$\sigma(a + b) \subseteq \sigma(a) + \sigma(b)$$

and

$$\sigma(ab) \subseteq \sigma(a)\sigma(b) .$$

10. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B} \subseteq \mathcal{A}$ be a sub-Banach-algebra (defined in the obvious way). Let $a \in \mathcal{B}$. Show that

$$\sigma_{\mathcal{A}}(a) \subseteq \sigma_{\mathcal{B}}(a) .$$

Find an example where the inclusion is strict.

11. Show that if $a^2 = a \in \mathcal{A}$ (i.e., an idempotent) then $\sigma(a) \subseteq \{0, 1\}$.
 12. Let $a \in \mathcal{A}$ and assume that $\sigma(a) = K_1 \sqcup K_2$ with $\text{dist}(K_1, K_2) > 0$. Let Γ_i encircle only K_i . Set

$$p_i := \frac{i}{2\pi} \oint (a - z\mathbf{1})^{-1} dz .$$

Show that p_i is an idempotent, i.e., that $p_i^2 = p_i$, show that $[p_i, a] = 0$.

13. Show that if $f, g : \tilde{\Omega} \rightarrow \mathbb{C}$ are holomorphic, $\Omega, \tilde{\Omega} \in \text{Open}(\mathbb{C})$, $a \in \mathcal{A}$ with $\sigma(a) \subseteq \Omega \subseteq \tilde{\Omega}$ and $f|_{\Omega} = g|_{\Omega}$ then $f(a) = g(a)$.
 14. Show that if R is the unilateral right shift on $\ell^2(\mathbb{N})$ then $r(R) = 1$.