MAT520 HW4

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1. Normed-closed convex subset K is weakly-cosed. To see this, for any $x_0 \in X \setminus K$, since K is normed-closed and convex and $\{x_0\}$ is (strongly-)compact and convex in X, apply the Hahn-Banach separation theorem (Theorem 3.4 in Rudin's Functional Analysis), there exists $\lambda \in X^*$ such that

$$\mathbb{Re}\{\lambda\}(x_0) < \gamma < \mathbb{Re}\{\lambda\}(y)$$

for some $\gamma \in \mathbb{R}$ and for all $y \in K$. In particular, we have $\{x \in X : |\lambda(x - x_0)| < \epsilon\} \subset X \setminus K$ for some ϵ small enough. If the closed unit ball B in X is weakly compact, then with $rK \subset B$ for r small by boundedness of K, we concude that rK and hence K is weakly-compact (note weak topology on X is Hausdorff). To show that B is weakly compact, we consider $X \cong X^{**}$ by reflexivity of X. In fact, with respect to the weak topology on X and weak-star topology on X^{**}, the spaces X and X^{**} are homeomorphic. Indeed, $x_\alpha \to x$ converges weakly in X if and only if $J(x_\alpha) \to J(x)$ in the weak-star sense, where $J : X \to X^{**}$ is the canonical map, since both translate to $\lambda(x_\alpha) \to \lambda(x)$ for all $x \in X^*$. Now $J(B)$ is the closed unit ball in X^{**} and hence is weak-star compact by the Banach-Alaoglu theorem. Thus B is weakly-compact.

2. (i.) (Use the Banach-Alaoglu theorem to exhibit an element of $(\ell^\infty)^*$ which is not in ℓ^1.) It is clear that $\mu_n \in (\ell^\infty)^*$ and $\|\mu_n\| \leq 1$ and we can apply the Banach-Alaoglu theorem on the sequence $\{\mu_n\}_{n=1}^\infty$ to find an element μ in the closed unit ball of $(\ell^\infty)^*$ such that for any weak-star neighborhood U of μ, we have $\mu_n \in U$ for infinitely many n. Let $e_j \in \ell^\infty$ be the vector that takes value 1 in the j-th position and zero otherwise. Since $\mu_n(e_j) \to 0$, we must have $\mu(e_j) = 0$; otherwise $\{\mu_n\} \cap \{\eta \in (\ell^\infty)^* | |(\eta - \mu)(e_j)| < \epsilon\}$ has finitely many terms. Let $a \in \ell^\infty$ be the all 1 vector. We have $\mu(a) = 1$ by similar reasoning. Now, consider the canonical map $J : \ell^1 \to (\ell^\infty)^*$ where $\{x_j\}$ is mapped to the functional $\lambda : \{a_j\} \mapsto \sum_j a_j x_j$. Suppose $\mu = J(x)$ for some $x \in \ell^1$. We have $x_j = J(x)(e_j) = \mu(e_j) = 0$ for all j. Thus $J(x) = 0$. However $\mu \neq 0$.

   (ii.) (Show that $\ell^\infty \cong (\ell^1)^*$.) Let $J : \ell^\infty \to (\ell^1)^*$ map $\{x_j\}$ to a functional $\lambda : \{a_j\} \mapsto \sum_j a_j x_j$. It is clear that J is injective. To show surjectivity, for $\lambda \in (\ell^1)^*$, let $x_j := \lambda(e_j)$, and we have $J(\{x_j\}) = \lambda$. Apply Hahn-Banach to show that J is isometric.

3. The dual of L^p for $p \in (1,\infty)$ is L^q where $1/p + 1/q = 1$. Since $L^q([-\pi,\pi]) \subset L^1([-\pi,\pi])$, we will show that for any $f \in L^1([-\pi,\pi])$, we have $\hat{f}(n) := \int_{-\pi}^{\pi} f(t)e^{int}dt \to 0$ as $n \to \infty$. We know that the trigonometric polynomials are dense in $C([-\pi,\pi])$ in sup norm, and $C([-\pi,\pi])$ is dense in $L^1([-\pi,\pi])$ in L^1 norm. For $f \in L^1$, find trigonometric polynomial p such that $\|f-p\|_\infty < \epsilon$ and find $g \in L^1$ such that $\|f-g\|_1 < \epsilon$. Then
$$|\hat{f}(n)| \le |\hat{f}(n) - \hat{g}(n)| + |\hat{g}(n) - \hat{p}(n)| + |\hat{p}(n)| \le 2\epsilon + |\hat{p}(n)|$$
since $\hat{p}(n) \to 0$, for sufficiently large n we have $|\hat{f}(n)| \le 2\epsilon$. Now if $f_n \to g$ in norm, then $g = 0$. However $\|f_n\|_p = 1$.

4. (Show $C([0,1])$ is dense in $L^\infty([0,1])$ with respect to the weak-star topology and not with respect to the norm topology.) Let η be the standard mollifier (see, e.g., Section C5 in Evans' Partial Differential Equation) and $\eta_\epsilon(x) = \frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$. If $f \in L^\infty$, we will show that $\int \eta_\epsilon * fg \to \int fg$ for all $g \in L^1$, and note that $\eta_\epsilon * f$ is smooth. Since $\int |\eta_\epsilon(x-y)f(y)||g(x)|dxdy \le \|\eta_\epsilon\|_\infty \|f\|_\infty \|g\|_{L^1}$, we can use Fubini's theorem to get $\int \eta_\epsilon * fg = \int \eta_\epsilon * gf$. Thus
$$\left| \int \eta_\epsilon * fg - \int fg \right| \le \int |f||\eta_\epsilon * g - g| \le \|f\|_\infty \|\eta_\epsilon * g - g\| \to 0$$
as $\epsilon \to 0$, since $\eta_\epsilon * g \to g$ in L^1. For the norm topology, we now that $C([0,1])$ is closed in $L^\infty([0,1])$ in this topology. Since $C([0,1]) \subsetneq L^\infty([0,1])$, it cannot be dense.

5. First we show that $B \subset \overline{S}$. Let $\|x_0\| < 1$. We need to show that
$$\{x : |\lambda_i(x - x_0)| < \epsilon\} \cap S$$
is nonempty for any $\lambda_1, \dots, \lambda_n \in X^*$ and $\epsilon > 0$. The map $(\lambda_1, \dots, \lambda_n) : X \to \mathbb{R}^n$ has nontrivial kernel; otherwise we will have the contradiction that $\dim X \le n$. Denote $y_0 \ne 0$ the be the vector such that $\lambda_i(y_0) = 0$ for all i. Since $\alpha \mapsto \|x_0 + \alpha y_0\|$ is continuous, and $\|x_0\| < 1$ and $\|x_0 + \alpha y_0\| \to \infty$ as $|\alpha| \to \infty$, by the intermediate value theorem, there is some α such that $\|x_0 + \alpha y_0\| = 1$. Thus $x_0 + \alpha y_0 \in S$ and $\lambda_i(x_0 + \alpha y_0 - x_0) = 0 < \epsilon$. To show $\overline{S} \subset B$, we note that B is weakly-closed since
$$B = \bigcap_{\|\lambda\|=1} \{x : |\lambda(x)| \le 1\}$$
which follows from $\|x\| = \sup_{\|\lambda\|=1} |\lambda(x)|$.

6. We have

$$|L_n(x_n) - L(x)| \leq |L_n(x_n) - L_n(x)| + |L_n(x) - L(x)|$$

The second term converges to zero since $L_n \to L$ in the weak-star sense. Also, since $|L_n(x)|$ is bounded for each $x \in X$, then $\|L_n\|$ is bounded by the uniform boundedness principle. Thus

$$|L_n(x_n) - L_n(x)| \leq \|L_n\|\|x_n - x\| \to 0$$

8. Use the Gelfand's formula for spectral radius.

9. $x^{-1}(xy) = y \in \mathcal{G}$.

10. One can construct left and right inverses for x and y.

11. $LR = \mathbb{1}$ and RL projects onto $n \geq 2$.

12. If $\lambda \neq 0$, then $\lambda - xy$ is invertible if and only if $\lambda - yx$ is invertible. This follows exactly the same as Problem 11. Take R and L from Problem 10. Then LR is invertible while RL is not.

14. If z is on the boundary of $\sigma(x)$, then there is a sequence $z_n \to z$ such that $x - z_n$ is invertible. In particular, any neighborhood balls of $x - z$ intersects $x - z_n$ for some n.

15. Take $x_n \to x$ where $x_n \in \mathcal{G}$. We have $\|x_n^{-1}\| \to \infty$. Indeed, xx_n^{-1} is not invertible and hence $1 \leq \|\mathbb{1} - xx_n^{-1}\|$. Thus

$$1 \leq \|\mathbb{1} - xx_n^{-1}\| = \|(x - x_n)x_n^{-1}\| \leq \|x - x_n\|\|x_n^{-1}\|$$

and $\|x_n^{-1}\| = 1/\|x - x_n\| \to \infty$. Let $y_n = x_n^{-1}/\|x_n^{-1}\|$. Then

$$\|xy_n\| = \frac{\|xx_n^{-1}\|}{\|x_n\|} = \frac{\|(x - x_n)x_n^{-1} + \mathbb{1}\|}{\|x_n^{-1}\|} \leq \|x - x_n\| + \frac{1}{\|x_n^{-1}\|} \to 0$$

If $\mathcal{A}$ is a Banach algebra whose nonzero elements are invertible, then by Gelfand-Mazur $\mathcal{A} = \mathbb{C}$, and 0 is the only topological divisor of 0.

16. Here $\ell^2(\mathbb{N})$ is a Hilbert space, and we can talk about the adjoint of T. It is not hard to find that T is unitary and $T^2 = -\mathbb{1}$, which implies $\sigma(T)$ belongs to the unit circle and $\sigma(T) \subset \{i, -i\}$, respectively. Thus $\sigma(T) = \{i, -i\}$ since T is not identically i or $-i$.

17. $r(x) = \inf_n \|x^n\|^{1/n} = 0$.

18. We need to show that $\{x \in \mathcal{A} : r(x) < \alpha\}$ is open for any $\alpha > 0$. If $r(x_0) < \alpha$, then $\sigma(x_0) \subset B(0, \alpha - \epsilon)$. We use Theorem 10.20 in Rudin's Functional Analysis to find $\delta > 0$ such that for all $\|x - x_0\| < \delta$, we have $\sigma(x) \subset B(0, \alpha - \epsilon)$. Thus $r(x) < \alpha$.

19. See Proposition 3.1, Chapter 1 in Stein and Shakarchi's Complex Analysis, and the discussion there for the invariance of reparametrization.

20. See Lemma 6.13 in LN.

21. See Rudin Theorem 3.31.

22. If f is strongly holomorphic, then it is clearly weakly holormohpci, i.e., $\phi \circ f(\cdot)$ is a complex-valued holomorphic function. Using Goursat's theorem (Theorem 1.1 in Stein and Shakarchi's Complex Analysis), we conclude that $\int_{\partial\Delta} \phi \circ f = 0$ for every triangle. Then we apply Problem 20 in this PSet to conclude.

23. See Chapter 3.5 in Stein and Shakarchi's Complex Analysis.

24. See Theorem 5.1, Chapter 2 in Stein and Shakarchi's Complex Analysis.

25. See Theorem 6.18 in LN.

26. See Theorem 6.20 in LN.

27. See Theorem 4.5, Chapter 3 in Stein and Shakarchi's Complex Analysis.

28. See Theorem 5.3, Chapter 2 in Stein and Shakarchi's Complex Analysis.