Functional Analysis
Princeton University MAT520
HW4, Assigned on Sep 26th 2025

September 27, 2025

Here's a list of problems I thought were nice, solve as many as you like and as many as doesn't cause you sleep deprivation.

1 Weak stuff

1. Prove that any norm-closed convex bounded subset of a reflexive Banach space is weakly compact.
2. Define the Banach space $\ell^\infty(\mathbb{N}) := \{ a : \mathbb{N} \to \mathbb{C} \mid \|a\|_\infty < \infty \}$ with the norm $\|a\|_\infty \equiv \sup_n |a_n|$. Also define
$$\ell^1(\mathbb{N}) := \{ a : \mathbb{N} \to \mathbb{C} \mid \|a\|_1 < \infty \}$$
with the norm $\|a\|_1 := \sum_{n\in\mathbb{N}} |a_n|$.
Our goal here is to use the Banach-Alaoglu theorem to exhibit an element of $(\ell^\infty)^*$ which is not in ℓ^1.
   (a) Define $\{ \mu_n \}_n \subseteq (\ell^\infty)^*$ via
$$\mu_n(a) := \frac{1}{n}\sum_{j=1}^{n} a_j \qquad (a \in \ell^\infty, n \in \mathbb{N}) .$$
Show that $\mu_n \in (\ell^\infty)^*$ indeed and $\|\mu_n\| \leq 1$.
   (b) Show there exists some $\mu \in (\ell^\infty)^*$ that is the limit of $\{ \mu_n \}_n$ (in the weak-star topology).
   (c) Show that $(\ell^1)^* = \ell^\infty$.
   (d) Hence we may think of $J(\ell^1) \subseteq (\ell^\infty)^*$ where J is the natural isometric injection. Show that the limit μ constructed above does *not* lie in $J(\ell^1)$. That is, show that for any $x \in \ell^1$, $J(x) \neq \mu$.
3. Let $\{ f_n \}_n$ be given by
$$f_n(t) := e^{int} \qquad (t \in [-\pi, \pi]) .$$
Show that if $p \in [1, \infty)$ then $f_n \to 0$ weakly in $L^p([-\pi, \pi])$, but not in the norm topology of $L^p([-\pi, \pi])$.
4. Consider $L^\infty([0,1])$ with its norm topology (the essential supremum norm), and, since $(L^1([0,1]))^* = L^\infty([0,1])$, the weak-star topology on $(L^1([0,1]))^*$, which is a topology on $L^\infty([0,1])$. Show that $C([0,1])$ (the space of all continuous functions) is dense in L^α but not in L^β, for either $(\alpha, \beta) = (1, \infty)$ or $(\alpha, \beta) = (\infty, 1)$.
5. Let X be an infinite-dimensional Banach space and define
$$S := \{ x \in X \mid \|x\| = 1 \} .$$
Show that the weak-closure of S is
$$B := \{ x \in X \mid \|x\| \leq 1 \} .$$
6. Let X be a Banach space, and $\{ L_n \}_n \subseteq X^*$ be a sequence which converges to some $L \in X^*$ in the weak-star sense. Assume that $\{ x_n \}_n \subseteq X$ converges to some $x \in X$ in norm. It is true that $L_n(x_n) \to L(x)$ in $\mathbb{C}$?
7. Find an example of a Banach space X for which there does *not* exist a Banach space Y such that $Y^* = X$.

2 Banach algebras

Here $\mathcal{A}$ is a Banach algebra and $x, y, \ldots$ are elements in it; $\mathcal{G}_{\mathcal{A}}$ is the set of invertible elements and $r : \mathcal{A} \to [0, \infty)$ is the spectral radius.

8. Use $(xy)^n = x\,(yx)^{n-1}\,y$ to show that $r\,(xy) = r\,(yx)$.

9. Show that if $x, xy \in \mathcal{G}_{\mathcal{A}}$ then $y \in \mathcal{G}_{\mathcal{A}}$.

10. Show that if $xy, yx \in \mathcal{G}_{\mathcal{A}}$ then $x, y \in \mathcal{G}_{\mathcal{A}}$.

11. On the Banach space $\ell^2\,(\mathbb{N} \to \mathbb{C})$, define the right shift operator $R \in \mathcal{B}\left(\ell^2\,(\mathbb{N} \to \mathbb{C})\right)$:

$$(Ra)_n := \begin{cases} a_{n-1} & n \geq 2 \\ 0 & n = 1 \end{cases}$$

and the right shift operator

$$(La)_n := a_{n+1} \qquad (n \in \mathbb{N})\;.$$

Calculate RL and LR. Conclude that one may have $xy = \mathbb{1}$ but $yx \neq \mathbb{1}$ in a Banach algebra.

12. Show that if $z \in \mathbb{C} \setminus \{\,0\,\}$ then $z \in \sigma\,(xy)$ iff $z \in \sigma\,(yx)$. I.e.,

$$\sigma\,(xy) \cup \{\,0\,\} = \sigma\,(yx) \cup \{\,0\,\}\;.$$

Find an example where $\sigma\,(xy) \neq \sigma\,(yx)$.

13. Define $\mathcal{A} := C^2\,([0,1] \to \mathbb{C})$, the space of functions with continuous second derivative. Define, for $a, b > 0$,

$$\|f\| := \|f\|_\infty + a\|f'\|_\infty + b\|f''\|_\infty\,.$$

Show that $\mathcal{A}$ is a Banach space. Show that $\mathcal{A}$ is a Banach algebra (with pointwise multiplication) iff $a^2 \geq 2b$. You may consider the functions $x \mapsto x$ and $x \mapsto x^2$.

14. Show that if $z \in \partial\sigma\,(x)$ then $x - z\mathbb{1} \in \partial\mathcal{G}_{\mathcal{A}}$.

15. Let $x \in \partial\mathcal{G}_{\mathcal{A}}$. Show there exists some $\{\,y_n\,\}_n \subseteq \mathcal{A}$ with $\|y_n\| = 1$ and

$$\lim_{n\to\infty} xy_n = \lim_{n\to\infty} y_n x = 0\,.$$

Try to characterize the type of Banach algebras in which there are such elements x (which are called *topological divisor of zero*).

16. On $\ell^2\,(\mathbb{N} \to \mathbb{C})$ define $T \in \mathcal{B}\left(\ell^2\,(\mathbb{N} \to \mathbb{C})\right)$ via

$$T\,(a_1, a_2, a_3, a_4, \ldots) := (-a_2, a_1, -a_4, a_3, \ldots)\;.$$

Calculate $\sigma\,(T)$.

17. Show that if $x \in \mathcal{A}$ is nilpotent (i.e. $\exists n \in \mathbb{N}$ with $x^n = 0$) then $\sigma\,(x) = \{\,0\,\}$.

18. Show that r is upper semicontinuous.

3 Banach-space-valued holomorphic functions and Cauchy integrals

Throughout, let X be a complex Banach space, $\Omega \subset \mathbb{C}$ open, and curves piecewise C^1. For $\gamma : [a,b] \to \Omega$ and continuous $f : \Omega \to X$, define the (Bochner) contour integral

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt,$$

whenever the Bochner integral exists (it does for continuous integrands).

19. **Bochner contour basics.**

(a) Show that $\int_\gamma (\alpha f + \beta g)\, dz = \alpha \int_\gamma f\, dz + \beta \int_\gamma g\, dz$ and that $\left\| \int_\gamma f\, dz \right\| \le \operatorname{len}(\gamma) \sup_{z\in\gamma} \|f(z)\|$.

(b) Prove invariance under C^1 reparametrization and orientation reversal.

(c) If γ and η are composable, show $\displaystyle\int_{\gamma * \eta} f\, dz = \int_\gamma f\, dz + \int_\eta f\, dz$.

20. **Scalarization principle.** Let $f : \Omega \to X$ be continuous. Prove that for every $\phi \in X^*$,

$$\phi\left(\int_\gamma f(z)\, dz \right) = \int_\gamma \phi(f(z))\, dz.$$

Deduce: if $\int_\gamma \phi(f) = 0$ for all $\phi \in X^*$, then $\int_\gamma f = 0$.

21. **Weak vs. strong holomorphy.** Define $f : \Omega \to X$ to be holomorphic if it is Fréchet differentiable at each point.

(a) Show: if $\phi \circ f$ is scalar holomorphic for every $\phi \in X^*$ and f is locally bounded, then f is holomorphic.

(b) Give an example showing local boundedness is necessary in (a).

22. **Cauchy's theorem (vector-valued).** Suppose $f : \Omega \to X$ is holomorphic. Show that for every triangle $\Delta \subset \Omega$, $\int_{\partial\Delta} f(z)\, dz = 0$. *Hint:* Apply (2) to $\phi \circ f$ and use the scalar Cauchy theorem.

23. **Path independence and primitives.**

(a) Prove: if Ω is simply connected and f holomorphic, then $\int_\gamma f$ depends only on the endpoints of γ.

(b) Show: if $\int_\gamma f = 0$ for every closed γ in Ω, then there exists $F : \Omega \to X$ with $F' = f$.

(c) Conversely, if $F' \equiv f$, prove $\displaystyle\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$.

24. **Morera's theorem (vector-valued).** Let $f : \Omega \to X$ be continuous and assume $\int_{\partial\Delta} f = 0$ for every triangle $\Delta \subset \Omega$. Show that f is holomorphic. *Hint:* Combine scalar Morera with (3a).

25. **Cauchy integral formula.** Let $D(a, r)$ be the open disc with $\overline{D(a, r)} \subset \Omega$ and set $\Gamma = \partial D(a, r)$ with positive orientation. For holomorphic f, prove

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - a}\, d\zeta, \qquad f^{(n)}(a) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta.$$

Hint: Test against $\phi \in X^*$ and use scalar Cauchy, then invoke (2).

26. **Cauchy estimates and Liouville.** With the notation of (7), show

$$\|f^{(n)}(a)\| \le \frac{n!}{r^n} \max_{|\zeta - a| = r} \|f(\zeta)\|.$$

Deduce: if $f : \mathbb{C} \to X$ is entire and bounded, then f is constant.

27. **Maximum modulus principle (norm version).** Suppose $f : \Omega \to X$ is holomorphic and $\|f\|$ attains a local maximum at $a \in \Omega$. Show that f is locally constant (hence constant on the component of Ω containing a). *Hint:* Use Hahn–Banach to find ϕ with $\|\phi\| = 1$ and $\phi(f(a)) = \|f(a)\|$, then apply the scalar maximum modulus principle to $\phi \circ f$.

28. **Uniform limits.** Let $f_n : \Omega \to X$ be holomorphic and $f_n \to f$ uniformly on compact subsets of Ω.

(a) Prove that f is holomorphic.

(b) Show $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets for every $k \ge 0$. *Hint:* Use Cauchy's formula on small circles and pass to the limit.