

MAT520 HW3

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1. See Theorem 4.5 in the lecture note.
3. Suppose X^* is reflexive but $X \neq X^{**}$. Denote $J : X \rightarrow X^{**}$ the canonical map. Let $f \in X^{**} \setminus J(X)$, and define $\ell : J(X) + \mathbb{C}f \rightarrow \mathbb{C}$ as $\ell(J(x) + tf) = t$. Note ℓ is bounded since $J(X)$ is closed in X^{**} , where $\inf_x \|J(x) + f\| = \delta > 0$ and $\inf_x \|J(x) + tf\| = |t|\delta$. Extend ℓ using Hahn-Banach to a bounded linear functional $\tilde{\ell}$ on X^{**} . Write $\psi : X^* \rightarrow X^{***}$. Since $X^* = X^{***}$, there is $\lambda \in X^*$ such that $\psi(\lambda) = \tilde{\ell}$. In particular $\psi(\lambda)(J(x)) = \tilde{\ell}(J(x)) = \ell(J(x)) = 0$, and this implies

$$0 = \psi(\lambda)(J(x)) = J(x)(\lambda) = \lambda(x)$$

for all $x \in X$. Thus $\lambda = 0$. However, $\tilde{\ell} \neq 0$, which is a contradiction.

4. This is similar to the previous one. Suppose $f^* : Y \rightarrow X^*$ is not surjective. Choose $\lambda_0 \in X^* \setminus f^*(Y)$, we define $\ell : f^*(Y) + \mathbb{C}\lambda_0$ as $\ell(f^*(y) + t\lambda_0) = t$. Extend ℓ using Hahn-Banach to $\tilde{\ell} \in X^{**}$. Since X is reflexive, there is x such that $J(x) = \tilde{\ell}$, where $J : X \rightarrow X^{**}$ is the canonical map. Then

$$J(x)(f^*(y)) = f^*(y)(x) = f(x)(y) = 0$$

for all $y \in Y$. Since $f : X \rightarrow Y^*$ is isometric and hence injective, then $f(x) = 0$ and hence $x = 0$. Thus $\tilde{\ell} = 0$, which is a contradiction. We must have $Y \cong X^*$ and $Y^* \cong X^{**} \cong X^*$.

7. Consider the set

$$F_n = \{f \in S \mid \|f\|_{1+\frac{1}{n}} \leq n\}$$

First, we argue that $S = \bigcup_n F_n$. If $f \in S$, then $f \in L^p([0, 1])$ for some $p > 1$. Since $L^p([0, 1]) \subseteq L^q([0, 1])$ for $p \leq q$, choose n large enough so that $n \geq \|f\|_p$ and $1 + \frac{1}{n} \leq p$, we have $\|f\|_{1+\frac{1}{n}} \leq \|f\|_p \leq n$, and hence $f \in F_n$. The set F_n is closed in $L^1([0, 1])$ and hence in S . Indeed, if $\{f_m\} \subset F_n$ and $f_m \rightarrow f$ in L^1 , there is a subsequence f_{m_k} that converges to f almost everywhere. Thus

$$\int |f|^{1+\frac{1}{n}} = \int \liminf_k |f_{m_k}|^{1+\frac{1}{n}} \leq \liminf_k \int |f_{m_k}|^{1+\frac{1}{n}} \leq n^{1+\frac{1}{n}}$$

Apply the Baire's category theorem to the Banach space S , there is F_{n_0} that has nonempty interior in S , i.e., there is $x_0 \in S$ and a ball $B(x_0, \epsilon)$ in L^1 such that

$$B(x_0, \epsilon) \cap S \subset F_{n_0}$$

If $x \in S$, then $rx + x_0 \in B(x_0, \epsilon) \cap S \subset F_{n_0}$ for some $r > 0$ small, and hence $rx + x_0 \in L^{1+\frac{1}{n_0}}$ and $x \in L^{1+\frac{1}{n_0}}$.

8. Define the subspace $M \subset \ell^\infty$ and the convex function p as in the problem. Let $\ell : M \rightarrow \mathbb{R}$ be the linear functional defined as $\ell(\psi) = \lim_n \Lambda_n \psi$. Use the Hahn-Banach theorem to extend ℓ to a linear functional $\Lambda : \ell^\infty \rightarrow \mathbb{R}$ such that $\Lambda(\psi) \leq p(\psi)$ for any $\psi \in \ell^\infty$. We have $L\psi - \psi \in M$ and hence $\Lambda(L\psi - \psi) = \ell(L\psi - \psi) = 0$ which implies $\Lambda L = \Lambda$. Indeed

$$\Lambda_n(L\psi - \psi) = \frac{\psi(2) + \cdots + \psi(n+1)}{n} - \frac{\psi(1) + \cdots + \psi(n)}{n} = -\frac{\psi(1)}{n} + \frac{\psi(n+1)}{n} \rightarrow 0$$

We show that $p(\psi) \leq \limsup_n \psi(n)$ for all ψ . Suppose for contradiction that $p(\psi) - \epsilon > \limsup_n \psi(n)$ for some ψ . Then $p(\psi) - \epsilon \geq \psi(n)$ for all $n \geq N$ for some N . Now

$$\frac{1}{n} \sum_{j=1}^n \psi(j) = \frac{1}{n} \sum_{j=1}^{N-1} \psi(j) + \frac{1}{n} \sum_{j=N}^n \psi(j) \leq \frac{1}{n} \sum_{j=1}^{N-1} \psi(j) + \frac{n-N}{n} (p(\psi) - \epsilon)$$

Thus $p(\psi) = \limsup \Lambda_n \psi \leq p(\psi) - \epsilon$ which is a contradiction.