MAT520 HW2

September 27, 2025

1. Recall the Cantor set is $C = \bigcap_{k=0}^{\infty} C_k$ where $C_0 = [0, 1]$ and $C_1 = [0, 1/3] \cup [2/3, 1]$ and so on. If the interior of C is nonempty, then there is an interval $I \subset C$, which is not possible. Indeed, for any two points x, y in the Cantor set such that $|x - y| \geq 1/3^k$, then x, y belongs to two different C_k and hence there is some point not in C but lies between x and y.

2. We show that $W \cap \bigcap V_j$ is nonempty for any nonempty open set W. Since V_1 is dense, it follows that $W \cap V_1$ is nonempty. Since X is a locally compact Hausdorff space, there exists an open set U_1 such that $U_1 \subset \overline{U}_1 \subset W \cap V_1$, and that $\overline{U}_1$ is compact. Similarly, choose and open set U_2 with $\overline{U}_2$ compact such that $U_2 \subset \overline{U}_2 \subset U_1 \cap V_1$, and so on. We obtain a nested sequence of nonempty compact sets $\overline{U}_1 \supset \overline{U}_2 \supset \cdots$, and hence $\bigcap \overline{U}_j$ is nonempty.

4. Let Y be a finite-dimensional subspace of X. We know that Y is closed in a TVS. Suppose Y contains some open set U. Pick $u \in U$. Since $U - u$ is absorbing, for any $x \in X$, we have $tx + u \in U$ for sufficiently small $t > 0$. It follows that $x \in Y$ and $X \subset Y$, which is a contradiction, since X is assumed to be infinite-dimensional. Thus Y is nowhere dense in X. In particular, X is of Baire's first category. For the second part, let X be an infinite-dimensional Banach space that has a countable Hamel basis $\{f_j\}_{j=1}^{\infty}$. Let Y_n be the span of $\{f_j\}_{j=1}^{n}$. Then $X = \bigcup_n Y_n$. However, Y_n is finite-dimensional and hence nowhere dense in X, implying that X is of first category, which contradicts the Baire's category theorem.

5. Let E_n be a Cantor-like set where at k^{th} stage we remove 2^{k-1} centrally situated open intervals each of length l_{nk} such that $\sum_{k=1}^{\infty} 2^{k-1} l_{nk} = 2^{-n}$. This can be achieved with $l_{nk} = 2^{-2k-n+1}$. Then $m(E_n) = 1 - 2^{-n}$ where m is the Lebesgue measure. We have $E_1 \subset E_2 \subset \cdots$ and let $E = \bigcup E_n$. Then $m(E) = \lim_n m(E_n) = 1$. In particular, each E_n is nowhere dense.

6. If f is twice continuously differentiable, then $\hat{f}(n) = O(1/|n|^2)$ as $|n| \to \infty$, and hence $\lim_n \Lambda_n f$ exists. This space is dense in $L^2(\mathbb{S}^1)$. For the second part, denote E to be the

set of $f \in L^2(\mathbb{S}^1)$ such that $\lim_n \Lambda_n f$ exists, and let E_N be the set of $f \in L^2(\mathbb{S}^1)$ such that $|\Lambda_n f| \leq N$. It is clear that $E \subset \bigcup_N E_N$ since convergent sequence is bounded. The set E_N is closed since Λ_n is linear and bounded $|\Lambda_n f| \leq \sqrt{2n+1}\|f\|_2$. It remains to show that E_N has no interior. Suppose E_N contains a ball B around f of radius $r > 0$. Let $g \in L^2(\mathbb{S}^1)$ corresponds to the Fourier coefficients $\{1/k\}_{k=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$. Now $f + \epsilon g \notin E_N$ for all $\epsilon > 0$, since $|\sum_{k=-n}^{n}(\hat{f}(k) + \epsilon/k)| \geq \epsilon|\sum_{k=-n}^{n} 1/k| - N$ can be made arbitrarily large. However, $f + \epsilon g \in B$ for ϵ sufficiently small.

7. If Y intersects with $Y + x$ for all $x \in X$, then we are done, using the fact that Y is a subspace. If Y does not intersect $Y + x$, then $Y + x \subset Y^c$ is of first category. This cannot be true since $X = Y \cup Y^c$ will then be of first category.

8. Let $x_n \to x$. Since K is compact, then there is a subsequence x_{n_k} for which $f(x_{n_k}) \to y$ converges. Since the graph of f is closed, it follows that $y = f(x)$.

11. See Corollary 3.3 in LN.

14. Using open mapping theorem, we have $A(B_X(0,1)) \supset B_Y(0,2c)$ for some $c > 0$. For $y \in Y$, there is $z \in X$ with $\|z\| < 1$ such that $Az = cy/\|y\|$. Let $x = z\|y\|/c$; then for any $y \in Y$, there is $x \in X$ with $\|x\| \leq \|y\|/c$ such that $Ax = y$. Let B be any bounded operator such that $\|A - B\| < c$. We show that B is surjective. Let y be arbitrary. Let $y_0 = y$ and $Ax_0 = y_0$ with $\|x_0\| \leq \|y_0\|/c$ and $y_1 = y_0 - Bx_0$ and so on. Let $x = \sum x_n$. Then

$$\|y_n\| \leq (\|A - B\|/c)^n$$

and

$$\sum \|x_n\| \leq \sum \|y_n\|/c \leq (\|A - B\|/c)^n/c$$

which converges. One verifies that $A \sum x_n = y$.