Functional Analysis
Princeton University MAT520
HW10, Assigned on Nov 23rd 2025

November 23, 2025

1. Provide an example for a non-normal operator $A \in \mathcal{B}(\mathcal{H})$ and a point in the resolvent set $z \in \rho(A)$ where
$$\left\|(A - z\mathbb{1})^{-1}\right\| \leq \frac{1}{\operatorname{dist}(z, \sigma(A))}$$
does *not* hold.

2. Let $\mathcal{H}$ be a separable Hilbert space. Prove that the *only* operator-norm-closed star-ideals in $\mathcal{B}(\mathcal{H})$ are $\{0\}$, $\mathcal{K}(\mathcal{H})$ (the compact operators) and $\mathcal{B}(\mathcal{H})$ itself.

3. Let K be compact.
   (a) Show that $\ker(\mathbb{1} + K)$ is finite dimensional.
   (b) Show that $\operatorname{im}(\mathbb{1} + K)$ is closed.
   (c) Show that $\operatorname{coker}(\mathbb{1} + K)$ is finite dimensional.
   (d) Show that $\ker(\mathbb{1} + K) \cong \ker(\mathbb{1} + K^*)$.

4. Show that for any Fredholm operator A, $\sigma_{\text{ess}}\left(|A|^2\right)$ and $\sigma_{\text{ess}}\left(|A^*|^2\right)$ are both bounded from below. Provide an example of A Fredholm where however $\sigma(A)$ contains the whole unit disc.

5. Define the Volterra operator $V : L^2([0,1]) \to L^2([0,1])$ as
$$(V\psi)(x) := \int_0^x \psi \qquad \left(x \in [0,1], \psi \in L^2([0,1])\right) .$$
Show that $\mathbb{1} - V$ is Fredholm and calculate its index.

6. On $\ell^2(\mathbb{N})$, let $\hat{R}$ be the *unilateral* right shift operator. Calculate $\ker \hat{R}$, $\ker \hat{R}^*$ and $\operatorname{im}\hat{R}$ and show that $\hat{R}$ is a Fredholm operator. Calculate its Fredholm index.

7. Show that on $\ell^2(\mathbb{N})$, $\frac{1}{X}$ where X is the position operator, is *not* a Fredholm operator by calculating $\operatorname{im}\frac{1}{X}$ and showing explicitly that it is not closed.

8. By finding an orthonormal Weyl sequence, show that
$$\sigma_{\text{ess}}(X) = [0,1]$$
where X is the position operator on on $L^2([0,1] \to \mathbb{C})$. If you are curious, even though we haven't properly defined unbounded operators and their domain, think about a Weyl sequence to show that
$$\sigma_{\text{ess}}(-\Delta) = [0,\infty)$$
where $-\Delta$ is the Laplacian on $L^2(\mathbb{R})$.

9. Provide an example of a diagonal Fredholm operator on $\ell^2(\mathbb{N})$ with non-zero index or prove it cannot exist.

10. [Krein-Widom-Davinatz] Let $f:\mathbb{S}^1\to\mathbb{C}\setminus\{\,0\,\}$ be continuous. Define $M_f\in\mathcal{B}\left(L^2\left(\mathbb{S}^1\right)\right)$ as

$$(M_f\psi)(z):=f(z)\,\psi(z)\ .$$

Let $\mathfrak{F}:L^2\left(\mathbb{S}^1\right)\to\ell^2(\mathbb{Z})$ be the Fourier series. Show that:

(a) The operator $\mathfrak{F}M_f\mathfrak{F}^*$ on $\ell^2(\mathbb{Z})$ is bounded.

(b) If $\Lambda:\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})$ is the self-adjoint projection defined on the standard basis by

$$\Lambda\delta_x:=\begin{cases}0 & x\leq 0\\ \delta_x & x>0\end{cases}\qquad (x\in\mathbb{Z})$$

and extended linearly to the entirety of $\ell^2(\mathbb{Z})$, then

$$\Lambda\mathfrak{F}M_f\mathfrak{F}^*\Lambda+\mathbb{1}-\Lambda$$

is a Fredholm operator.

(c) Show that

$$\operatorname{index}\left(\Lambda\mathfrak{F}M_f\mathfrak{F}^*\Lambda+\mathbb{1}-\Lambda\right)=-\operatorname{Wind}(f)\ .$$

(d) If U is the polar part of $\mathfrak{F}M_f\mathfrak{F}^*$ in the polar decomposition, show that

$$\operatorname{index}\left(\Lambda\mathfrak{F}M_f\mathfrak{F}^*\Lambda+\mathbb{1}-\Lambda\right)=\operatorname{tr}\left(U^*\left[\Lambda,U\right]\right)$$

by using Fedosov's formula.

11. [The Atiyah-Singer $\mathbb{Z}_2$-index] Let $\mathcal{H}$ be a separable Hilbert space. Assume further that there is a bounded anti-$\mathbb{C}$-linear operator $J:\mathcal{H}\to\mathcal{H}$ which is anti-unitary, i.e.,

$$\langle J\psi,J\varphi\rangle=\langle\varphi,\psi\rangle\qquad(\varphi,\psi\in\mathcal{H})\ .$$

We assume $J^2=-\mathbb{1}$. Consider now the following class of operators

$$\mathcal{F}_J(\mathcal{H}):=\{\,F\in\mathcal{F}(\mathcal{H})\mid FJ=JF^*\,\}\ .$$

(a) Show that

$$\operatorname{index}\left(\mathcal{F}_J(\mathcal{H})\right)\subseteq\{\,0\,\}\ .$$

(b) Show that $\operatorname{index}_2:\mathcal{F}_J(\mathcal{H})\to\mathbb{Z}_2$ given by

$$\operatorname{index}_2(F):=\dim\ker F\mod 2$$

is norm-continuous.

(c) Show that if $K\in\mathcal{K}(\mathcal{H})$ and $F,F+K\in\mathcal{F}_J(\mathcal{H})$ then

$$\operatorname{index}_2(F)=\operatorname{index}_2(F+K)\ .$$