Functional Analysis
Princeton University MAT520
October 2024 Take-home Midterm Exam

October 9, 2024

Instructions: This is a take-home exam, meaning you are expected to do it "at home" by yourself in a quiet environment with no other person aiding you (this is not a group assignment). You may very well use any source material which is "not interactive" such as books / notes / lecture notes / etc. But you may *not* communicate with other people, nor post any questions online. Pretend you were in a classroom and could bring with you whichever *non-electronic offline* reference material you needed. Obviously I have no way of policing that, so we shall rely on Princeton's honor code.

- The exam has six questions to yield a maximum of 100 points.
- You are expected to complete the exam in six hours, starting from whenever you open it on Gradescope. You may take breaks as needed but not communicate with others about the exam during that time.
- If you can't submit the assignment through Gradescope just send it to my email (shapiro@math.princeton.edu). I trust you that you time yourself correctly for the six hours.
- During your exam it is possible to email us (shapiro@math.princeton.edu or juihui@princeton.edu) if there is any doubt about the phrasing of the questions (but don't expect to get help or hints, in the interest of fairness).
- The exam is meant to be easy and straightforward and there are no trick questions. If something seems too easy maybe it's just easy.

1. (15 points) Let X, Y be two Banach spaces. Define on the Cartesian product

$$X \times Y$$

coordinate-wise addition and scalar multiplication. For $p \in [1, \infty]$, define

$$\|(x,y)\|_p := \begin{cases} \max\left(\{ \|x\|_X, \|y\|_Y \}\right) & p = \infty \\ \left(\|x\|_X^p + \|y\|_Y^p\right)^{\frac{1}{p}} & p \in [1,\infty) \end{cases} .$$

   (a) Show that with these definitions, $X \times Y$ is a Banach space (i.e., show it is a complete normed vector space).
   (b) Show that all p-norms are equivalent on $X \times Y$.

2. (15 points) Let $M : X \to Y$ be a continuous map between *real* normed spaces X, Y such that

$$M(0) = 0$$

and

$$M\left(\frac{1}{2}(x + \tilde{x})\right) = \frac{1}{2}M(x) + \frac{1}{2}M(\tilde{x}) \qquad (x, \tilde{x} \in X) .$$

Show that M is $\mathbb{R}$-linear.

3. (15 points) Provide an example (no further explanation or proof is necessary) for each of the following:
   (a) A normed vector space which is not a Banach space.
   (b) A linear functional that is not continuous.
   (c) A topological vector space which is not locally convex.

(d) A Banach space whose closed unit ball is compact.

(e) A Banach space which is not reflexive.

4. (15 points) Show that if X, Y are Banach spaces and $A \in \mathcal{B}(X \to Y)$ then if $x_n \to x$ weakly in X then $Ax_n \to Ax$ weakly in Y.

5. (20 points) Let X be a Banach space. Let $K \subseteq X^*$. Show that K is weak-star compact iff K is bounded (in the norm of X^*) and weak-star closed.

6. (20 points) Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ is called an *idempotent* if $a^2 = a$. Set

$$\Omega^+ = \{z \in \mathbb{C} : \mathbb{Re}(z) > 1/2\}, \quad \Omega^- = \{z \in \mathbb{C} : \mathbb{Re}(z) < 1/2\}, \quad \Omega = \Omega^+ \cup \Omega^- .$$

An element $a \in \mathcal{A}$ is called a *quasi-idempotent* if $\sigma(a) \subseteq \Omega$. Define a holomorphic function $h : \Omega \to \mathbb{C}$ by

$$h(z) = \begin{cases} 0, & z \in \Omega^- \\ 1, & z \in \Omega^+ \end{cases} .$$

Prove the following.

(a) For every idempotent $a \in \mathcal{A}$, we have $\sigma(a) \subseteq \{0, 1\} \subseteq \Omega$.

(b) $h(a)$ is an idempotent for each quasi-idempotent $a \in \mathcal{A}$.

(c) $h(a) = a$ for each idempotent $a \in \mathcal{A}$.

(d) Let $a \in \mathcal{A}$ be a quasi-idempotent. For each $t \in [0, 1]$, define a holomorphic function $h_t : \Omega \to \mathbb{C}$ by

$$h_t(z) = \begin{cases} tz, & z \in \Omega^- \\ t(z-1)+1, & z \in \Omega^+ \end{cases}$$

and set $a_t = h_t(a)$. Verify that the map

$$[0,1] \ni t \mapsto a_t \in \mathcal{A}$$

is continuous, a_t is a quasi-idempotent for each $t \in [0, 1]$, $a_0 = h(a)$ and $a_1 = a$.

(e) Let a and b be idempotents in $\mathcal{A}$. Let

$$[0,1] \ni t \mapsto a_t \in \mathcal{A}$$

be a continuous map in which a_t is a quasi-idempotent for each $t \in [0, 1]$, with $a_0 = a$ and $a_1 = b$. Let $e_t = h(a_t)$. Verify that the map

$$[0,1] \ni t \mapsto e_t \in \mathcal{A}$$

is continuous, e_t is an idempotent for each $t \in [0, 1]$, $e_0 = a$ and $e_1 = b$. (You may use the fact that there exists a compact subset $K \subseteq \Omega$ such that $\sigma(a_t) \subseteq K$ for all $t \in [0, 1]$.)