# Functional Analysis
# Princeton University MAT520
# Lecture Notes

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**Abstract**

These lecture notes correspond to a course given in the Fall semester of 2024 in the math department of Princeton University.

## Contents

# Syllabus

- Main source of material for the lectures: this very document (to be published and weekly updated on the course website–please do not print before the course is finished and the label "final version" appears at the top).
- Official course textbook: No one, main official text will be used but in preparing these notes; I will probably make heavy use of [Rud91] as well as [RS80] and [BB89].
- Other books one may consult are [HN01, Sch01, Bre10, Dou98, LL01, Lax14, Con19, BS18, Yos12, Sim15, Sim10, HS12].
- Two lectures per week: Tue and Thur, 3:00pm–4:20pm in Fine Hall 110.
- People involved:
  - Instructor: Jacob Shapiro shapiro@math.princeton.edu
    Office hours: Fine 603, Thursdays 5:30pm-6:30pm, or, by appointment.
  - Assistant: Jui-Hui Chung (juihui@princeton.edu)
    Office hours: Fine 216, TBA, or, by appointment.
- HW to be submitted weekly on Friday evening either via email to shapiro@math.princeton.edu or in hard copy to my mailbox (at my door on Fine 603) or on Gradescope.
  Submission by Sunday evening will not harm your grade but is not recommended (i.e. you get an automatic extension of two days always).
  HW may be worked together in groups but needs to be written down and submitted separately for each student.
  10% automatic extra credit on HW (up to a maximum of 100%) if you write *legibly* and *coherently*.
  You may use LaTeX or LyX to submit if you like but it is *not* a requirement.
- Grade: 40% HW, 20% Midterm, 40% (take home) Final.
- Attendance policy: extra credit to students who attend lectures regularly and ask questions or point out problems.
- Anonymous Ed discussion enabled. Use it to ask questions or to raise issues (technical or academic) with the course.
- If you alert me about typos and mistakes in this manuscript (unrelated to the sections marked [todo]) I'll grant you some extra credit. In doing so, please refer to a version of the document by the date of typesetting.
  - Thanks goes to: Janine Roshardt ($\times 2$), Akshat Agarwal ($\times\infty$), Brandon Cho ($\times 4$), Jui-Hui Chung, Elie Belkin, Colby Riley, Julian Shah ($\times 5$).

# Semester plan

List of (big) theorems and topics aimed at being included:

- The Banach-Steinhaus theorem, open-mapping theorem, closed-graph theorem.
- Hahn-Banach theorem and convexity.
- Weak topologies and Banach-Alaoglu theorem.
- Duality in Banach spaces.
- Polar decomposition.
- Bounded operators on Hilbert space: spectra.
- Spectral theorem for bounded self-adjoint operators and the functional calculus.
- Spectral theorem for unbounded self-adjoint operators, the Hille-Yosida theorem, semigroups.
- The trace ideals, Schatten classes, etc.
- Fredholm theory: Atkinson's, Dieudonné's, Fedosov, Atiyah-Jänich, Kuiper, etc.
- Mathematical quantum mechanics.

Semester plan by date:

1. Sep 3rd: Topological vector spaces.
2. Sep 5th: Topological vector spaces.
3. Sep 10th: Banach spaces.
4. Sep 12th: Banach spaces and Baire category.
5. Sep 17th: Open, Closed mapping theorems.
6. Sep 19th: Hahn-Banach and Duality.
7. Sep 24th: Weak topology and Banach Alaoglu.
8. Sep 26th: Banach algebras and invertible elements.
9. Oct 1st: The spectrum of elements in a Banach algebra.
10. Oct 3rd: The holomorphic functional calculus for elements in Banach algebras.

# 1 Soft introduction

Functional analysis is the branch of mathematics that is obtained when one marries together *topology* (or *point set topology*) and *linear algebra*, both of which are assumed the reader is very well familiar with (as well as measure theory). The first order of business, is why you should want to combine these two?

> *Remark* 1.1. For notational simplicity we will almost always assume that our vector spaces are over $\mathbb{C}$, sometimes some statements may be recast for $\mathbb{R}$, certainly some care should be taken for general fields. Hence from now on please read the phrase "vector space" as "$\mathbb{C}$-vector-space".

In a vague sense (and there will be more about this in the homework) all finite dimensional vector spaces are "equivalent" and hence the question of their topology only becomes interesting when the dimension of the vector space goes to infinity. To be a bit more explicit, in $n \in \mathbb{N}$ dimensions, there is only one vector space, and we denote it by $\mathbb{C}^n$. We measure distances between vectors in Euclidean fashion, which induces a topology, and we have a notion of a *standard basis* $\{ e_j \}_{j=1}^n$ which gives us a way to concretely write down matrices and vectors as tables and columns. This class is about what happens when $n$ goes to infinity. Then, one has to make some additional choices which concern the interplay between topology (and more concretely, analysis) and the vector space structure, and one has to contend with topological and analytical questions of convergence that essentially build the heart of what is functional analysis. These types of infinite dimensional vector spaces usually arise in applications as spaces of functions, which is the reason for the name of the field "functional analysis": we will do analysis on functions, whereas so far we have done analysis on numbers.

*Remark* 1.2 (Motivation for physicists). To motivate a bit more the situation for physicists (see [Gie00]), among many things we want to understand is the notion of *continuous spectrum*. After all, the development of quantum mechanics and functional analysis are intimately related. Consider then the hydrogen atom and its "spectrum": we know it has bound states of negative energy and scattering states of positive energy. The bound states are "eigenstates" in the sense that they obey the eigenequation

$$H\psi = \lambda\psi$$

for some wave function $\psi$ and (negative) eigenvalue $\lambda < 0$. However, what about the scattering states? Do they also obey a similar equation? How do we characterize that part of the spectrum?

In many physics textbooks we usually read about the position basis as that basis of Dirac delta "functions"

$$\mathbb{1} = \int_{x\in\mathbb{R}} |x\rangle \langle x| \, \mathrm{d}x \, .$$

However we know that the Dirac delta is not really a function (but a measure, or a distribution).

Everyone knows the canonical commutation relation, which states that

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar\mathbb{1} \, .$$

Taking the trace of this equation leads to a putative contradiction using the cyclic property of the trace:

$$\mathrm{tr}\left([\hat{x}, \hat{p}]\right) = \mathrm{tr}\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right) = \mathrm{tr}\left(\hat{x}\hat{p}\right) - \mathrm{tr}\left(\hat{p}\hat{x}\right) \overset{\star}{=} \mathrm{tr}\left(\hat{x}\hat{p}\right) - \mathrm{tr}\left(\hat{x}\hat{p}\right) = 0 \overset{???}{=} \mathrm{tr}\left(\mathrm{i}\hbar\mathbb{1}\right) \, .$$

We usually think of $\hat{p} \equiv -\mathrm{i}\hbar\nabla$ as a self-adjoint operator. It should then be the case that

$$\langle\psi, \hat{p}\varphi\rangle = \langle\hat{p}\psi, \varphi\rangle \, .$$

To prove it one usually invokes integration by parts via

$$\begin{aligned} \langle\psi, \hat{p}\varphi\rangle &= \int_{x\in\mathbb{R}} \overline{\psi(x)} \left(-\mathrm{i}\hbar\right) \varphi'(x) \, \mathrm{d}x \\ &= -\mathrm{i}\hbar \, \overline{\psi(x)}\varphi(x)\Big|_{x=\pm\infty} + \mathrm{i}\hbar \int_{x\in\mathbb{R}} \overline{\psi'(x)}\varphi(x) \, \mathrm{d}x \, . \end{aligned}$$

Now one usually says that $\varphi, \psi$ are wave-functions so they must vanish at infinity, hence the boundary term vanishes and we indeed find that $\hat{p}$ is self-adjoint. However, we define the Hilbert space of wave functions as those functions which are square summable. Ignoring for the moment the much more basic fact that such square-summable functions need not be differentiable (and so what do we mean by $\hat{p}$ acting on them?) we also note that being square-summable does *not* imply the function vanishes at infinity. Consider for instance

$$x \mapsto x^2 \exp\left(-x^8 \left(\sin(x)\right)^2\right) \, .$$

This is essentially a shrinking comb. Is $\hat{p}$ actually self-adjoint???

Let us delve a bit more on the finite-dimensional setting. In what sense do we mean things are equivalent? Surely on $\mathbb{C}$ we may furnish two different topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $(\mathbb{C}, \mathcal{T}_1)$ and $(\mathbb{C}, \mathcal{T}_2)$ are not homeomorphic[1].

**Example 1.3.** If we define $\mathcal{T}_1$ to be the Euclidean topology on $\mathbb{C}$ (associated with the Euclidean distance $\mathbb{C} \ni z \mapsto |z|$) and $\mathcal{T}_2$ to be the topology associated to the *French metro metric*

$$d(z, w) := \begin{cases} |z - w| & z = \alpha w \exists \alpha \in \mathbb{R} \\ |z| + |w| & \text{else} \end{cases} \qquad (z, w \in \mathbb{C}) \, .$$

(one first shows this is indeed a metric). Then $(\mathbb{C}, \mathcal{T}_1)$ and $(\mathbb{C}, \mathcal{T}_2)$ are not homeomorphic.

[1] homeomorphism is the appropriate notion of equivalence in the category of topological spaces

*Proof.* Fix any $z \in \mathbb{C} \setminus \{ 0 \}$ and let $r < |z|$. Then in the French metro metric, $B_r(z)$ is an open line segment of length $2r$ on the ray defined by $z$, and centered at $z$. That ball is by definition, an open ball in the topology $\mathcal{T}_2$. If there were a continuous map $f : (\mathbb{C}, \mathcal{T}_1) \to (\mathbb{C}, \mathcal{T}_2)$, it better be the case that $f^{-1}(B_r(z))$ is open in the Euclidean topology, but we know it is not. ☐

Of course one could also take for $\mathcal{T}_2$ the trivial topology (with only the empty set and the whole set as open) to get a counter-example, albeit a bit too silly since it is not even Hausdorff.

This example worked in $\mathbb{C}$, which is *one* dimensional, and we have seen two ways to furnish $\mathbb{C}$ with inequivalent topologies. So what's all this talk about only needing functional analysis in infinite dimensions?

Enter the concept of *topological vector spaces*.

Whenever we are doing mathematics with two (or more) mathematical structures (i.e., categories) we need to make sure they are *compatible*. A familiar example should be that of a Lie group, which is a group which is also a manifold, but moreover and crucially, the two structures are compatible in the sense that the group operations respect the manifold structure: they must be smooth functions on the manifold. We want to achieve the same thing when we combine vector spaces with topological spaces.

In category theory, to preserve a structure means to be a morphism in a given category. So in the category of topological spaces, it is the continuous functions which are "topological space morphisms".

**Definition 1.4** (Topological vector spaces)**.** Let $X$ be a vector space furnished with a $T_1$-topology $\text{Open}(X)$. $X$ is a *topological vector space* iff the two vector space operations, vector addition $+ : X^2 \to X$, $(u, v) \mapsto u + v$ and scalar multiplication $m : \mathbb{C} \times X \to X$, $(\alpha, v) \mapsto \alpha v$, are *continuous* with respect to $\text{Open}(X)$ (and the standard topology on $\mathbb{C}$).

*Remark* 1.5. Recall that for a topology to be $T_1$ means that all singletons are closed. Presumably it is not very useful to talk about topological vector spaces where $\text{Open}(X)$ is not $T_1$ [Rud91].

*Claim* 1.6. In Example 1.3, $(\mathbb{C}, \mathcal{T}_1)$ is a topological vector space (indeed $\mathbb{C}^n$ with its standard topology is, for any $n \in \mathbb{N}$) whereas $(\mathbb{C}, \mathcal{T}_2)$ is *not*.

*Proof.* [TODO: fix this proof.] We first show that $\mathbb{C}^n$ is a topological vector space. Clearly the Euclidean topology has the property that singletons are closed, and is hence $T_1$. We now want to show that vector addition $+ : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is continuous. Suffice to work with bases, so let $\varepsilon > 0$ and $z \in \mathbb{C}^n$ and WTS that $+^{-1}(B_\varepsilon(z))$ is open in $\mathbb{C}^n \times \mathbb{C}^n$. This set, however, may be written as the (arbitrary) union of open sets:

$$+^{-1}(B_\varepsilon(z)) = \bigcup_{u \in \mathbb{C}^n} B_\varepsilon(u) \times \{ v \in \mathbb{C}^n \mid \|u + v - z\| < \varepsilon \} . \tag{1.1}$$

Similarly, we want to show that for $\cdot : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ which maps $(\alpha, z) \mapsto \alpha z$ is continuous, which proceeds in a similar way.

We leave the fact that $(\mathbb{C}, \mathcal{T}_2)$ is not a topological vector space as an exercise to the reader. ☐

*Claim* 1.7. If $X, Y$ are two topological vector spaces of the same finite dimension then they are both (topologically) homeomorphic and (linearly) isomorphic, i.e., they are isomorphic in the category of topological vector spaces. Hence for a given dimension $n \in \mathbb{N}$, $\mathbb{C}^n$ with its Euclidean topology is the *only* topological vector space.

*Proof.* See Claim 2.16 below. ☐

It is in this sense that to find additional interesting spaces we branch out to infinite dimensions.

**Example 1.8.** Consider the space $\ell^p(\mathbb{N} \to \mathbb{C})$ of $p$-summable sequences, i.e.,

$$\ell^p(\mathbb{N}) \equiv \left\{ a : \mathbb{N} \to \mathbb{C} \;\middle|\; \sum_{n \in \mathbb{N}} |a(n)|^p < \infty \right\} .$$

It is a vector space (with pointwise addition and scalar multiplication), and we furnish it with a topology which is associated with a metric, which is associated with the norm

$$\ell^p(\mathbb{N}) \ni a \mapsto \|a\|_p \equiv \left(\sum_{n\in\mathbb{N}} |a(n)|^p\right)^{\frac{1}{p}} .$$

We observe that for different values of $p$, the space itself (as a vector space) is very similar and the main thing that changes is the topology! One may verify $\ell^p(\mathbb{N})$ is a TVS, and moreover, it is infinite dimensional. We claim that $\ell^p(\mathbb{N})$ is not homeomorphic to $\ell^{p'}(\mathbb{N})$ if $p \neq p'$. Note that only if $p \geq 1$ are these spaces actually Banach.

*Proof.* It is clear that $\ell^p(\mathbb{N})$ is indeed a vector space, so one merely has to show it is a TVS. This follows similarly to (1.1). To see that $\ell^p(\mathbb{N})$ is infinite dimensional we may observe it has a basis $\{\delta_n\}_n \subseteq \ell^p(\mathbb{N})$ where $\delta_n(m) \equiv \delta_{nm}$.

The proof that $\ell^p$ is not isomorphic to $\ell^{p'}$ is somewhat delicate. We will contend ourselves to the fact that if $p < 1$ and $p' > 1$ then these can't be isomorphic, as the former is not metrizable and the latter is. ☐

# 2 Topological vector spaces

We have already seen above in Definition 1.4 the definition of a TVS. Let us get our hands dirty with a few examples and properties.

**Example 2.1** (Some TVS which cannot be realized via norms)**.** The following spaces are examples of topological vector spaces which *cannot* be realized as Banach spaces–a complete normed vector space–an important structure we'll get to later.

1. For any $\Omega \in \mathrm{Open}(\mathbb{R}^n)$, $C(\Omega \to \mathbb{C})$ (often times we'll write $C(\Omega)$ when the codomain is implicitly understood as $\mathbb{C}$ if not indicated) is the space of all continuous functions $\Omega \to \mathbb{C}$. It is an infinite dimensional TVS. [TODO: define the topology on this].
2. For any $\Omega \in \mathrm{Open}(\mathbb{C})$, $H(\Omega)$ is the space all holomorphic functions $\Omega \to \mathbb{C}$. [TODO: define the topology on this]

**Definition 2.2** (Bounded, balanced and absorbing sets)**.** A subset $S$ of a TVS $X$ is said to be *bounded* if for every neighborhood $N$ of 0 in $X$ there is some $t > 0$ such that

$$S \subseteq sN \qquad (s > t) .$$

$S$ is said to be *balanced* iff $\alpha S \subseteq S$ for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Think of balanced like "star-shaped", which need not be convex.

$S$ is said to be *absorbing* iff for any $x \in X$, there is some $t > 0$ with $x \in tS$ for some $t$ sufficiently large.

*Remark* 2.3 (Caution!). This notion of boundedness need not necessarily agree with the usual notion of boundedness in metric spaces (that where there is a sufficiently large $R > 0$ such that the set is covered by $B_R(0)$), actually (see Rudin 1.29). When the metric is induced by a norm, however, the two notions will agree [TODO].

*Remark* 2.4 (Caution!). It might happen that $A + A \neq 2A$. Example: $A = \{0, 1\}$ then $A + A = \{0, 1, 2\}$ whereas $2A = \{0, 2\}$.

Recall that a *local base* at a point $\psi \in X$ is a collection of neighborhoods $\gamma$ (open sets which contain $\psi$) such that every neighborhood of $\psi$ contains an element of $\gamma$. Clearly a local base induces a base for a topology. Recall a base $\mathcal{B}$ for a topology $\mathcal{T}$ is a subset $\mathcal{B} \subseteq \mathcal{T}$ such that for any $U \in \mathcal{T}$ and $x \in U$, there exists some $B \in \mathcal{B}$ such that

$$x \in B \subseteq U .$$

Hence if for every $\psi \in X$, $\gamma_\psi$ is a local base at $\psi$, then

$$\bigcup_{\psi \in X} \gamma_\psi$$

is a base for $\text{Open}(X)$. Conversely, if $\mathcal{B}$ is a base for $\text{Open}(X)$ and $\psi \in X$, then

$$\{ B \in \mathcal{B} : \psi \in B \}$$

is a local base at $\psi$. We re-emphasize that $\text{Open}(X)$ is itself determined by a basis $\mathcal{B}$ since the other defining property of a basis is that any $U \in \text{Open}(X)$ may be written as

$$U = \bigcup_{\alpha} B_\alpha$$

for some $\{ B_\alpha \}_\alpha \subseteq \mathcal{B}$.

Now, in a TVS $X$, for any fixed $\psi \in X$ we define the *translation* map

$$\begin{aligned} T_\psi : X &\to X \\ \varphi &\mapsto \psi + \varphi \end{aligned}$$

whose inverse is $T_{-\psi}$ and for any fixed scalar $\lambda \in \mathbb{C}$, we define the *multiplication* map

$$\begin{aligned} M_\lambda : X &\to X \\ \varphi &\mapsto \lambda\varphi \end{aligned}$$

whose inverse (if $\lambda \neq 0$) is $M_{\frac{1}{\lambda}}$ with which it is clear that $T_\psi$ and $M_\lambda$ (for $\lambda \neq 0$) are *homeomorphisms* from $X \to X$ (in fact $M_\lambda$ is a TVS isomorphism because it also respects the linear structure whereas $T_\psi$ is merely affine). Since $T_\psi$ is a homeomorphism, the topology $\text{Open}(X)$ is translation invariant, i.e., $S \in \text{Open}(X)$ iff

$$T_{-\psi}^{-1}(S) \equiv \{ \varphi \in X \mid \varphi - \psi \in S \} =: S + \psi \in \text{Open}(X)$$

for any $\psi \in X$. Hence it suffices to determine $\text{Open}(X)$ by studying *any* local basis, e.g., that which is defined at the origin $0_X$ of $X$. We may thus characterize $\text{Open}(X)$ by specifying its local basis at zero: a collection $\mathcal{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathcal{B}$, and $\text{Open}(X)$ is comprised of unions of translates of members of $\mathcal{B}$.

Thus, given a local basis $\mathcal{B}$ at the origin $0_X$, we might replace our intuition of "open $\varepsilon$-ball around $x$" for some $\varepsilon > 0$ from metric spaces, with $x + U$ where $U \in \mathcal{B}$.

**Definition 2.5.** A metric $d$ on $X$ is *translation-invariant* iff

$$d(\psi + \eta, \varphi + \eta) = d(\psi, \varphi) \qquad (\psi, \varphi, \eta \in X) .$$

We give the following special terminology for properties of TVS:

1. $X$ is *locally convex* iff $\exists$ a local base $\mathcal{B}$ of $0_X$ all of whose elements are convex. I.e., if $B \in \mathcal{B}$ and $x, y \in B$,

$$tx + (1-t)y \in B \qquad (t \in [0,1]) .$$

2. $X$ is *locally bounded* iff $0_X$ has a bounded neighborhood.
3. $X$ is *locally compact* iff $0_X$ has a neighborhood whose closure is compact.
4. $X$ is *metrizable* iff $\text{Open}(X)$ arises from some metric $d$ and $X$ is an $F$-space iff its topology arises via a complete translation-invariant metric (sometimes called Fréchet space).
5. $X$ is *normable* iff there is a norm on $X$ which induces a metric which induces $\text{Open}(X)$.
6. $X$ has the Heine-Borel property iff every closed and bounded subset of $X$ is compact.

**Lemma 2.6.** *If $W \in \text{Nbhd}(0_X)$ then $\exists U \in \text{Nbhd}(0_X)$ which is symmetric, i.e., $U = -U$ and such that $U + U \subseteq W$.*

*Proof.* Since addition is continuous and $0+0=0$, there are $V_1, V_2 \in \text{Nbhd}(0)$ such that $V_1 + V_2 \subseteq W$. Indeed, let $a : X^2 \to X$ be vector addition. Then $a^{-1}(W)$ is open by definition of continuity. Define then

$$U := V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$$

and observe it has the desired properties: it is clearly symmetric by construction, and

$$U + U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2) + V_1 \cap V_2 \cap (-V_1) \cap (-V_2) \subseteq V_1 + V_2 \subseteq W\,.$$

□

Note that this may be iterated to get a symmetric $U$ with $U+U+U+U \subseteq W$.

**Theorem 2.7** (Separation property)**.** *Let $K, C \subseteq X$ where $X$ is a TVS and such that $K$ is compact, $C$ is closed, and $K \cap C = \varnothing$. Then $\exists V \in \text{Nbhd}(0_X)$ such that*

$$(K+V) \cap (C+V) = \varnothing\,.$$

We note that $K + V = \bigcup_{x\in K} x + V$ and each $x+V$ is open (translation invariance), so that $K+V$ is in fact open, and contains $K$. By the same reasoning $C+V$ is open, and it is also true that

$$\left(\overline{K+V}\right) \cap (C+V) \quad = \quad \varnothing\,.$$

Indeed, this is equivalent to

$$K+V \quad \subseteq \quad (C+V)^c$$

but the closure is the *smallest* closed set which contains $K+V$.

**Corollary 2.8** (Special case)**.** *If we take $K = \{\, 0_X \,\}$, we find that in particular that for any closed $C$ which does not contain the origin, there exists some $V \in \text{Nbhd}(0_X)$ such that*

$$\overline{V} \cap (C+V) = \varnothing\,.$$

*Now, for a given $U \in \text{Nbhd}(0)$, $U^c$ is closed and does not contain zero, and hence, there is some $V \in \text{Nbhd}(0)$ with $\overline{V} \cap (U^c + V) = \varnothing$ which implies $\overline{V} \cap U^c = \varnothing$, i.e., $\overline{V} \subseteq U$.*

*Proof of Theorem 2.7.* . Assume WLOG that $K \neq \varnothing$ (otherwise there is nothing to prove, with $\varnothing + V = \varnothing$). Let $x \in K$ then, and $x \notin C$. Then $C^c - x \in \text{Nbhd}(0)$ so that by the above lemma, we have some symmetric $V_x \in \text{Nbhd}(0)$ such that $V_x + V_x + V_x \subseteq C^c - x$, i.e.,

$$(x + V_x + V_x + V_x) \cap C = \varnothing\,.$$

Since $V_x$ is symmetric (apply $-V_x$ on both sides of the equation...), this implies that

$$(x + V_x + V_x) \cap (C + V_x) = \varnothing\,.$$

Since $K$ is compact, there are finitely many $\{\, x_1, \dots, x_n \,\}$ such that

$$K \subseteq \bigcup_{j=1}^{n} \left(x_j + V_{x_j}\right)\,.$$

Define $V := \bigcap_{j=1}^{n} V_{x_j}$ so that

$$K + V \subseteq \bigcup_{j=1}^{n} \left(x_j + V_{x_j} + V\right) \subseteq \bigcup_{j=1}^{n} \left(x_j + V_{x_j} + V_{x_j}\right)$$

and all of these terms in the union do not intersect $C+V$. □

**Lemma 2.9.** *If $X$ is a TVS and $A \subseteq X$ then $\overline{A} = \bigcap_{U \in \mathrm{Nbhd}(0)} (A + U)$.*

*Proof.* By definition, $x \in \overline{A}$ iff $(x + U) \cap A \neq \varnothing$ for every $U \in \mathrm{Nbhd}(0)$, iff $x \in A - U$ for every $U \in \mathrm{Nbhd}(0)$, But $U$ is a neighborhood of zero iff $-U$ is. So we find the desired claim. ☐

*Claim* 2.10. If $E$ is bounded then so is $\overline{E}$.

*Proof.* By Theorem 2.7, for a given $V \in \mathrm{Nbhd}(0_X)$, we pick some $W \in \mathrm{Nbhd}(0_X)$ such that $\overline{W} \subseteq V$. Since $E$ is bounded, $E \subseteq tW$ for all $t$ sufficiently large. For such $t$,

$$\overline{E} \subseteq t\overline{W} \subseteq tV$$

so that $\overline{E}$ is bounded too. ☐

**Theorem 2.11.** *In a TVS $X$, (1) For any $U \in \mathrm{Nbhd}(0_X)$ there exists some $V \in \mathrm{Nbhd}(0_X)$ which is balanced such that $V \subseteq U$ and (2) For any $U \in \mathrm{Nbhd}(0_X)$ which is convex there exists some $V \in \mathrm{Nbhd}(0_X)$ which is balanced and convex such that $V \subseteq U$.*

*Proof.* [TODO: fill this in from Rudin 1.14] [SKIP] Let $U \in \mathrm{Nbhd}(0_X)$ be given. Thanks to the fact that scalar multiplication is continuous, there is some $\delta > 0$ and a $W \in \mathrm{Nbhd}(0_X)$ such that $\alpha W \subseteq U$ for all $0 < \alpha < \delta$. Set $V := \bigcup_{\alpha \in (0,\delta)} \alpha W$. Then $V$ obeys the desired criteria. Indeed, clearly $V$ is balanced, contained in $U$ and is a neighborhood of 0.

[SKIP] Next, suppose that further that $U$ is convex. Set

$$A := \bigcap_{\alpha \in \mathbb{C} : |\alpha| = 1} \alpha U$$

and let $V$ be balanced and constructed as above. ☐

**Theorem 2.12.** *If $X$ is a TVS and $U \in \mathrm{Nbhd}(0_X)$, then*

1. *For $\{ r_n \}_{n \in \mathbb{N}} \subseteq (0, \infty)$ with $\lim_n r_n = \infty$,*

$$X = \bigcup_{n=1}^{\infty} r_n U \,.$$

*I.e., any neighborhood of zero is* absorbing.

2. *If $K \subseteq X$ is compact then $K$ is bounded.*

3. *If $\{ \delta_n \}_n \subseteq (0, \infty)$ are such that $\delta_j > \delta_{j+1}$ and $\lim_n \delta_n = 0$, and if furthermore $U$ is bounded, then $\{ \delta_n U \}_{n \in \mathbb{N}}$ is a local base at $0_X$ for $X$.*

*Proof.* Fix $x \in X$, and observe the map $\mathbb{C} \ni \alpha \mapsto \alpha x$ is continuous, and so, $(\cdot x)^{-1}(U) \in \mathrm{Open}(\mathbb{C})$ and contains the origin (since $0x = 0 \in U$) and so contains $\frac{1}{r_n}$ for $n$ sufficiently large. That is, $\frac{1}{r_n} x \in U$ for $n$ sufficiently large, which is $x \in r_n U$.

Next, let $K \subseteq X$ be compact, and pick some $W \in \mathrm{Nbhd}(0_X)$ which is balanced and contained in $U$. By the first part,

$$K \subseteq \bigcup_{n=1}^{\infty} nW$$

But this is an open cover, so there is some finite subset $\{ n_1, \dots, n_M \} \subseteq \mathbb{N}$ with

$$K \subseteq \bigcup_{n \in \{ n_1, \dots, n_M \}} nW \subseteq n_M W$$

assuming $n_M$ is the largest one. This last inclusion follows by the balanced assumption on $W$. Now, if $t > n_M$, then $K \subseteq tW \subseteq tU$.

Finally, let $V$ be a neighborhood of 0. Since $U$ is assumed to be bounded, then there exists some $s > 0$ such that $U \subseteq tV$ for all $t > s$. Pick $n$ sufficiently large so that $s\delta_n < 1$. Then $U \subseteq \frac{1}{\delta_n}V$, so that $V$ contains all but finitely many of the set $\delta_n U$. ☐

## 2.1 Properties of linear maps between TVS

*Claim* 2.13. Let $\Lambda : X \to Y$ be a linear map between two TVS $X, Y$. Assume that $\Lambda$ is continuous at 0. Then $\Lambda$ is continuous globally.

*Proof.* Let $U \in \text{Open}(Y)$. We want to show that $\Lambda^{-1}(U) \in \text{Open}(X)$. If $\Lambda^{-1}(U) = \varnothing$ we are finished. Otherwise, let $x \in \Lambda^{-1}(U)$, that is

$$\begin{array}{rcl} \Lambda x & \in & U \\ & \updownarrow & \\ 0_Y & \in & U - \Lambda x \\ & \downarrow & \\ U - \Lambda x & \in & \text{Nbhd}(0_Y) \, . \end{array}$$

Since $\Lambda$ is linear, $\Lambda 0_x = 0_Y$, and hence, the fact that $\Lambda$ is continuous *at* $0_X$ implies that $\Lambda^{-1}(U - \Lambda x) \in \text{Nbhd}(0_x)$. But that means that $\Lambda^{-1}(U - \Lambda x)$ is open, and moreover,

$$\begin{aligned} \Lambda^{-1}(U - \Lambda x) & \equiv \{ z \in X \mid \Lambda z \in U - \Lambda x \} \\ & = \{ z \in X \mid \Lambda(z + x) \in U \} \\ & = \left\{ z \in X \mid z + x \in \Lambda^{-1}(U) \right\} \\ & = \Lambda^{-1}(U) - x \, . \end{aligned}$$

But translations are homeomorphisms, so $\Lambda^{-1}(U)$ is open indeed. ☐

**Lemma 2.14.** *Let $\Lambda : V \to \mathbb{C}$ be a linear functional from a TVS $V$ such that $\ker(\Lambda) \neq V$. Then TFAE:*

1. *$\Lambda$ is continuous.*
2. $\ker(\Lambda) \in \text{Closed}(V)$.
3. $\ker(\Lambda)$ *is not dense in* $V$.
4. $\exists U \in \text{Nbhd}(0)$ *such that* $\Lambda|_U : U \to \mathbb{C}$ *is bounded.*

*Proof.* Continuity may just as well be characterized by saying the pre-images of closed sets are closed. Since $\ker(\Lambda) \equiv \Lambda^{-1}(\{0\})$ and $\{0\} \in \text{Closed}(\mathbb{C})$, we find that (1) implies (2).

To be dense, we'd have to have $\overline{\ker(\Lambda)} = V$. But $\ker(\Lambda)$ is already closed, so that would mean $\ker(\Lambda) = V$ which is false by hypothesis. So (2) implies (3).

Let us now assume (3). Then that means $\ker(\Lambda)$ has a complement which has a non-empty interior. I.e., for some $x \in V$,

$$(x + U) \cap \ker(\Lambda) = \varnothing \tag{2.1}$$

where $U \in \text{Nbhd}(0)$. Now, WLOG we may assume that $U$ is balanced via Theorem 2.11. Note that linearity implies that $\Lambda U$ is a balanced subset of $\mathbb{C}$. If $\Lambda U$ happens to be bounded then we have (4). Otherwise, because $\Lambda U$ is balanced and unbonded, $\Lambda U = \mathbb{C}$, in which case, there is some $y \in U$ such that $\Lambda y = -\Lambda x$. But the $\Lambda(x + y) = 0$ so that $x + y \in \ker(\Lambda)$, in contradiction with (2.1).

Now, assume (4). Then $|\Lambda\psi| < M$ for all $\psi \in U$ for some $U \in \text{Nbhd}(0)$ and some $M < \infty$. Now, if $r > 0$, $W := \frac{r}{M}U$, then by linearity, $|\Lambda\psi - \Lambda 0| = |\Lambda\psi| < r$ for any $\psi \in W$, so that $\Lambda$ is continuous at the origin, and is hence continuous globally via Claim 2.13. ☐

## 2.2 Finite-dimensional spaces

*Claim* 2.15. If $X$ is a TVS and $f:\mathbb{C}^n\to X$ is linear then $f$ is continuous.

*Proof.* By linearity, we may write for any $z\in\mathbb{C}^n$ that

$$f(z) \quad = \quad \sum_{j=1}^{n} z_j f(e_j)$$

where $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{C}^n$ and $z\equiv(z_1,\dots,z_n)$. Since $z\mapsto z_j$ is continuous for each $j$ (by definition of product topology) and addition and scalar multiplication are continuous in $X$, we find that $f$ is continuous. ☐

*Claim* 2.16. If $Y\subseteq X$ is a vector space of a TVS $X$ such that $\dim(Y)<\infty$ then $Y\in\operatorname{Closed}(X)$ and every VS isomorphism $\mathbb{C}^n\to Y$ is a TVS isomorphism.

*Proof.* We start by showing the second property. Let $f:\mathbb{C}^n\to Y$ be a VS isomorphism. We have just seen that that means $f$ is continuous. Let us denote by

$$S:=\left\{ z\in\mathbb{C}^n \,\middle|\, \sum_{j=1}^{n}|z_j|^2=1 \right\}.$$

Clearly $S$ is compact as it is topologically $\mathbb{S}^{2n-1}$, and hence, with $f$ being continuous, $f(S)$ is compact too. Since $f$ is injective and $f0_{\mathbb{C}^n}=0$, and $0_{\mathbb{C}^n}\notin S$, we must have $0_X\notin f(S)$. Hence there is some *balanced* $V\in\operatorname{Nbhd}(0_X)$ such that $V\cap f(S)=\varnothing$ (indeed, $f(S)^c\in\operatorname{Nbhd}(0_X)$ and apply Theorem 2.11). Hence

$$E \quad := \quad f^{-1}(V)=f^{-1}(V\cap Y)\subseteq\mathbb{C}^n$$

is disjoint from $S$. Since $V$ is balanced it is connected (indeed, path-connected) and one may varify that $f$ being linear implies that $E\equiv f^{-1}(V)$ is balanced and connected too. Hence $E\subseteq B_1(0)$ as $0\in E$. So the linear map $f^{-1}$ takes $V\cap Y$ into $B_1(0)$. Hence the linear map $f^{-1}$ is bounded, and so each of its coordinates is a bounded functional, so that by Lemma 2.14 we learn that $f^{-1}$ is continuous. Hence $f$ is a topological homeomorphism, and since we have assumed it is a vector space isomorphism, we conclude all together it is a TVS isomorphism.

Next, for the first property, we want to show that $\overline{Y}\subseteq Y$, which is equivalent to $Y^c\subseteq\left(\overline{Y}\right)^c$. Let $x\in Y^c$. Then $Z:=\operatorname{span}(x,Y)$ is a also finite dimensional, and by the first part, linearly homeomorphic to $\mathbb{C}^{n+1}$. Hence, since this property holds in $\mathbb{C}^{n+1}\cong Z$, we have that $x\in\left(\overline{Y}\right)^c$ where now we mean closure and complement in $Z$ and not in $X$. By definition of the subspace topology,

$$\operatorname{closure}_Z(Y)\equiv\operatorname{closure}_X(Y)\cap Z$$

and hence $\left(\overline{Y}\right)^c$ with closure and complement in $Z$ equals

$$\begin{aligned} Z\setminus\operatorname{closure}_Z(Y) &= Z\setminus(\operatorname{closure}_X(Y)\cap Z)\\ &= Z\setminus\operatorname{closure}_X(Y)\\ &\subseteq X\setminus\operatorname{closure}_X(Y)\,. \end{aligned}$$

Hence, $x\notin\operatorname{closure}_X(Y)$ which is what we wanted to show. ☐

**Theorem 2.17.** *Every locally compact TVS $X$ has finite dimension.*

*Proof.* Assume that there is some $V \in \text{Nbhd}(0_X)$ whose closure $\overline{V}$ is compact. By Theorem 2.12, $V$ is bounded, and $\{ 2^{-n}V \}_n$ forms a local base (at 0) for $X$. Since

$$\bigcup_{x\in X} x + \frac{1}{2}V$$

is an open cover for $\overline{V}$, by compactness, there are some $\{ x_1, \dots, x_m \}$ such that

$$\overline{V} \subseteq \bigcup_{x\in\{ x_1,\dots,x_m \}} x + \frac{1}{2}V$$

Define $Y := \text{span}(\{ x_1, \dots, x_m \})$, so that $\dim(Y) \leq m$, and so by the above, $Y \in \text{Closed}(X)$. We have $\overline{V} \subseteq Y + \frac{1}{2}V$. Indeed, for any $v \in \overline{V}$, there is some $j = 1, \dots, m$ with $v \in x_j + \frac{1}{2}V \subseteq Y + \frac{1}{2}V$. But $V \subseteq \overline{V}$, so actually

$$V \quad\subseteq\quad Y + \frac{1}{2}V\,.$$

Using the fact that $\lambda Y = Y$ for any $\lambda \neq 0$ (since $Y$ is a vector subspace) we find

$$\frac{1}{2}V \quad\subseteq\quad Y + \frac{1}{4}V$$

and hence

$$V \subseteq \bigcap_{n=1}^{\infty} \left(Y + 2^{-n}V\right)\,.$$

Now since $\{ 2^{-n}V \}_n$ is a local base, we use Lemma 2.9 to find $\bigcap_{n=1}^{\infty}(Y + 2^{-n}V) = \overline{Y} = Y$, so $V \subseteq Y$. This implies that $kV \subseteq Y$ for all $k \in \mathbb{N}$ (recall $kY = Y$) So $Y = X$ via Theorem 2.12. □

**Theorem 2.18.** *If $X$ is a locally bounded TVS with the Heine-Borel property then $X$ is finite-dimensional.*

*Proof.* By assumption, there is some $V \in \text{Nbhd}(0_X)$ which is bounded. We claim that $\overline{V}$ is also bounded too (via Claim 2.10). Hence by the Heine-Borel property $\overline{V}$ is compact. Thus $X$ is locally compact, and thus finite dimensional by the above theorem. □

## 2.3 Metrization

**Theorem 2.19.** *If $X$ is a TVS with a* countable *local base at $0_X$ then there is a metric $d : X^2 \to [0, \infty)$ such that:*

1. *$d$ is compatible with* $\text{Open}(X)$.
2. *$B_\varepsilon(0_X)$ is balanced for any $\varepsilon > 0$.*
3. *$d$ is translation-invariant.*

*Proof.* See Rudin 1.24. □

## 2.4 Cauchy-Sequences

**Definition 2.20.** A sequence $x : \mathbb{N} \to X$ where $X$ is a TVS (with a local base at zero $\mathcal{B}$) is called *Cauchy* iff for any $V \in \mathcal{B}$, there exists some $N_V \in \mathbb{N}$ such that

$$x_n - x_m \in V \qquad (n, m > N_V)\,.$$

*Claim* 2.21. This definition does not depend on the choice of local base $\mathcal{B}$.

*Claim* 2.22. If $X$ is a TVS with $\text{Open}(X)$ induced by some metric $d$ then sequences are Cauchy iff they are Cauchy in the usual sense from metric spaces.

## 2.5 Bounded linear maps

**Definition 2.23.** If $X, Y$ are two TVS and $\Lambda : X \to Y$ is linear, $\Lambda$ is called *bounded* iff $\Lambda$ maps bounded sets into bounded sets.

Note that this is not the same thing as a *bounded function,* whose range is a bounded set (i.e. it maps the whole domain to a bounded set). But no linear function could ever be bounded in this way.

**Theorem 2.24.** *If $X, Y$ are two TVS and $\Lambda : X \to Y$ is linear then (1)->(2)->(3).*

1. *$\Lambda$ is continuous.*
2. *$\Lambda$ is bounded.*
3. *If $x : \mathbb{N} \to X$ has $\lim_n x_n = 0$ then $\{ \Lambda x_n \}_n$ is bounded.*

It turns out that if $X$ is metrizable then (3) implies (1) too, though we will not show this.

*Proof.* Assume $\Lambda$ is continuous. We try to show it is bounded. So let $E$ be a bounded set of $X$, and we need to show $\Lambda E$ is a bounded set of $Y$. To that end, let $W \in \text{Nbhd}(0_Y)$. Continuity of $\Lambda$ implies that there is some $V \in \text{Nbhd}(0_X)$ with $\Lambda V - \Lambda 0 = \Lambda V \subseteq W$. Now since $E$ is bounded, $E \subseteq tV$ for large $t$, so

$$\Lambda E \subseteq \Lambda tV = t\Lambda V \subseteq tW .$$

Next, we claim convergent sequences are bounded. Indeed, use the fact that for a convergent sequence $x_n \to x$, the set $\{ x_n \}_n \cup \{ x \}$ is compact (any open cover has a finite sub-cover) and then employ Theorem 2.12. But now since $\Lambda$ is bounded, $\{ \Lambda x_n \}_n$ is bounded and we get (3) out of (2). □

## 2.6 Seminorms

**Definition 2.25** (Seminorm)**.** A *seminorm* on a vector space $X$ is map $p : X \to [0, \infty)$ which has all the axioms of a norm (see Definition D.7) except it is allowed that $p(x) = 0$ for some $x \neq 0$. I.e.,

1. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.
2. $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{C}$ and $x \in X$.

**Lemma 2.26.** *If $p$ is a seminorm on a vector space $X$ then: (1) $p(0) = 0$, (2) $|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$, (3) $\ker(p)$ is a vector subspace of $X$ and (4) $B := \{ x \in X \mid p(x) < 1 \}$ is convex, balanced and absorbing.*

*Proof.* For (1), use $p(\alpha x) = |\alpha| p(x)$ with $\alpha = 0$. From the triangle inequality, we have

$$p(x) = p(x - y + y) \leq p(x - y) + p(y) .$$

Since this holds with $x, y$ interchanged and $p(x - y) = p(y - x)$, we have (2) (which actually implies automatically $p \geq 0$ with $y = 0$). Next, to show $\ker(p)$ is a subspace, let $x, y \in \ker(p)$ and $\alpha \in \mathbb{C}$. Then

$$0 \leq p(\alpha x + y) \leq |\alpha| p(x) + p(y) = 0$$

so $\alpha x + y \in \ker(p)$ as well.

Let us show that $B$ is balanced: let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Then

$$\begin{aligned}\alpha B &\equiv \{\, \alpha x \mid p(x) < 1 \,\} \\ &= \left\{ \alpha x \,\middle|\, \frac{1}{|\alpha|} p(\alpha x) < 1 \right\} \\ &= \{\, x \mid p(x) < |\alpha| \,\} \\ &= \{\, x \mid p(x) < 1 \,\} \,.\end{aligned}$$

If $x, y \in B$ and $t \in (0,1)$ then

$$p(tx + (1-t)y) \quad \leq \quad tp(x) + (1-t)p(y) < t + 1 - t = 1$$

so that $B$ is convex. Finally, to see $B$ is absorbing, let $x \in X$. Pick some $s \in (0, \infty)$ with $s > p(x)$. Then

$$p\left(s^{-1}x\right) = s^{-1}p(x) < 1$$

so that $s^{-1}x \in B$ or $x \in sB$. □

Next, we discuss *families* of seminorms

**Definition 2.27** (Separating family)**.** A family $\mathcal{P}$ of seminorms on $X$ is called separating iff $\forall x \in X \setminus \{\, 0 \,\}$ there exists some $p \in \mathcal{P}$ with $p(x) \neq 0$.

**Definition 2.28** (The Minkowski functional)**.** For any convex absorbing set $A \subseteq X$, define its Minkowski functional $\mu_A$as

$$\mu_A(x) := \inf(\{\, t > 0 \mid x \in tA \,\}) \qquad (x \in X) \,.$$

The fact that $A$ is absorbing implies $\mu_A : X \to [0, \infty)$.

*Claim* 2.29. Note that if $p$ is a seminorm and $B \equiv \{\, x \in X \mid p(x) < 1 \,\}$ then as above, $B$ is convex and absorbing, and $\mu_B = p$.

*Proof.* From the above we have seen that for any $x \in X$ and $s > p(x)$, $x \in sB$ which implies $\mu_B(x) \leq s$ and hence $\mu_B \leq p$. Conversely, if $t \in (0, p(x)]$ then $p\left(\frac{1}{t}x\right) = \frac{1}{t}p(x) \geq 1$ and hence $x \notin tB$. Thus $p(x) \leq \mu_B(x)$. □

**Theorem 2.30.** *Assume $\mathcal{P}$ is a separating family of seminorms on a VS $X$ (note* Open$(X)$ *is not assumed to exist). To each $p \in \mathcal{P}$ and $n \in \mathbb{N}$ define*

$$V_n(p) := \left\{ x \in X \,\middle|\, p(x) < \frac{1}{n} \right\} .$$

*Let $\mathcal{B}$ be the collection of all finite intersections of $\{\, V_n(p) \,\}_{n \in \mathbb{N}, p \in \mathcal{P}}$. Then $\mathcal{B}$ is a convex, balanced local base for a topology on $X$ which makes it a locally convex TVS with (1) any $p \in \mathcal{P}$ is continuous and, (2) $E \subseteq X$ is bounded iff for every $p \in \mathcal{P}$, $p|_E$ is bounded.*

*Proof.* Define $A \in \text{Open}(X)$ iff $A$ is the union of translates of members of $\mathcal{B}$, which automatically defines a translation-invariant topology on $X$, for which $\mathcal{B}$ is a local-base.

Let us see that $\text{Open}(X)$ is $T_1$: Since $\mathcal{P}$ is separating, for $x \neq 0$, $p(x) > 0$ for some $p \in \mathcal{P}$. Hence $p(x) > \frac{1}{n}$ for *some $n$ sufficiently large,* whence $x \notin V_n(p)$, i.e., $0 \notin x - V_n(p)$, that is, $x$ is not in $\overline{\{\, 0 \,\}}$. This is equivalent to the fact that $\{\, 0 \,\} \in \text{Closed}(X)$, and since $\text{Open}(X)$ is translation-invariant, we are $T_1$.

Let us see that addition is continuous. For any $U \in \text{Nbhd}(0)$, we must have by the definition of the local base,

$$V_{n_1}(p_1) \cap \cdots \cap V_{n_m}(p_m) \subseteq U \tag{2.2}$$

for some finite $p_1, \ldots, p_m \in \mathcal{P}$ and $n_1, \cdots, n_m \in \mathbb{N}$. Define

$$V := V_{2n_1}(p_1) \cap \cdots \cap V_{2n_m}(p_m) \,.$$

Then

$$\begin{aligned} V+V &= V_{2n_1}(p_1)\cap\cdots\cap V_{2n_m}(p_m)+V_{2n_1}(p_1)\cap\cdots\cap V_{2n_m}(p_m) \\ &\overset{\text{subadd.}}{\subseteq} V_{n_1}(p_1)\cap\cdots\cap V_{n_m}(p_m) \\ &\subseteq U\,. \end{aligned}$$

This shows that addition is continuous.

To see that scalar multiplication is continuous: for $x\in X$ and $\alpha\in\mathbb{C}$, let $U,V$ be as above. Then thanks to $V$ being absorbing (why is it?), $x\in sV$ for some $s>0$. Define $t:=\frac{s}{1+|\alpha|s}$. If $y\in x+tV$ and $|\alpha-\beta|<\frac{1}{s}$ then

$$\beta y-\alpha x=\beta(y-x)+(\beta-a)x$$

which lies in

$$|\beta|tV+|\beta-\alpha|sV\subseteq V+V\subseteq U$$

since $|\beta|t\leq 1$ (why)? and $V$ is balanced.

Hence $X$ is a locally convex TVS. From the definition of $V_n(p)$, every $p\in\mathscr{P}$ is continuous at zero, and so on the whole of $X$ thanks to $|p(x)-p(y)|\leq p(x-y)$.

Now let $E\subseteq X$ be bounded. For any $p\in\mathscr{P}$, $V_1(p)\in\text{Nbhd}(0)$ so that $E\subseteq kV_1(p)$ for some $k$ sufficiently large as $E$ is presumed bounded. Hence $p(x)<k$ for every $x\in E$. Hence $p|_E$ is bounded.

Conversely, if $p|_E$ is bounded for any $p\in\mathscr{P}$, we need to show that $E$ is bounded. Let $U\in\text{Nbhd}(0)$. Thus $p_i|_E\leq M_i$ for all $i=1,\dots,m$ where $m$ is the same as the finite intersection (2.2) above. If $n>M_i n_i$ for all $i=1,\dots,m$ then $E\subseteq nU$, so that $E$ is indeed bounded. □

## 2.7 Quotient spaces

**Definition 2.31.** For any vector space $X$ and a vector subspace $N\subseteq X$, we define

$$\pi(x):=x+N\qquad(x\in X)\,.$$

The space $\{\,\pi(x)\,\}_{x\in X}$ is called $X/N$, the quotient of $X$ modulu $N$, with addition and multiplication defined via

$$\pi(x)+\pi(y)\equiv\pi(x+y)\,,\qquad\qquad \alpha\pi(x)\equiv\pi(\alpha x)\,.$$

One shows that since $N$ is a vector subspace, $X/N$ is a vector space, and that

$$\pi:X\to X/N$$

is a linear mapping which is surjective and has $N$ has its kernel. Recall also that considered as topological spaces, $\text{Open}(X)$ induces a topology $\text{Open}(X/N)$ (the *final topology* associated with $\pi$: $E\in\text{Open}(X/N)$ iff $\pi^{-1}(E)\in\text{Open}(X)$).

**Theorem 2.32.** *Let $N$ be a* closed *subspace of a TVS $X$. Then*

1. *$X/N$ is also a TVS with respect to the quotient topology, and $\pi:X\to X/N$ is linear, continuous and open.*
2. *If $\mathcal{B}$ is a local base for* $\text{Open}(X)$ *then $\pi(\mathcal{B})$ is a local base for* $\text{Open}(X/N)$.
3. *The following properties of $X$ are inherited by $X/N$: local convexity, local boundedness, metrizability, normability, $F$-space and Banach space.*

*Proof.* See Rudin 1.41. □

### 2.7.1 Seminorm quotient

If $p$ is a seminorm on a TVS $X$, set $N:=\ker(p)$, which as we've seen is a subspace. Define then $\tilde{p}:X/N\to[0,\infty)$ via

$$\tilde{p}(\pi(x)):=p(x)\qquad(x\in X)\,.$$

Note that $\tilde{p}$ is well-defined. Indeed, if $\pi(x)=\pi(y)$ then $x+N=y+N$ or $x-y\in N$, i.e., $p(x-y)=0$. But since $|p(x)-p(y)|\leq p(x-y)$ this implies $p(x)=p(y)$ and hence well-definedness. One may furthermore verify that $\tilde{p}$ is a *norm* on $X/N$.

### 2.8 Examples

**Example 2.33.** Let $r \in [1,\infty)$ and define

$$L^r([0,1]) \equiv \{ f : [0,1] \to \mathbb{C} \mid f \text{ is msrbl. and } p_r(f) < \infty \}$$

with

$$p_r(f) \equiv \left( \int_{t\in[0,1]} |f(t)|^r \, \mathrm{d}t \right)^{\frac{1}{r}} .$$

We claim that $p_r$ is a seminorm on $L^r([0,1])$ (indeed, $p_r(f) = 0$ if $f{=}0$ almost-everywhere, but may not be identically zero). However, $L^r([0,1])/N$ with $N \equiv \ker(p_r)$ yields a Banach space on which $\tilde{p}_r$ is a norm.

**Example 2.34.** Let $\Omega \in \mathrm{Open}(\mathbb{R}^n) \setminus \{ \varnothing \}$. Then we know that

$$\Omega = \bigcup_{n\in\mathbb{N}} K_n$$

with each $K_n \neq \varnothing$ compact, which may be chosen such that $K_n \subseteq \mathrm{interior}(K_{n+1})$. On $C(\Omega)$, we define a separating family of seminorms

$$p_n(f) := \sup(\{ |f(x)| \mid x \in K_n \}) .$$

Via Theorem 2.30 this yields a convex topology. Actually this topology is metrizable since the basis is countable, and the metric is given by

$$d(f,g) = \max_n \frac{2^{-n} p_n(f-g)}{1 + p_n(f-g)} .$$

One may show that $d$ is a *complete* metric. However, $C(\Omega)$ is not locally bounded, and is hence not normable!

**Example 2.35.** Let $\Omega \in \mathrm{Open}(\mathbb{C}) \setminus \{ \varnothing \}$, define $C(\Omega)$ as above and let $H(\Omega)$ be the subspace of holomorphic functions. Actually sequences of holomorphic functions that converge uniformly on compact sets have holomorphic limits, so $H(\Omega)$ is a *closed* subspace of $C(\Omega)$.

# 3 Completeness in the context of Banach spaces

In this chapter, we shall delve into the notion of completeness. However, before doing so, let us switch gears and specify to the case of Banach spaces (instead of TVS). Strictly speaking we don't *have* to do this (indeed, see Rudin Chapter 2), but it seems like the discussion would be smoother this way. To that end

### 3.1 Banach spaces

**Definition 3.1** (Norm)**.** A vector space $V$ is called *a normed vector space* iff there is a map

$$\|\cdot\| : V \to [0,\infty)$$

which obeys the following axioms:

1. Absolute homogeneity:
$$\|\alpha\psi\| = |\alpha| \, \|\psi\| \qquad (\alpha \in \mathbb{C}, \psi \in V) .$$
2. Triangle inequality:
$$\|\psi + \varphi\| \leq \|\psi\| + \|\varphi\| \qquad (\psi, \varphi \in V) .$$
3. Injectivity at zero: If $\|\psi\| = 0$ for some $\psi \in V$ then $\psi = 0$.

**Example 3.2.** Of course the first example of a normed vector space is simply $\mathbb{C}^n$, with the Euclidean norm:

$$\mathbb{C}^n \ni z \mapsto \|z\| \equiv \sqrt{\sum_{j=1}^{n} |z_j|^2}\,.$$

To show this is a norm we only need to establish the triangle inequality (the other two properties being easy). To that end, From the Cauchy-Schwarz inequality:

$$|\langle z, w\rangle_{\mathbb{C}^n}| \leq \|z\|\|w\|$$

we get

$$\begin{aligned} \|z+w\|^2 &\equiv \langle z+w, z+w\rangle \\ &= \|z\|^2 + \|w\|^2 + 2\,\mathbb{R}\mathrm{e}\,\{\langle z, w\rangle\} \\ &\leq \|z\|^2 + \|w\|^2 + 2\,|\langle z, w\rangle| \\ &\overset{\text{C.S.}}{\leq} \|z\|^2 + \|w\|^2 + 2\|z\|\|w\| \\ &= (\|z\| + \|w\|)^2\,. \end{aligned}$$

Hence we merely need to show the Cauchy-Schwarz inequality. To that end, if $w = 0$ there is nothing to prove. So define

$$\tilde{z} := z - \frac{\langle z, w\rangle}{\|w\|^2} w\,.$$

By construction, $\langle \tilde{z}, w\rangle = 0$ so

$$\|z\|^2 = \left\|\tilde{z} + \frac{\langle z, w\rangle}{\|w\|^2} w\right\|^2 = \|\tilde{z}\|^2 + \frac{|\langle z, w\rangle|^2}{\|w\|^4}\|w\|^2 \geq \frac{|\langle z, w\rangle|^2}{\|w\|^4}\|w\|^2\,.$$

*Remark* 3.3. Be careful that in the foregoing example we have used the inner-product structure of $\mathbb{C}^n$, but more generally, a norm need not be associated with an inner product.

**Definition 3.4** (Inner product space)**.** An inner-product space is a vector space $\mathcal{H}$ along with a sesquilinear map

$$\langle \cdot, \cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

such that

1. Conjugate symmetry:
$$\langle \psi, \varphi\rangle = \overline{\langle \varphi, \psi\rangle} \qquad (\varphi, \psi \in \mathcal{H})\,.$$
2. Linearity in second argument:
$$\langle \psi, \alpha\varphi + \tilde{\varphi}\rangle = \alpha\,\langle \psi, \varphi\rangle + \langle \psi, \tilde{\varphi}\rangle \qquad (\varphi, \tilde{\varphi}, \psi \in \mathcal{H}, \alpha \in \mathbb{C})\,.$$
3. Positive-definite:
$$\langle \psi, \psi\rangle > 0 \qquad (\psi \in \mathcal{H} \setminus \{\,0\,\})\,.$$

**Example 3.5.** Of course $\mathbb{C}^n$ with

$$\langle z, w\rangle_{\mathbb{C}^n} \equiv \sum_{j=1}^{n} \overline{z_j} w_j$$

is an inner-product space.

To every inner product one immediately may associate a norm, via

$$\|\psi\| := \sqrt{\langle \psi, \psi\rangle} \qquad (\psi \in \mathcal{H})\,.$$

The converse, however, hinges on the norm obeying the parallelogram law

*Claim* 3.6. If a norm satisfies the parallelogram law:

$$\|\psi+\varphi\|^2+\|\psi-\varphi\|^2 \leq 2\|\psi\|^2+2\|\varphi\|^2 \qquad (\varphi,\psi\in\mathcal{H})$$

then

$$\langle\psi,\varphi\rangle := \frac{1}{4}\left[\|\psi+\varphi\|^2-\|\psi-\varphi\|^2+\mathrm{i}\|\mathrm{i}\psi-\varphi\|^2-\mathrm{i}\|\mathrm{i}\psi+\varphi\|^2\right]$$

defines an inner product whose associated norm is $\|\cdot\| \equiv \sqrt{\langle\cdot,\cdot\rangle}$. Conversely if the parallelogram law is violated then *no* inner-product may be defined compatible with that norm.

*Proof.* Left as an exercise to the reader. □

**Example 3.7** (Normed vector space which is not an inner product space)**.** Consider the space $\mathbb{C}^n$, but now with the $L^1$ norm

$$\|z\|_1 := \sum_{j=1}^{n} |z_j| \,.$$

Convince yourself that it is indeed a norm, and furthermore, that it violates the parallelogram law and hence cannot be associated with any inner product.

Another example we will see later is that the space of bounded linear operators on a Hilbert space is a normed vector space which is not an inner-product space.

We will continue with inner product spaces a little later, but for now we focus on *normed* vector spaces.

To any norm $\|\cdot\|$ a metric is associated via

$$\begin{array}{rcl} d:V^2 & \to & [0,\infty) \\ (\psi,\varphi) & \mapsto & \|\psi-\varphi\|\,. \end{array}$$

Hence every normed vector space is also a metric space automatically. Recall that a metric space is termed *complete* if every Cauchy sequence on it converges.

**Definition 3.8** (Banach space)**.** If a normed vector space $(V,\|\cdot\|)$ is complete when regarded as a metric space, then we refer to it as a *Banach* space.

**Example 3.9.** It is clear that $\mathbb{C}^n$ as a TVS is also a Banach space with the Euclidean norm.

**Example 3.10** (Counter-example)**.** Let $X := \{\, f:[0,1]\to\mathbb{C} \mid f \text{ is continuous}\,\}$. On $X$ define pointwise addition and multiplication, which makes it into a VS. We furthermore define on it the $L^2$-norm

$$\|f\|_2 := \sqrt{\int_{t\in[0,1]} |f(t)|^2\,\mathrm{d}t}\,.$$

One shows that on $X$, $\|\cdot\|_2$ is indeed a norm. However, one may find Cauchy-sequences in $X$ which converge to discontinuous functions (i.e. do not converge in $X$) and thus $X$ is incomplete. Contrast this with $(X,\|\cdot\|_\infty)$ which *is* a Banach space.

Here is an $L^2$-Cauchy sequence of continuous functions converging to a discontinuous function:

$$f_n(t) := \chi_{\left[\frac{1}{2}+2^{-n},1\right]}(t)+\chi_{\left[\frac{1}{2}-2^{-n},\frac{1}{2}+2^{-n}\right]}(t)\left(2^{n-1}t-2^{n-2}+\frac{1}{2}\right) \qquad (t\in[0,1]\,,n\in\mathbb{N})\,.$$

One shows that

$$\|f_n-f_m\|_2 \quad\leq\quad 2^{-n} \qquad (m\geq n)$$

and the sequence is hence Cauchy. But alsom, $\left\|\chi_{\left[\frac{1}{2},1\right]}-f_n\right\|_2 \to 0$ and $\chi_{\left[\frac{1}{2},1\right]}\notin X$.

**Definition 3.11** (Dense subsets)**.** If $(V, \|\cdot\|)$ is a Banach space and $S \subseteq V$ then we say $S$ is *dense* in $V$ iff for any $\psi \in V$ and any $\varepsilon > 0$ there exists some $\varphi \in S$ such that

$$d(\psi, \varphi) < \varepsilon .$$

This definition agrees with the topological one ($S$ is dense iff $\overline{S} = V$).

**Definition 3.12** (Separable spaces)**.** If $(V, \|\cdot\|)$ is a Banach space which contains a countable, dense subset then $V$ is called *separable*.

**Proposition 3.13.** *Any Banach space is also a TVS.*

*Proof.* We need to show that addition and multiplication are continuous w.r.t. the metric induced by the norm. Let $(X, \|\cdot\|)$ be a Banach space then. Given any $\varepsilon > 0$ and $x, y \in X$, we want to show that there are $\delta_1, \delta_2 > 0$ (which depend on $\varepsilon$) such that if $\tilde{x} \in B_{\delta_1}(x)$ and $\tilde{y} \in B_{\delta_2}(y)$ then

$$(\tilde{x} + \tilde{y}) \in B_\varepsilon(x + y) .$$

We solve the constraint backwards via the triangle inequality:

$$\begin{aligned} \|\tilde{x} + \tilde{y} - x - y\| &\leq \|\tilde{x} - x\| + \|\tilde{y} - y\| \\ &\leq \delta_1 + \delta_2 . \end{aligned}$$

So it is clear that picking $\delta_1 = \delta_2 = \frac{1}{2}\varepsilon$ will do the job.

For scalar multiplication one follows a similar route: given any $\varepsilon > 0$ and $x \in X, \lambda \in \mathbb{C}$, we seek some $\delta_1, \delta_2 > 0$ such that if $\tilde{x} \in B_{\delta_1}(x)$ and $\tilde{\lambda} \in B_{\delta_2}(\lambda)$ then

$$\tilde{\lambda}\tilde{x} \in B_\varepsilon(\lambda x) .$$

We have

$$\begin{aligned} \left\|\tilde{\lambda}\tilde{x} - \lambda x\right\| &= \left\|\tilde{\lambda}\tilde{x} - \lambda\tilde{x} + \lambda\tilde{x} - \lambda x\right\| \\ &\leq \left|\tilde{\lambda} - \lambda\right| \|\tilde{x}\| + |\lambda| \|\tilde{x} - x\| \\ &\leq \left|\tilde{\lambda} - \lambda\right| (\|\tilde{x} - x\| + \|x\|) + |\lambda| \|\tilde{x} - x\| \\ &\leq \delta_2 (\delta_1 + \|x\|) + |\lambda| \delta_1 \\ &\leq \delta_2 (1 + \|x\|) + |\lambda| \delta_1 \end{aligned}$$

Hence if we pick $\delta_1 := \min\left(\left\{ \frac{1}{|\lambda|}\frac{1}{2}\varepsilon, 1 \right\}\right)$ and $\delta_2 := \frac{1}{1+\|x\|}\frac{1}{2}\varepsilon$ we satisfy the constraint.

Lastly we want to show that $X$ is indeed $T_1$, i.e., that $\{ x \} \in \text{Closed}(X)$ for any $x \in X$. This is a property that is always true for metric spaces: indeed, we can show $\{ x \}^c$ is open by taking any open ball around any point in $\{ x \}^c$ thanks to the non-degeneracy of the metric: if $x \neq y$ then $d(x, y) > 0$. ☐

### 3.1.1 The operator norm

Recall from the discussion on TVS that $S \subseteq X$ is bounded iff for *any* $N \in \text{Nbhd}(0_X)$,

$$S \subseteq tN$$

for all $t$ sufficiently large.

*Claim* 3.14. On a Banach space $X$, $S \subseteq X$ is bounded iff

$$\sup_{x \in S} \|x\| < \infty .$$

*Proof.* All this requires is actually that the metric $d$ is homogeneous:

$$d(\alpha x, \alpha y) = |\alpha| d(x,y) \qquad (\alpha \in \mathbb{C}, x, y \in X) .$$

This property is automatically true for metrics induced by a norm, thanks to the homogeneity of the norm.

First assume that $S$ is bounded in the TVS sense. We may pick the neighborhood of zero as $N := B_1(0_X)$. That means that there exists some $t_0 > 0$ such that if $t \geq t_0$ then

$$S \subseteq tB_1(0_x) .$$

Elements in $tB_1(0_x)$ are of the form

$$\begin{aligned} tB_1(0_x) &\equiv t\{ x \in X \mid \|x\| < 1 \} \\ &= \{ x \in X \mid \|x\| < t \} \\ &= B_t(0_x) . \end{aligned}$$

I.e., we have

$$\sup_{x\in S}\|x\| \leq t_0 .$$

Conversely, assume that $S$ is bounded in the metric space sense, which we may take to mean that $S \subseteq B_M(0_X)$ for some $M \in (0,\infty)$ and let $N \in \mathrm{Nbhd}(0_X)$ be given. Since the open balls at zero form a local basis, we must have $B_\varepsilon(0_X) \subseteq N$ for some $\varepsilon > 0$. Then

$$\begin{aligned} S &\subseteq B_M(0_X) \\ &= \frac{M}{\varepsilon} B_\varepsilon(0_X) \\ &\subseteq \frac{M}{\varepsilon} N . \end{aligned}$$

□

Given any two Banach spaces $X, Y$, we may consider a *continuous linear* map

$$A : X \to Y .$$

We have seen above that such maps are automatically bounded Theorem 2.24: If $A$ is continuous then $A$ maps bounded sets of $X$ to bounded sets of $Y$. In light of Claim 3.14, we rephrase this as saying: If $A : X \to Y$ is continuous, then

$$A(B_r(0_X)) \subseteq B_M(0_Y) .$$

In other words,

$$\sup_{\|x\|_X \leq r} \|Ax\|_Y < \infty .$$

An extremely useful notion in this regard for continuous linear maps is that of the

**Definition 3.15** (The operator norm)**.** Given a linear map $A : X \to Y$ between Banach spaces, we define its *operator norm* as

$$\|A\|_{\mathcal{B}(X\to Y)} := \sup(\{ \|Ax\|_Y \mid x \in X : \|x\| \leq 1 \})$$

and $\mathcal{B}(X \to Y)$ as the space of all *bounded linear maps.* I.e., the operator norm gives us the maximal scaling of the unit ball in the domain.

*Claim* 3.16. The "operator norm" is indeed a norm.

*Proof.* Absolute homogeneity is clear. Now if $\|A\|_{\mathcal{B}(X\to Y)}=0$ then $\|Ax\|_Y=0$ for all $\|x\|\leq 1$, which implies that $Ax=0$ for all $x$, and hence $A=0$. Finally, the triangle inequality follows by that of $\|\cdot\|_Y$:

$$\|(A+B)\,x\|_Y\leq\|Ax\|_Y+\|Bx\|_Y\qquad(\|x\|\leq 1)\ .$$

Take now the supremum over $\|x\|\leq 1$ of both sides to obtain

$$\begin{aligned}\sup_{\|x\|\leq 1}\|(A+B)\,x\|_Y &\leq \sup_{\|x\|\leq 1}\left[\|Ax\|_Y+\|Bx\|_Y\right]\\ &\leq \left(\sup_{\|x\|\leq 1}\|Ax\|_Y\right)+\sup_{\|x\|\leq 1}\|Bx\|_Y\,.\end{aligned}$$

□

Summarizing the above succinctly, we have seen in Theorem 2.24 is that if $A:X\to Y$ is linear and continuous, then

$$\|A\|_{\mathcal{B}(X\to Y)}<\infty\,.$$

*Claim* 3.17. If $A:X\to Y$ is a linear map between two Banach spaces and if $\|A\|_{\mathcal{B}(X\to Y)}<\infty$ then $A$ is continuous.

*Proof.* This could have been demonstrated in Theorem 2.24 if we assumed that $X,Y$ are merely metrizable, but for Banach spaces (with a homogeneous metric) things are even simpler. Indeed, given $x\in X$ and $\varepsilon>0$, we show continuity at $x$ as follows: for any $\tilde{x}\in B_{\frac{\varepsilon}{\|A\|}}(x)$, we have (using Lemma 3.18 right below)

$$\begin{aligned}\|Ax-A\tilde{x}\| &= \|A\,(x-\tilde{x})\|\\ &\leq \|A\|\|x-\tilde{x}\|\\ &= \varepsilon\,.\end{aligned}$$

□

**Lemma 3.18.** *If $A:X\to Y$ is a bounded linear map between two Banach spaces then*

$$\|Ax\|_Y\leq\|A\|_{\mathcal{B}(X\to Y)}\|x\|_X\,.$$

*Proof.* We write thanks to the homogeneity of the norm,

$$\begin{aligned}\|Ax\|_Y &= \frac{\|Ax\|_Y}{\|x\|_X}\|x\|_X\\ &= \left\|A\frac{x}{\|x\|_X}\right\|_Y\|x\|_X\,.\end{aligned}$$

But since

$$\left\|\frac{x}{\|x\|_X}\right\|_X=1$$

we must have

$$\left\|A\frac{x}{\|x\|_X}\right\|_Y\leq\|A\|_{\mathcal{B}(X\to Y)}\,.$$

□

**Lemma 3.19.** *The operator norm is submultiplicative: If $A,B:X\to X$ then*

$$\|AB\|_{\mathcal{B}(X)}\leq\|A\|_{\mathcal{B}(X)}\|B\|_{\mathcal{B}(X)}\,.$$

*Proof.* We have thanks to the above

$$\|ABx\| \leq \|A\|_{\mathcal{B}(X)} \|Bx\|$$

taking the supremum over $\|x\| \leq 1$ on both sides we obtain the result. □

*Claim* 3.20 (R&S Thm. III.2). If $X, Y$ are two Banach spaces then $\mathcal{B}(X, Y)$ together with the operator norm is itself a Banach space.

*Proof.* Thanks to Claim 3.16 we know that $\left(\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{B}(X,Y)}\right)$ is indeed a normed vector space (with pointwise addition and scalar multiplication). To show it is a Banach space we need to show it is complete. Let $\{A_n\}_n$ be Cauchy. Then that means that $\|A_n - A_m\|_{\mathcal{B}(X,Y)}$ is small as $n, m$ are large. This implies that for any $x \in X$,

$$\|(A_n - A_m) x\|_Y = \|A_n x - A_m x\|$$

is small. I.e., the sequence $\{A_n x\}_x$ is Cauchy in $Y$. Since $Y$ itself is a Banach space (and is hence complete) that means it converges to some $y \in Y$. Define a new operator, $B$, via

$$X \ni x \mapsto \lim_{n\to\infty} A_n x \in Y$$

which is clearly linear too since the limit is linear.

From the triangle inequality we have

$$\|A_n - A_m\| \geq |\|A_n\| - \|A_m\||$$

so that $\{\|A_n\|\}_n$ is a Cauchy sequence of real numbers, and so converges to some $\alpha \in \mathbb{R}$. Hence, by definition of $B$,

$$\begin{aligned} \|Bx\|_Y &= \lim_{n\to\infty} \|A_n x\|_Y \\ &\leq \lim_{n\to\infty} \|A_n\|_{\mathcal{B}(X\to Y)} \|x\|_X \\ &= \alpha \|x\|_X . \end{aligned}$$

Hence $B$ is bounded, and so continuous. We want to show that $\lim_n A_n = B$ *in operator norm*. We have, by definition of $B$,

$$\|(B - A_m) x\|_Y = \lim_{n\to\infty} \|(A_n - A_m) x\|_Y$$

so that for $\|x\| \leq 1$ we have

$$\|(B - A_m) x\|_Y \quad \leq \quad \lim_{n\to\infty} \|A_n - A_m\|_{\mathcal{B}(X\to Y)}$$

which implies

$$\|B - A_m\|_{\mathcal{B}(X\to Y)} \leq \lim_{n\to\infty} \|A_n - A_m\|_{\mathcal{B}(X\to Y)} .$$

The right hand side however becomes arbitrarily small for large $m$. □

**Definition 3.21.** A linear map $A : X \to Y$ between Banach spaces is called an *isometry* iff $\|Ax\|_Y = \|x\|_X$ for any $x \in X$.

*Claim* 3.22. A closed vector subspace of a Banach space is itself a Banach space.

## 3.2 Completeness

### 3.2.1 Baire category

**Definition 3.23.** If $S$ is a topological space, $E \subseteq S$ is called *nowhere dense* iff $\overline{E}$ has an empty interior:

$$\text{int}\left(\text{cl}\left(E\right)\right) = \varnothing\,.$$

Then,

1. Sets "of the first category" or "meager" are subsets of $S$ which are the countable unions of nowhere dense sets.
2. Sets "of the second category" or "nonmeager" are subsets of $S$ which are not of the first category.

**Example 3.24** (Nowhere dense set)**.** $\mathbb{Z}$ is nowhere dense within $\mathbb{R}$. On the other hand, $\mathbb{Q}$ is *not* nowhere dense within $\mathbb{R}$. Also, $(0,1)$ is not nowhere dense within $\mathbb{R}$. Moreover, $\mathbb{R}$ is nowhere dense in $\mathbb{C}$. $\varnothing$ is nowhere dense, and it is the only one in a discrete space. Any vector subspace in a TVS is either dense or nowhere dense. The Cantor set is nowhere dense within $[0,1]$.

*Claim* 3.25. Let $S$ be a topological space.

1. If $A \subseteq B$ and $B$ is of the first category, so is $A$.
2. Any countable union of sets of the first category is of the first category.
3. Any $E \in \text{Closed}\left(S\right)$ such that $\text{int}\left(E\right) = \varnothing$ is of the first category.
4. If $h : S \to S$ is a topological isomorphism (homeomorphism) and $B \subseteq S$ then $B$ and $h\left(B\right)$ are of the same category.

*Proof.* For (1), we have $B = \bigcup_{j\in\mathbb{N}} N_j$ where $N_j$ are nowhere dense. Hence $A = B \cap A = \bigcup_{j\in\mathbb{N}} N_j \cap A$, but each $N_j \cap A$ is nowhere dense:

$$\text{cl}\left(N_j \cap A\right) \subseteq \text{cl}\left(N_j\right) \cap \text{cl}\left(A\right) \subseteq \text{cl}\left(N_j\right)$$

and so

$$\text{int}\left(\text{cl}\left(N_j \cap A\right)\right) \subseteq \text{int}\left(\text{cl}\left(N_j\right)\right) = \varnothing\,.$$

For (2), we note that the countable union of a countable union is itself a countable union, and (3) follows from the definition. For (4), suppose $B$ is of the first category so that $B = \bigcup_{j\in\mathbb{N}} N_j$ where $N_j$ are nowhere dense. Then,

$$\begin{aligned} h\left(B\right) &= h\left(\bigcup_{j\in\mathbb{N}} N_j\right) \\ &= \bigcup_{j\in\mathbb{N}} h\left(N_j\right)\,. \end{aligned}$$

But each $h\left(N_j\right)$ is nowhere dense. Indeed, homeomorphisms map closure and interiors to closures and interiors resp. ☐

**Theorem 3.26** (Baire's category theorem)**.** *If $S$ is either a* complete metric space *or a* locally compact Hausdorff space *and $\left\{ V_j \right\}_{j\in\mathbb{N}}$ are open dense sets then $\bigcap_{j\in\mathbb{N}} V_j$ is dense in $S$ itself. In particular, $S$ is of the second category.*

*Proof.* We first show the "in particular": Note that if $\left\{ E_j \right\}_j \subseteq \mathcal{P}\left(S\right)$ are nowhere dense, we have the following equivalent chain:

$$\begin{aligned} \text{int}\left(\text{cl}\left(E_j\right)\right) &= \varnothing \\ \text{int}\left(\text{cl}\left(E_j\right)\right)^c &= S \\ \text{cl}\left(\left(\text{cl}\left(E_j\right)\right)^c\right) &= S \\ \left(\text{cl}\left(E_j\right)\right)^c &\text{ is dense}\,. \end{aligned}$$

Hence Baire's conclusion says that

$$\begin{aligned}\bigcap_j (\mathrm{cl}\,(E_j))^c &\neq \varnothing \\ \bigcup_j \mathrm{cl}\,(E_j) &\neq S\end{aligned}$$

whence it is impossible that $\bigcup_j E_j = S$, i.e., $S$ is of the second category.

We now tend to the proof of the actual claim. We only show the claim if $S$ is a complete metric space, the other case left as an exercise. Let $\{ V_j \}_j$ be a collection of dense open subsets. Suffice to show that given some $W \in \mathrm{Open}\,(S)$, $W \cap \bigcap_{j\in\mathbb{N}} V_j \neq \varnothing$. Since $V_1$ is dense, $W \cap V_1$ is non-empty. Let $x_1 \in W \cap V_1$. Since $W \cap V_1$ is open, there is some $r_1 \in \left(0, \frac{1}{2}\right)$ such that

$$\overline{B_{r_1}(x_1)} \subseteq W \cap V_1\,.$$

We proceed inductively to define $x_j \in B_{r_{j-1}}(x_{j-1}) \cap V_j$, $r_j \in \left(0, \frac{1}{2^j}\right)$ such that

$$\overline{B_{r_j}(x_j)} \subseteq B_{r_{j-1}}(x_{j-1}) \cap V_j\,.$$

Hence

$$\begin{aligned} x_j &\in B_{r_{j-1}}(x_{j-1}) \cap V_j \\ &\subseteq B_{r_{j-2}}(x_{j-2}) \cap V_{j-1} \cap V_j \\ &\subseteq B_{r_1}(x_1) \cap \bigcap_{l=2}^{j} V_l \\ &\subseteq W \cap \bigcap_{l=1}^{j} V_l\,. \end{aligned}$$

We now claim that $\{ x_j \}$ is Cauchy. Indeed, if $\varepsilon > 0$ then pick any $N \in \mathbb{N}$ such that $2^{-N} < \frac{1}{2}\varepsilon$. Then if $n, m \geq N$ we get

$$x_n, x_m \in B_{r_N}(x_N)$$

so that

$$d(x_n, x_m) < \varepsilon\,.$$

But since $S$ is complete, we conclude $x_n \to x$ for some $x$. But for every $N \in \mathbb{N}$,

$$x_n \in \overline{B_{r_N}(x_N)} \qquad (n \geq N)$$

so that (since the limit must lie in the closure of the set where all terms lie)

$$x \in \overline{B_{r_N}(x_N)} \subseteq B_{r_{N-1}}(x_{N-1}) \cap W \subseteq V_N \cap W\,.$$

Since this is true for every $N$, we conclude that $x \in W \cap \bigcap_j V_j$, i.e., that intersection is not empty. □

For fun, here is an example of an application of the category theorem

*Claim* 3.27. $[0,1]$ is uncountable.

*Proof.* Assume otherwise, so that $[0,1] = \bigcup_{x\in[0,1]} \{ x \}$ is a countable union. Since each $\{ x \}$ is nowhere dense, $[0,1]$ is realized as the countable union of nowhere dense sets, i.e., it is of the first category. But $[0,1]$ is a complete metric space, so it is of the second category! □

**The Banach-Steinhaus theorem: the principle of uniform boundedness**

**Theorem 3.28** (Banach-Steinhaus)**.** *Let $X$ be a Banach space and $Y$ be a normed vector space. Denote by $\mathcal{B}(X \to Y)$ the Banach space of all* continuous *linear operators $X \to Y$. For $F \subseteq \mathcal{B}(X \to Y)$, if*

$$\sup_{A\in F}\|Ax\|_Y < \infty \qquad (x \in X)$$

*then*

$$\sup_{A\in F}\|A\|_{\mathcal{B}(X\to Y)} < \infty\,.$$

We have seen before that continuous implies bounded for linear maps (see Theorem 2.24), i.e.,

$$\|A\|_{\mathcal{B}(X\to Y)} < \infty \qquad (A \in F)\ .$$

The content of the theorem is that, for continuous families, pointwise uniform (in $F$) boundedness implies *uniform (*in both $F$ and $X$) boundedness.

*Proof.* Define

$$X_n := \left\{\, x \in X \;\middle|\; \sup_{A\in F}\|Ax\|_Y \leq n \,\right\} \qquad (n \in \mathbb{N})\ .$$

This is a closed set with $\bigcup_{n\in\mathbb{N}} X_n = X$. Since $X$ is of the second category, Theorem 3.26 implies that there must be some $n_0 \in \mathbb{N}$ for which $X_{n_0}$ is not nowhere dense, i.e., for which $\operatorname{int}(X_{n_0}) \neq \varnothing$: $\exists x_0 \in X_{n_0}$ and $\varepsilon > 0$ with

$$\overline{B_\varepsilon(x_0)} \subseteq X_{n_0}\,.$$

Now, let $u \in X$ with $\|u\| \leq 1$ and $A \in F$. Then

$$\begin{aligned}\|Au\|_Y &= \frac{1}{\varepsilon}\|A(x_0+\varepsilon u) - Ax_0\|_Y \\ &\leq \frac{1}{\varepsilon}\|A(x_0+\varepsilon u)\| + \frac{1}{\varepsilon}\|Ax_0\| \\ &\leq \frac{1}{\varepsilon}(n_0+n_0)\end{aligned}$$

which yields the result. □

As an example application, here is the following

*Claim* 3.29. Let $X, Y, Z$ be Banach spaces and $B : X \times Y \to Z$ a separately-continuous bilinear mapping (i.e., for each $x \in X$, $B(x,\cdot) : X \to Z$ is continuous and the same for the other slot). Then $B$ is actually jointly continuous in its two slots.

*Proof.* Since $B(x,\cdot)$ is continuous for any $x \in X$, there is some $c_x \in (0,\infty)$ such that

$$\|B(x,y)\| \leq c_x\|y\| \qquad (y \in Y)\ .$$

Hence

$$\|B(x,y)\| \leq c_x \qquad (y \in \partial B_1(0_Y))\ .$$

We now consider the family $\mathcal{F} := \{\, B(\cdot,y) : X \to Z \mid y \in \partial B_1(0_Y) \,\}$. Since this is a family of continuous maps, the uniform boundedness principle implies there is some $M > 0$ such that

$$\|B(\cdot,y)\|_{\mathcal{B}(X\to Z)} \leq M \qquad (y \in \partial B_1(0_Y))\ .$$

Hence if $y \neq 0$,

$$\begin{aligned}\|B(x,y)\|_Z &= \|y\|_Y \left\| B\left(x, \frac{y}{\|y\|}\right)\right\|_Z \\ &\leq \|y\|_Y \left\| B\left(\cdot, \frac{y}{\|y\|}\right)\right\|_{\mathcal{B}(X\to Z)} \|x\|_X \\ &\leq M\|x\|\|y\| .\end{aligned}$$

☐

*Proof without directly referencing the uniform boundedness principle.* It suffices to show that if $x_n \to 0, y_n \to 0$ then $B(x_n, y_n) \to 0$. Let us define

$$T_n y := B(x_n, y) \qquad (y \in Y) .$$

Each $T_n$ is bounded by continuity of $B(x_n, \cdot)$. We have $\{ \|T_n y\| \}_n$ bounded pointwise in $y$. Indeed, this follows from $B(\cdot, y)$ being bounded and $x_n \to 0$. But then, since boundedness implies continuity,

$$\|T_n y\| \leq C\|y\|_Y \qquad (n \in \mathbb{N}, y \in Y)$$

for some $C < \infty$. Hence

$$\|B(x_n, y_n)\| = \|T_n y_n\| \leq C\|y_n\| \to 0 .$$

☐

Contrast this with how horribly things can go for *non*linear functions:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} . \end{aligned}$$

### 3.2.2 The open mapping theorem

**Definition 3.30.** A map $f : X \to Y$ between two topological spaces is open iff $\forall U \in \operatorname{Open}(X)$, $f(U) \in \operatorname{Open}(Y)$.

*Claim* 3.31. A linear map $f : X \to Y$ between two TVS is open iff the image of any $N \in \operatorname{Nbhd}(0_X)$ contains some $M \in \operatorname{Nbhd}(0_Y)$.

*Proof.* First assume that $f : X \to Y$ is indeed open and let $N \in \operatorname{Nbhd}(0_X)$. Since $f$ is open, $f(N) \in \operatorname{Open}(Y)$. Since $f$ is linear, $f(0_X) = 0_Y$ and hence $0_Y \in f(N)$, i.e., $f(N) \in \operatorname{Nbhd}(0_Y)$.

Conversely, let $U \in \operatorname{Open}(X)$ and try to show $f(U) \in \operatorname{Open}(Y)$. To that end, let $y \in f(U)$ and try to show that there is some $M \in \operatorname{Nbhd}(y)$ with $M \subseteq f(U)$. Since $y \in f(U)$, there is some $x \in U$ with $f(x) = y$. That means that $0_X \in U - x$, i.e., $(U - x) \in \operatorname{Nbhd}(0_X)$. Applying the hypothesis on $U - x$ we find $L \subseteq f(U - x)$ for some $L \in \operatorname{Nbhd}(0_Y)$. By linearity, this implies $L + y \subseteq f(U)$, which is precisely what we wanted to show. ☐

**Theorem 3.32** (Open mapping (R&S III.10))**.** *Let $X, Y$ be Banach spaces and $A \in \mathcal{B}(X \to Y)$ is surjective. Then $A$ is an open map.*

*Proof.* Since the open balls are a local basis for $\operatorname{Open}(X)$, by the above, it suffices to show that there is some $r > 0$ such that $AB_r(0_X)$ contains some neighborhood of $0_Y$. In fact, it will suffices to only show that: (1) $\overline{AB_r(0_X)}$ has non-empty interior for some $r > 0$ and (2) $\overline{AB_r(0_X)} \subseteq AB_{2r}(0_X)$ for all $r > 0$.

*[TODO: replace this with a simpler version] Step 0: Show the that (1) and (2) together imply that $A$ is open.* Indeed, if (1) holds then $\exists y \in Y, \varepsilon > 0$ such that $B_\varepsilon(y) \subseteq \overline{A(B_r(0_X))}$, then since $y \in \overline{A(B_r(0_X))}$, there is some $\{ y_n \}_n \subseteq A(B_r(0_X))$ which converges to $y$. This corresponds to some sequence $x_n \in B_r(0_X)$ which converges to some $x$ which obeys $Ax = y$. Let now $\tilde{y} \in B_\varepsilon(0_Y)$. Then $\tilde{y} + y \in B_\varepsilon(y) \subseteq \overline{AB_r(0_X)}$ so that again there is some

sequence $\{\tilde{x}_n\}_n \subseteq B_r(0_X)$ which converges to some $\tilde{x}$ which obeys $A\tilde{x} = \tilde{y} + y$. Hence

$$\begin{aligned} A(\tilde{x} - x) &= A\tilde{x} - Ax \\ &= \tilde{y} + y - y \\ &= \tilde{y} \end{aligned}$$

and moreover, $\|\tilde{x} - x\| \leq 2r$ so that $\tilde{y} \in A\overline{B_{2r}(0_X)} = \overline{AB_{2r}(0_X)}$ (the last inclusion follows by continuity of $A$). Since $\tilde{y}$ was arbitrary, we conclude that there exists some $\varepsilon > 0$ such that

$$B_\varepsilon(0_Y) \subseteq \overline{AB_{2r}(0_X)}\,. \tag{3.1}$$

Using (2) now, we have $\overline{AB_{2r}(0_X)} \subseteq AB_{4r}(0_X)$ so that $B_\varepsilon(0_Y) \subseteq AB_{4r}(0_X)$, i.e., $AB_{4r}(0_X)$ contains a neighborhood of zero.

*Step 1: Show (1)*: We use the fact that

$$X = \bigcup_{k\in\mathbb{N}} kB_1(0_X) = \bigcup_{k\in\mathbb{N}} B_k(0_X)$$

thanks to Theorem 2.12. Since $A$ is surjective we have

$$Y = AX = A\left(\bigcup_{k\in\mathbb{N}} B_k(0_X)\right) = \bigcup_{k\in\mathbb{N}} AB_k(0_X)$$

where we used the fact that the image of a union is the union of the images. Since $Y$ is a Banach space, it is of the *second category*, i.e., it can't be that for all $k \in \mathbb{N}$, $A(B_k(0_X))$ is nowhere dense. I.e., there exists some $k \in \mathbb{N}$ such that

$$\left(\overline{AB_k(0_X)}\right)^\circ \neq \varnothing\,.$$

That means that $\overline{AB_k(0_X)}$ has non-empty interior, and (1) is established.

*Step 2: Show (2)*: Let $y \in \overline{AB_r(0_X)}$. By the defining property of the closure, for any $\varepsilon > 0$ there exists some $x_1(\varepsilon) \in B_r(0_X)$ with $Ax_1(\varepsilon) \in B_\varepsilon(y) = B_\varepsilon(0_Y) + y$, i.e., $y - Ax_1(\varepsilon) \in B_\varepsilon(0_Y)$. Since this is true for *any* $\varepsilon > 0$, pick that one $\varepsilon$ for which (3.1) holds with $2r$ replaced by $\frac{1}{2}r$ (i.e., for which $B_\varepsilon(0_Y) \subseteq \overline{AB_{\frac{1}{2}r}(0_X)}$). We find some $x_1 \in B_r(0_X)$ for which

$$y - Ax_1 \in B_\varepsilon(0_Y) \subseteq \overline{AB_{\frac{1}{2}r}(0_X)}\,.$$

We now apply this procedure on $y - Ax_1$ to obtain some $x_2 \in B_{\frac{1}{2}r}(0_X)$ for which

$$y - Ax_1 - Ax_2 \in B_{\frac{1}{2}\varepsilon}(0_Y) \subseteq \overline{AB_{\frac{1}{4}r}(0_X)}\,.$$

Note here in the last inclusion we are using the scaling property of the balls rather than picking a new $\varepsilon$. Repeating this process inductively $n$ times we find $x_n \in B_{2^{1-n}r}(0_X)$ such that

$$y - \sum_{j=1}^n Ax_j \in B_{2^{1-n}\varepsilon}(0_Y) \subseteq \overline{AB_{2^{-n}r}(0_X)}\,.$$

As a result, $\sum_{j=1}^n x_j$ exists and lies in $B_{2r}(0_X)$, since

$$\left\|\sum_{j=1}^n x_j\right\| \leq \sum_{j=1}^n \|x_j\| \leq \sum_{j=1}^n 2^{1-j} r = 2\left(1 - 2^{-n}\right) r \to 2r < \infty\,.$$

Since $A$ is continuous,

$$A\sum_{j=1}^n x_j = \sum_{j=1}^n Ax_j \in B_{2^{1-n}\varepsilon}(y) \Longrightarrow \lim_{n\to\infty}\sum_{j=1}^n Ax_j = y \Longrightarrow A\sum_{j=1}^\infty x_j = y\,.$$

so that $y \in AB_{2r}(0_X)$. We conclude

$$\overline{AB_r(0_X)} \subseteq B_{2r}(0_X) \qquad (r > 0) .$$

□

**Corollary 3.33** (Inverse mapping theorem). *If $A : X \to Y$ is a continuous bijection between Banach spaces then it has a continuous inverse (and so it is a homeomorphism).*

*Proof.* Since $A$ is a bijection, it is surjective. Hence it is open, which is tantamount to $A^{-1}$ being continuous. □

**Proposition 3.34.** *A linear map $A : X \to Y$ between Banach spaces is bounded if*

$$\left[A^{-1}\left(\overline{B_1(0_Y)}\right)\right]^{\circ} \neq \varnothing .$$

*Proof.* By hypothesis, there exists some $x_0, \varepsilon > 0$ such that

$$B_\varepsilon(x_0) \subseteq A^{-1}\left(\overline{B_1(0_Y)}\right) .$$

If $x \in X$ has $\|x\| < \varepsilon$, then $x_0 + x \in B_\varepsilon(x_0)$ so that

$$\|Ax\| \leq \|A(x + x_0)\| + \|Ax_0\| \leq 1 + \|Ax_0\| .$$

Hence for general $x \in X$ with $\|x\| \leq 1$, we always have

$$\left\|\frac{1}{2}\varepsilon x\right\| \leq \frac{1}{2}\varepsilon$$

so that

$$\begin{aligned} \|Ax\| &= \frac{2}{\varepsilon}\left\|A\frac{1}{2}\varepsilon x\right\| \\ &\leq \frac{2}{\varepsilon}(1 + \|Ax_0\|) \end{aligned}$$

so that $\|A\| < \infty$. □

### 3.2.3 The closed graph theorem

**Definition 3.35** (Graph of an operator). Let $f : X \to Y$ be a mapping between two normed spaces. The *graph* of $f$, denoted by $\Gamma(f)$, is given by

$$\Gamma(f) := \{ (x, y) \in X \times Y \mid f(x) = y \} .$$

**Proposition 3.36.** *For a* linear *map $f : X \to Y$, its graph $\Gamma(f)$ is a sub-vector space of the Banach space $X \times Y$ and if $\Gamma(f) \in \text{Closed}(X \times Y)$ then $\Gamma(f)$ is a Banach space in its own right.*

**Theorem 3.37** (Closed graph theorem (R&S Thm. III.12)). *Let $A : X \to Y$ be a linear map between two Banach spaces. Then $A$ is bounded iff $\Gamma(A) \in \text{Closed}(X \times Y)$.*

*Proof.* We use the sequential characterization of a closed set. First assume that $A$ is bounded. Then it is continuous. Hence, given any sequence $\{ (x_j, Ax_j) \}_{j \in \mathbb{N}} \subseteq \Gamma(A)$ which converges to some $(x, y) \in X \times Y$, with $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the two projections (continuous by definition of the product topology) we have

$$x_j = p_1((x_j, Ax_j)) \to p_1((x, y)) = x$$

and

$$Ax_j = p_2\left(\left(x_j, Ax_j\right)\right) \to p_2\left(\left(x, y\right)\right) = y$$

by continuity of the projections. But $A$ is continuous, so

$$Ax = \lim_j Ax_j = y\,.$$

Hence $(x, y) \in \Gamma(A)$ so that $\Gamma(A)$ is indeed closed.

Conversely, if $\Gamma(A)$ is closed, since it is a closed vector subspace of $X \times Y$, it is a Banach space. Let $\tilde{A} : X \to \Gamma(A)$ be given by

$$\tilde{A}x := (x, Ax)\ .$$

Clearly $\tilde{A}$ is a bijection. Its inverse is $p_1|_{\Gamma(A)}$. The restriction of a continuous map is continuous as well, so that the inverse mapping theorem states it has a continuous inverse, i.e., $\tilde{A}$ is continuous, so that $A = p_2 \circ \tilde{A}$ is continuous too. □

**Example 3.38** (Grothendieck)**.** Let $p \in (1, \infty)$ and

1. $\mu$ is a probability measure on some measure space $\Omega$.
2. $S \in \text{Closed}\left(L^p(\mu)\right)$ is a subspace.
3. $S \subseteq L^\infty(\mu)$.

Then $S$ is finite-dimensional.

*Proof.* Let $j : S \to L^\infty(\mu)$ be the injection map. Let $S$ have the subspace topology from $L^p(\mu)$, i.e., given by the norm

$$\|f\|_p := \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}$$

which is complete (verify..., strictly speaking this is only a norm on the quotient space, when quotioning by the functions which are zero a.e.).

Now if $\{ f_n \}_n$ is a sequence in $S$ such that $f_n \to f$ in $S$, and $f_n \to g$ in $L^\infty$, then $f = g$ a.e.. Indeed, we have

$$\|f - g\|_p \leq \|f - f_n\|_p + \|f_n - g\|_p$$

and

$$\|f_n - g\|_p \leq \|f_n - g\|_\infty$$

from Holder's inequality with $x \mapsto 1$. Hence $j$ satisfies the hypothesis of the closed graph theorem, i.e., that $\Gamma(j) \in \text{Closed}\left(S \times L^\infty\right)$. Hence, $\|j\|_{\mathcal{B}(S \to L^\infty)}$ is bounded, i.e., there exists some $K \in (0, \infty)$ such that

$$\|f\|_\infty \quad \leq \quad K\|f\|_p \qquad (f \in S)\ .$$

If $p \leq 2$ then $\|f\|_p \leq \|f\|_2$ by virtue of Jensen's inequality and $\alpha \mapsto \alpha^{\frac{2}{p}}$ being convex. If $p > 2$, use

$$|f|^p \leq \|f\|_\infty^{p-2} |f|^2$$

and integrate it to find that $\|f\|_\infty \leq K^{\frac{p}{2}}\|f\|_2$. In either case, $\exists M \in (0, \infty)$ such that

$$\|f\|_\infty \leq M\|f\|_2 \qquad (f \in S)\ .$$

I.e., the embedding $L^2 \to L^\infty$ is continuous.

Let $\{ \varphi_1, \ldots, \varphi_n \}$ be an orthonormal set in $S$ regarded as a subspace of $L^2$. Let $Q$ be a countable dense subset of

$$B := B_1\left(0_{\mathbb{C}^n}\right)\ .$$

Define

$$\begin{aligned}\psi : B &\to S \\ c &\mapsto \sum_{j=1}^{n} c_j \varphi_j\end{aligned}$$

Then clearly

$$\|\psi(c)\|_2 \leq \|c\|_{\mathbb{C}^n} = 1$$

so by the above,

$$\|\psi(c)\|_\infty \leq M\,.$$

Since $Q$ is countable, there is some $\Omega' \subseteq \Omega$ with $\mu(\Omega') = 1$ such that $|\psi(c)(x)| \leq M$ for every $c \in Q$ and every $x \in \Omega'$. For fixed $x$, the map

$$B \ni c \mapsto |\psi(c)(x)|$$

is a continuous function. Hence $|\psi(c)(x)| \leq M$ whenever $c \in B$ and $x \in \Omega'$. Hence $\sum_j |\varphi_j(x)|^2 \leq M^2$ for every $x \in \Omega'$. Integrating this inequality yield $n \leq M^2$ so that $\dim(S) \leq M^2$ and hence the proof of this theorem. □

# 4 Convexity

We are going to make use of *Zorn's lemma* for which we need to set up some basic terminology about *order*.

**Definition 4.1** (Partial order)**.** Given a set $X$, a relation $R \subseteq X^2$ is a *partial order* iff it is

1. Reflexive: $(a,a) \in R$ for all $a \in X$.
2. Anti-symmetric: $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$ for all $a, b \in X$.
3. Transitive: $(a,b), (b,c) \in R$ implies $(a,c) \in R$ for all $a,b,c \in X$.

The word *partial* implies that it may be the case that given $a, b \in X$, neither $(a,b) \in R$ nor $(b,a) \in R$. If this *does* happen to be the case, $R$ is called *linearly ordered*.

**Definition 4.2.** In a partial order, if there is some $m \in X$ such that

$$[(m,x) \in R \Longrightarrow x = m] \qquad (x \in X)$$

then $m$ is called a *maximal element*. If $Y \subseteq X$ and $(y,p) \in R$ for all $y \in Y$ then $p$ is called *an upper bound on* $Y$.

**Lemma 4.3** (Zorn)**.** *If $X$ is a nonempty partially ordered set such that every linearly ordered subset has an upper bound in $X$, then each linearly ordered set has some upper bound which is also a maximal element of $X$.*

## 4.1 Hahn-Banach

**Theorem 4.4** ($\mathbb{R}$-Hahn-Banach theorem)**.** *Let $X$ be an $\mathbb{R}$-vector space, $p : X \to \mathbb{R}$ be given such that*

$$p(\alpha x + (1-\alpha) y) \leq \alpha p(x) + (1-\alpha) p(y) \qquad (x, y \in X; \alpha \in [0,1])\,.$$

*Let $\lambda : Y \to \mathbb{R}$ linear where $Y \subseteq X$ is a subspace, and such that*

$$\lambda(x) \leq p(x) \qquad (x \in Y)\,.$$

*Then there exists $\Lambda : X \to \mathbb{R}$ linear such that $\Lambda|_Y = \lambda$ and such that*

$$\Lambda(x) \leq p(x) \qquad (x \in X)\,.$$

Note that the point of this theorem is not that an extension exists, but that an extension *which is also dominated by the convex function* exists.

*Proof.* Let $z \in X \setminus Y$ and $\tilde{Y} := \operatorname{span}(Y, z)$. Define $\tilde{\lambda} : \tilde{Y} \to \mathbb{R}$ via

$$\tilde{\lambda}(az + y) := a\tilde{\lambda}(z) + \lambda(y) \qquad (y \in Y, a \in \mathbb{R})$$

with $\tilde{\lambda}(z)$ to be determined soon.

Now we show we may choose $\tilde{\lambda}(z)$ so that the desired properties are obeyed.

Let $y_1, y_2 \in Y$ and $\alpha, \beta > 0$. Then

$$\begin{aligned} \beta\lambda(y_1) + \alpha\lambda(y_2) &= \lambda(\beta y_1 + \alpha y_2) = (\alpha + \beta)\lambda\left(\frac{\beta}{\alpha+\beta}y_1 + \frac{\alpha}{\alpha+\beta}y_2\right) \\ &\leq (\alpha+\beta)p\left(\frac{\beta}{\alpha+\beta}(y_1 - \alpha z) + \frac{\alpha}{\alpha+\beta}(y_2 + \beta z)\right) \\ &\leq \beta p(y_1 - \alpha z) + \alpha p(y_2 + \beta z) \end{aligned}$$

so that

$$\frac{1}{\alpha}\left[-p(y_1 - \alpha z) + \lambda(y_1)\right] \quad \leq \quad \frac{1}{\beta}\left[p(y_2 + \beta z) - \lambda(y_2)\right] .$$

Hence we have some number $q \in \mathbb{R}$ obeying

$$\sup_{y_1 \in Y,\, \alpha > 0} \frac{1}{\alpha}\left[-p(y_1 - \alpha z) + \lambda(y_1)\right] \leq q \leq \inf_{y_2 \in Y,\, \beta > 0} \frac{1}{\beta}\left[p(y_2 + \beta z) - \lambda(y_2)\right] .$$

Define $\tilde{\lambda}(z) := q$. We verify $\tilde{\lambda}(az + y) \leq p(az + y)$: Assume $a > 0$. Then applying the above inequality with $\beta = a$, $y_2 = y$ yields

$$\tilde{\lambda}(z) \quad \leq \quad \frac{1}{a}\left[p(y + az) - \lambda(y)\right]$$

which is the desired inequality.

Let now $\mathcal{E}$ be the collection of extensions $e$ of $\lambda$ which satisfy $e \leq p$ on the subspace where they are defined. Define a partial order $R$ on $\mathcal{E}$ by declaring $(e_1, e_2) \in R$ iff $e_2$ is defined on a larger subset than $e_1$ and $e_1 = e_2$ where both are defined, i.e., if $e_2$ extends $e_1$. Now if $\{ e_\alpha \}_{\alpha \in A}$ is a linearly ordered subset of $\mathcal{E}$, with $X_\alpha$ the subspace on which $e_\alpha$ is defined, define $e$ on $\bigcup_{\alpha \in A} X_\alpha$ by $e(x) := e_\alpha(x)$ if $x \in X_\alpha$ (this is unambiguous thanks to the extension property). Clearly $(e_\alpha, e) \in R$ so each linearly ordered set has an upper bound. Now apply Zorn's lemma to find that $\mathcal{E}$ has a maximal element $\Lambda$ defined on some $X' \subseteq X$ with $\Lambda(x) \leq p(x)$ for $x \in X'$. It must be that $X' = X$ since otherwise we could extend $\Lambda$ outside of $X'$ as above and then it would not be a maximal element. □

**Theorem 4.5** (Hahn-Banach theorem)**.** *Let $X$ be a vector space, $p : X \to \mathbb{R}$ be given such that*

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) \qquad (x, y \in X; \alpha, \beta \in \mathbb{C} : |\alpha| + |\beta| = 1) .$$

*Let $\lambda : Y \to \mathbb{C}$ linear where $Y \subseteq X$ is a subspace, and such that*

$$|\lambda(x)| \leq p(x) \qquad (x \in Y) .$$

*Then there exists $\Lambda : X \to \mathbb{C}$ linear such that $\Lambda|_Y = \lambda$ and such that*

$$|\Lambda(x)| \leq p(x) \qquad (x \in X) .$$

*Proof.* Define $\ell(x) := \mathbb{Re}\{\lambda(x)\}$. Then $\ell$ is a real linear functional on $Y$ and

$$\ell(\mathrm{i}x) = \mathbb{Re}\{\lambda(\mathrm{i}x)\} = \mathbb{Re}\{\mathrm{i}\lambda(x)\} = -\mathbb{Im}\{\lambda(x)\}$$

we have

$$\lambda(x) = \ell(x) - \mathrm{i}\ell(\mathrm{i}x) \qquad (x \in X) .$$

Applying the above $\mathbb{R}$-Hahn-Banach theorem on the $\mathbb{R}$-valued functional $\ell$ we find an extension $L : X \to \mathbb{R}$ with $L(x) \leq p(x)$. Define

$$\Lambda(x) := L(x) - \mathrm{i}L(\mathrm{i}x) \ .$$

Now, $\Lambda$ extends $\lambda$ and is $\mathbb{R}$-linear. $\Lambda(\mathrm{i}x) = \mathrm{i}\Lambda(x)$ so $\Lambda$ is $\mathbb{C}$-linear as well. So we now must show $|\Lambda(x)| \leq p(x)$. Note that if $|\alpha| = 1$ we have $p(\alpha x) = p(x)$ by hypothesis. Then

$$\begin{aligned} |\Lambda(x)| &= \mathrm{e}^{-\mathrm{i}\arg(\Lambda(x))}\Lambda(x) = \Lambda\left(\mathrm{e}^{-\mathrm{i}\arg(\Lambda(x))}x\right) \\ &= \mathbb{Re}\left\{\Lambda\left(\mathrm{e}^{-\mathrm{i}\arg(\Lambda(x))}x\right)\right\} \\ &= L\left(\mathrm{e}^{-\mathrm{i}\arg(\Lambda(x))}x\right) \\ &\leq p\left(\mathrm{e}^{-\mathrm{i}\arg(\Lambda(x))}x\right) \\ &= p(x) \ . \end{aligned}$$

□

# 5 Duality

**Definition 5.1.** Given a Banach space $X$, its dual, denoted by $X^*$ is the space $\mathcal{B}(X \to \mathbb{C})$, i.e., of continuous linear functionals $X \to \mathbb{C}$, which is also a Banach space. The norm on $X^*$ is of course the operator norm:

$$\|\lambda\|_{\mathrm{op}} \equiv \sup_{x \in X : \|x\| \leq 1} |\lambda(x)| \qquad (\lambda \in X^*) \ .$$

**Example 5.2.** Given some $g \in L^p(\mathbb{R})$ and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we define a *functional* $G$ on $L^q(\mathbb{R})$ as

$$G(f) := \int_{\mathbb{R}} \overline{g} f \ .$$

By Hölder's inequality, we have

$$\begin{aligned} \left|\int_{\mathbb{R}} \overline{g} f\right| &\leq \int_{\mathbb{R}} |gf| \\ &\equiv \|gf\|_1 \\ &\leq \|g\|_p \|f\|_q \end{aligned}$$

so that

$$\|G\|_{\mathrm{op}} \leq \|g\|_p \ .$$

Actually this is an equality (exercise). In fact, all bounded linear functionals on $L^q$ arise in this way, and different $g$'s give rise to different functionals. Hence, the map $L^p \ni g \mapsto G \in (L^q)^*$ is an isometric isomorphism (if we take the co-domain of all spaces as $\mathbb{R}$; otherwise these are $\mathbb{C}$-vector-spaces in which case this map is clearly $\mathbb{C}$-anti-linear). Hence we identify $L^p$ as the dual of $L^q$ for $\frac{1}{q} + \frac{1}{p} = 1$. By symmetry, it is clear that $L^p = (L^q)^* = \left((L^p)^*\right)^*$, so that $L^p$ is its own double-dual.

For $p = \infty$ this is different: the dual of $L^1$ is $L^\infty$, but the dual of $L^\infty$ is much larger than $L^1$. So it is not always true that a space is its own double dual!

**Theorem 5.3** ($X$ may be regarded as a subset of $X^{**}$)**.** *Let $X$ be a Banach space, and define*

$$J : X \to X^{**}$$

*via*

$$X \ni x \mapsto (X^* \ni \lambda \mapsto \lambda(x)) \in \mathcal{B}(X^* \to \mathbb{C}) \equiv X^{**} \ .$$

*Then $J$ is isometric and injective, and its range is a (possibly proper) subspace of $X^{**}$.*

*Proof.* For $x \in X, \lambda \in X^*$ we have $J(x) \in \mathcal{B}(X^* \to \mathbb{C})$ and the estimate

$$|J(x)(\lambda)| \equiv |\lambda(x)| \leq \|\lambda\|_{\text{op}} \|x\|_X$$

so that $\|J(x)\|_{\mathcal{B}(X^* \to \mathbb{C})} \leq \|x\|_X$. Fixing $x_0$, apply the Hahn-Banach theorem on the one-dimensional subspace $\mathbb{C}x_0$ and the linear functional

$$X \supseteq \mathbb{C}x_0 \ni \alpha x_0 \mapsto \alpha \|x_0\|_X \tag{5.1}$$

which is bounded by the convex function $p(y) := \|y\|$, to get some $\lambda_0 \in X^*$ so that

$$\begin{aligned} \|\lambda_0\|_{\text{op}} &\equiv \sup(\{ |\lambda_0(y)| \mid \|y\| \leq 1 \}) \\ &\leq \sup(\{ p(y) \mid \|y\| \leq 1 \}) \\ &= 1 . \end{aligned}$$

and $\lambda_0(x_0) = \|x_0\|_X$ (since it is an extension). Hence,

$$\|J(x_0)\|_{\mathcal{B}(X^* \to \mathbb{C})} \equiv \sup_{\tilde{\lambda} \in X^* : \|\tilde{\lambda}\|_{\text{op}} \leq 1} \left|\tilde{\lambda}(x_0)\right| \geq |\lambda_0(x_0)| = \|x_0\|_X .$$

We conclude that $\|J(x)\|_{\mathcal{B}(X^* \to \mathbb{C})} = \|x\|_X$ for all $x \in X$ so that $J$ is indeed an isometry. Clearly $J$ is linear, so its range is a subspace of its codomain. □

If it happens that $J$ is surjective we call $X$ reflexive (in which case $X^{**} \overset{J}{\cong} X$). As we have seen $L^p$ are reflexive for $p > 1$ but $L^1$ is *not*.

*Remark* 5.4. As Rudin points out in the bottom of pp. 45 of [Rud91], it is not enough to have $X^{**} \cong X$ with *any* isometric isomorphism to be called *reflexive*. We must have $X^{**} \overset{J}{\cong} X$. We shall encounter the James space as a counter-example of a space isometrically isomorphic to its double dual, but *not* reflexive.

**Lemma 5.5.** *Let $X$ be a Banach space and $Y \subseteq X$ a subspace. If $\lambda \in Y^*$ then there exists some extension $\Lambda \in X^*$ for which $\|\Lambda\|_{op} = \|\lambda\|_{op}$.*

*Proof.* Use the Hahn-Banach theorem with the dominating convex function $p(x) = \|\lambda\|_{\text{op}} \|x\|_X$ which is easily convex. Now, clearly

$$|\lambda(y)| \leq p(y) \qquad (y \in Y) .$$

Hence we get some $\Lambda : X \to \mathbb{C}$ with

$$|\Lambda(x)| \leq \|\lambda\|_{\text{op}} \|x\|_X$$

which immediately implies $\|\Lambda\|_{\text{op}} \leq \|\lambda\|_{\text{op}}$. Since $\Lambda$ extends $\lambda$, we must have also

$$\begin{aligned} \|\Lambda\|_{\text{op}} &\equiv \sup_{x \in X : \|x\| \leq 1} |\Lambda(x)| \\ &\geq \sup_{x \in Y : \|x\| \leq 1} |\Lambda(x)| \\ &= \sup_{x \in Y : \|x\| \leq 1} |\lambda(x)| \\ &\equiv \|\lambda\|_{\text{op}} . \end{aligned}$$

□

**Lemma 5.6** (Another characterization of the operator norm)**.** *If $X$ is a Banach space then*

$$\|x\| = \sup\left(\left\{ |\lambda(x)| \,\middle|\, \lambda \in X^* : \|\lambda\|_{op} \leq 1 \right\}\right) .$$

*Moreover, the supremum is attained.*

*Proof.* Let

$$\alpha := \sup\left(\left\{ |\lambda(x)| \,\middle|\, \lambda \in X^* : \|\lambda\|_{\text{op}} \leq 1 \right\}\right) .$$

Clearly $\alpha \leq \|x\|_X$ by $|\lambda(x)| \leq \|\lambda\|_{\text{op}}\|x\|_X$.

For the reverse direction, defining, for each $x_0 \in X$,

$$\mathbb{C}x_0 \ni \beta x_0 \overset{\eta}{\mapsto} \beta\|x_0\|_X$$

which has operator norm equal to 1:

$$\begin{aligned} \|\eta\|_{\text{op}} &\equiv \sup_{\beta x_0 \in \mathbb{C}x_0 : \|\beta x_0\|_X \leq 1} |\eta(\beta x_0)| \\ &\equiv \sup_{\beta x_0 \in \mathbb{C}x_0 : \|\beta x_0\|_X \leq 1} |\beta\|x_0\|| \\ &= \sup_{\beta x_0 \in \mathbb{C}x_0 : \|\beta x_0\|_X \leq 1} \|\beta x_0\| \\ &= 1 . \end{aligned}$$

So we may use the above lemma to find some $\lambda_0 \in X^*$ with $\|\lambda_0\|_{\text{op}} = 1$ and $\lambda_0(x_0) = \|x_0\|$ (since it is an extension of $\eta$) so that $\|x_0\| \leq \alpha$. The fact we exhibited $\lambda_0$ means the supremum is attained. ☐

## 5.1 Weak topologies on Banach spaces

**Definition 5.7.** Let $X$ be a Banach space with dual $X^*$. The *weak topology* on $X$ is the *weakest* (i.e., coarsest) topology on $X$ in which *each* functional $\lambda \in X^*$ is continuous. I.e., *the weak topology is the initial topology generated by $X^*$.*

Some remarks:

- Recall that $\mathcal{T}_1$ is coarser (weaker, smaller) than $\mathcal{T}_2$ iff

$$\mathcal{T}_1 \subseteq \mathcal{T}_2 .$$

  In this case we also say that $\mathcal{T}_2$ is finer (stronger, larger) than $\mathcal{T}_1$.

- Recall in particular that this topology is generated by the *sub-basis*

$$\left\{ \lambda^{-1}(E) \,\middle|\, \lambda \in X^*, E \in \text{Open}(\mathbb{C}) \right\} .$$

  So a generic open set in the weak topology is of the form

$$\bigcup_{\alpha} \bigcap_{j=1}^{n_\alpha} \lambda_{\alpha,j}^{-1}(E_{\alpha,j})$$

  with $\lambda_{\alpha,j} \in X^*$ and $E_{\alpha,j} \in \text{Open}(\mathbb{C})$, and about any point $x$ in a weakly-open set $U$ we have

$$x \in \bigcap_{j=1}^{n} \lambda_j^{-1}\left(B_{\varepsilon_j}(z_j)\right) \subseteq U .$$

  We have $\lambda_j(x) \in B_{\varepsilon_j}(z_j)$, so we may take some $\varepsilon > 0$ so that

$$B_\varepsilon(\lambda_j(x)) \subseteq B_{\varepsilon_j}(z_j) \qquad (j = 1, \ldots, n)$$

which the inverse image respects, so we get

$$x \in \bigcap_{j=1}^{n} \lambda_j^{-1}\left(B_\varepsilon\left(\lambda_j\left(x\right)\right)\right) \subseteq U\,. \tag{5.2}$$

*Remark* 5.8. A somewhat subtle point is the fact that as a Banach space, $(X, \|\cdot\|_X)$ has its topology induced by the metric which is induced by the norm. This data automatically generates

$$\text{Open}\left(X\right) \equiv \text{Open}_{\|\cdot\|_X}\left(X\right)$$

which in turns generates $X^*$ (the space of all linear *continuous* maps, where continuity is w.r.t. $\text{Open}_{\|\cdot\|_X}(X)$). Now to define the weak topology $\text{Open}_{\text{weak}}(X)$ we use the set of maps in $X^*$ (whose definition stemmed from $\text{Open}_{\|\cdot\|_X}(X)$!). Hence $\text{Open}_{\|\cdot\|_X}(X)$ *induces* $\text{Open}_{weak}(X)$. This is somewhat confusing since all $\lambda \in X^*$ are *already* continuous w.r.t. $\text{Open}_{\|\cdot\|_X}(X)$. However, $\text{Open}_{\text{weak}}(X)$ is defined as *the weakest topology on $X$ so that* all $\lambda \in X^*$ *remain* continuous. In particular,

$$\text{Open}_{\text{weak}}\left(X\right) \subseteq \text{Open}_{\|\cdot\|_X}\left(X\right)$$

so that a set open in the weak topology is necessarily open in the norm topology, and if $f : X \to Z$ is continuous in the weak topology, then for any $U \in \text{Open}(Z)$, $f^{-1}(U) \in \text{Open}_{\text{weak}}(X)$ so $f^{-1}(U) \in \text{Open}_{\|\cdot\|_X}(X)$ and hence any weakly-continuous map $f : X \to Z$ is norm continuous too.

**Lemma 5.9.** *If $X$ is an infinite dimensional Banach space then every $U \in \text{Open}_{weak}(X)$ is unbounded in $\|\cdot\|_X$.*

*Proof.* Let $x_0 \in U \in \text{Open}_{\text{weak}}(X)$. Then there is some finite collection $\lambda_1, \dots, \lambda_n \in X^*$ and $\varepsilon > 0$ such that

$$x_0 \in \bigcap_{j=1}^{n} \lambda_j^{-1}\left(B_\varepsilon\left(\lambda_j\left(x_0\right)\right)\right) \subseteq U\,.$$

By linearity, we may write

$$\begin{aligned}
\lambda_j^{-1}\left(B_\varepsilon\left(\lambda_j\left(x_0\right)\right)\right) &\equiv \left\{\, x \in X \mid \lambda_j\left(x\right) \in B_\varepsilon\left(\lambda_j\left(x_0\right)\right) \,\right\} \\
&= \left\{\, x \in X \mid \left|\lambda_j\left(x\right) - \lambda_j\left(x_0\right)\right| < \varepsilon \,\right\} \\
&= \left\{\, x \in X \mid \left|\lambda_j\left(x - x_0\right)\right| < \varepsilon \,\right\} \\
&= \left\{\, x + x_0 \in X \mid \left|\lambda_j\left(x\right)\right| < \varepsilon \,\right\} \\
&= x_0 + \left\{\, x \in X \mid \left|\lambda_j\left(x\right)\right| < \varepsilon \,\right\} \\
&= x_0 + \lambda_j^{-1}\left(B_\varepsilon\left(0_{\mathbb{C}}\right)\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\bigcap_{j=1}^{n} \lambda_j^{-1}\left(B_\varepsilon\left(\lambda_j\left(x_0\right)\right)\right) &= \bigcap_{j=1}^{n} \left(x_0 + \lambda_j^{-1}\left(B_\varepsilon\left(0_{\mathbb{C}}\right)\right)\right) \\
&= x_0 + \bigcap_{j=1}^{n} \lambda_j^{-1}\left(B_\varepsilon\left(0_{\mathbb{C}}\right)\right)
\end{aligned}$$

where the last inequality follows from $(x + A) \cap (x + B) = x + A \cap B$. Since

$$\begin{aligned}
\ker\left(\lambda_j\right) &\equiv \lambda_j^{-1}\left(\left\{\, 0_{\mathbb{C}} \,\right\}\right) \\
&\subseteq \lambda_j^{-1}\left(B_\varepsilon\left(0_{\mathbb{C}}\right)\right)
\end{aligned}$$

we get

$$x_0 \in x_0 + \bigcap_{j=1}^{n} \ker\left(\lambda_j\right) \subseteq U\,.$$

Now let

$$\begin{aligned}\eta : X &\to \mathbb{C}^n \\ x &\mapsto (\lambda_1(x), \dots, \lambda_n(x)) \ .\end{aligned}$$

Clearly, $\ker(\eta) = \bigcap_{j=1}^n \ker(\lambda_j)$. If $\ker(\eta) = \{ 0_X \}$ we obtain a linear injection $\eta$ into $\mathbb{C}^n$ which implies that $\dim(X) \le n < \infty$. So it must be that there must be some $v \in \left(\bigcap_{j=1}^n \ker(\lambda_j)\right) \setminus \{ 0_X \}$. Then by linearity we find, in particular, that for all $\alpha \in \mathbb{C}$,

$$x_0 + \alpha v \in U$$

and so clearly $U$ cannot be bounded in norm since we can make $x_0 + \alpha v$ arbitrarily large in norm:

$$\begin{aligned}\|x_0 + \alpha v\| &\ge |\alpha| \|v\| - \|x_0\| \\ &\to \infty \qquad (|\alpha| \to \infty) \ .\end{aligned}$$

□

**Lemma 5.10.** *If $X$ is an infinite dimensional Banach space then the weak topology* $\mathrm{Open}_{weak}(X)$ *does* not *arise from a metric.*

*Proof.* Assume otherwise. Then there is some metric $d : X^2 \to [0, \infty)$ which generates $\mathrm{Open}_{\mathrm{weak}}(X)$. Let then

$$U_n := \left\{ x \in X \,\middle|\, d(0_X, x) < \frac{1}{n} \right\} \qquad (n \in \mathbb{N}) \ .$$

Each of these sets is open w.r.t. the metric and hence weakly open. By Lemma 5.9 it follows that $U_n$ is norm unbounded, so that for each $n \in \mathbb{N}$ there must be some $x_n \in U_n$ with $\|x_n\| \ge n$. This however contradicts the fact that $x_n \to 0_X$ in the metric $d$, and hence weakly by hypothesis, so that $\{ x_n \}_n$ is bounded. □

**Lemma 5.11.** *$X$ is a TVS also with respect to* $\mathrm{Open}_{weak}(X)$*. Since* $\mathrm{Open}_{\|\cdot\|_X}(X)$ *is arises from the metric induced by $\|\cdot\|_X$ and* $\mathrm{Open}_{weak}(X)$ *is not metrizable if $X$ is infinite dimensional, we obtain two non-homeomorphism TVS structures on $X$ starting from merely $(X, \|\cdot\|_X)$.*

*Proof.* First we need the $T_1$ property, i.e., that singletons are closed. We shall show that $\{ 0_X \}$ is closed which suffices by translation invariance. To that end we take any point in $\{ 0_X \}^c$ and show there is a weakly-open subset around it which is entirely within $\{ 0_X \}^c$. I.e., let $x \in X \setminus \{ 0_X \}$. Then there exists some $\lambda \in X^*$ for which $\lambda(x) > 0$ (for example take the construction from (5.1)). Let $\varepsilon > 0$ so that $\lambda(x) > \varepsilon$. Then

$$x \notin \lambda^{-1}(B_\varepsilon(0_{\mathbb{C}}))$$

so

$$0_X \notin x - \lambda^{-1}(B_\varepsilon(0_{\mathbb{C}})) \in \mathrm{Nbhd}(x) \ .$$

So $x - \lambda^{-1}(B_\varepsilon(0_{\mathbb{C}}))$ is the open set about $x$ which entirely contained within $\{ 0_X \}^c$. Thus

$$\{ 0_X \} \in \mathrm{Closed}_{\mathrm{weak}}(X)$$

making this topology $T_1$.

We leave the demonstration of continuity of addition and scalar multiplication as an exercise to the reader (it follows very similarly to Rudin's Theorem 1.37). Note that the topology $\mathrm{Open}_{\mathrm{weak}}(X)$ arises from the separating family of seminorms

$$X^2 \ni (x, y) \mapsto |\lambda(x) - \lambda(y)| =: p_\lambda(x, y) \qquad (\lambda \in X^*) \ .$$

□

**Lemma 5.12** (Another characterization of the weak topology)**.** *$x_n \to x$ in $\operatorname{Open}_{weak}(X)$ iff $\lambda(x_n) \to \lambda(x)$ in $\operatorname{Open}(\mathbb{C})$ for* every *$\lambda \in X^*$.*

*Proof.* Assume that $x_n \to x$ in $\operatorname{Open}_{\text{weak}}(X)$ and let $\lambda \in X^*$. By definition, for any $U \in \operatorname{Nbhd}_{\text{weak}}(x)$ there is some $N_U \in \mathbb{N}$ with

$$x_n \in U \qquad (n \geq N_U) \ .$$

Now let $V \in \operatorname{Nbhd}_{\operatorname{Open}(\mathbb{C})}(\lambda(x))$. We want some $N_V \in \mathbb{N}$ such that

$$\lambda(x_n) \in V \qquad (n \geq N_V) \ .$$

Since $\lambda$ is weakly continuous, we know $\lambda^{-1}(V) \in \operatorname{Open}_{\text{weak}}(X)$. Hence apply the above on $U := \lambda^{-1}(V)$ to get that

$$x_n \in \lambda^{-1}(V) \qquad \left(n \geq N_{\lambda^{-1}(V)}\right)$$

which is precisely what we wanted.

Conversely, we try to show that $x_n \to x$ in $\operatorname{Open}_{\text{weak}}(X)$ given that $\lambda(x_n) \to \lambda(x)$ in $\operatorname{Open}(\mathbb{C})$ for *every* $\lambda \in X^*$. To that end, let $U \in \operatorname{Nbhd}_{\text{weak}}(x)$. We seek some $N_U \in \mathbb{N}$ such that if $n \geq N_U$ then $x_n \in U$. By the above proofs, we know we may find some $\varepsilon > 0$ and $\lambda_1, \dots, \lambda_m \in X^*$ so that

$$x \in x + \bigcap_{j=1}^{m} \lambda_j^{-1}\left(B_\varepsilon\left(0_{\mathbb{C}}\right)\right) \subseteq U \,.$$

Since we have $\lambda_j(x_n) \overset{n\to\infty}{\to} \lambda_j(x)$ for each $j$, we have for each $j$, some $N_j$ so that if $n \geq N_j$

$$\lambda_j(x_n) \quad \in \quad \lambda_j(x) + B_\varepsilon\left(0_{\mathbb{C}}\right) \ .$$

Let $N_U := \max_{j=1,\dots,m} N_j$ so that, whence for all $n \geq N_U$ we have

$$x_n - x \quad \in \quad \bigcap_{j=1}^{m} \lambda_j^{-1}\left(B_\varepsilon\left(0_{\mathbb{C}}\right)\right)$$

which implies the condition we wanted to satisfy. □

**Proposition 5.13** (Reed & Simon Chapter IV)**.** *Every weakly convergent sequence is norm bounded.*

*Proof.* Let $x_n \to x$ in $\operatorname{Open}_{\text{weak}}(X)$. For each $n \in \mathbb{N}$, let $\eta_n \in X^{**}$ be defined by

$$\eta_n(\lambda) := \lambda(x_n)$$

i.e., in the notation of Theorem 5.3, $\eta_n = J(x_n)$. Via Lemma 5.12, for any $\lambda \in X^*$, the sequence $\{\, \lambda(x_n) \,\}_n \subseteq \mathbb{C}$ converges, so $\{\, \eta_n(\lambda) \,\}_n$ is a bounded set. Hence by the uniform boundedness principle Theorem 3.28 applied on the pointwise (in $\lambda$) bounded family of operators $\eta_n : X^* \to \mathbb{C}$ we find that

$$\sup_n \|\eta_n\|_{\text{op}} < \infty \,.$$

But $\eta_n = J(x_n)$ and $J$ is an isometry (again by Theorem 5.3), so

$$\sup_n \|x_n\|_X < \infty \,.$$

□

**Definition 5.14** (Weak-star topology)**.** Let $X^*$ be the dual of some Banach space. The weak-star topology on $X^*$ is the weakest topology on $X^*$ in which all functions (determined by all $x \in X$)

$$X^* \ni \lambda \mapsto \lambda(x) \in \mathbb{C}$$

are continuous. Since we have the injective isometry $J : X \to X^{**}$ via Theorem 5.3, we say that the weak-star topology is the weakest topology on $X^*$ making all elements in $J(X)$ continuous, i.e., it is the initial topology on $X^*$ generated by $J(X) \subseteq X^{**}$.

*Claim* 5.15. If $X$ is reflexive (i.e., $J(X) = X^{**}$) then the weak topology on $X^*$ and the weak-star topology on $X^*$ coincide.

*Proof.* The weak topology on $X^*$ is characterized by the initial topology generated by $X^{**}$ whereas the weak-star topology is characterized by the initial topology generated by $J(X)$. □

Actually the converse is also true, see [Con19, pp. 131 Theorem 4.2 Banach Spaces, Reflx revisited]. It is thus clear that if $J(X) \neq X^{**}$ (i.e., if $X$ is not reflexive) then the weak-star topology on $X^*$ is weaker than the weak-topology on $X^*$ (since it demands less continuity).

## 5.2 Banach-Alaoglu

We start with an observation.

**Lemma 5.16** (Riesz)**.** *Let $X$ be a Banach space and $Y \subseteq X$ a subspace which is* not *dense. Then for any $r < 1$ there exists some $x \in X$ with $\|x\| = 1$ and*

$$\inf_{y \in Y} \|x - y\| \geq r\,.$$

*Proof.* Since $Y$ is not dense, there exists some $z \in \left(\overline{Y}\right)^c \in \operatorname{Open}(X)$. Then

$$R := \inf_{y \in Y} \|y - z\| > 0\,.$$

Now for any $\varepsilon > 0$, let $y_\varepsilon \in Y$ be such that $\|y_\varepsilon - z\| < R + \varepsilon$ (by the approximation property of the infimum). Define

$$x := \frac{z - y_\varepsilon}{\|z - y_\varepsilon\|}$$

so clearly $\|x\| = 1$ and

$$\begin{aligned}
\inf_{y \in Y} \|x - y\| &= \inf_{y \in Y} \left\| y - \frac{z}{\|z - y_\varepsilon\|} + \frac{y_\varepsilon}{\|z - y_\varepsilon\|} \right\| \\
&= \inf_{y \in Y} \left\| \frac{y}{\|z - y_\varepsilon\|} - \frac{z}{\|z - y_\varepsilon\|} + \frac{y_\varepsilon}{\|z - y_\varepsilon\|} \right\| \\
&= \frac{\inf_{y \in Y} \|y - z\|}{\|z - y_\varepsilon\|} \geq \frac{R}{R + \varepsilon}
\end{aligned}$$

where the first two inequalities follow from the fact $Y$ is a subspace. Since $\varepsilon > 0$ was arbitrary, we may arrange for $\frac{R}{R+\varepsilon}$ to be as close to 1 as we need. □

*Claim* 5.17. Let $X$ be a Banach space. Iff $\dim(X) = \infty$ then $B := \{\, x \in X \mid \|x\| \leq 1 \,\}$ is not compact.

*Proof.* If $\dim(X) < \infty$ then $X \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$. In this case, we know that $B$ is both closed and bounded, from which it follows (by Heine-Borel) that it is compact.

Conversely, construct a sequence $\{\, x_n \,\}_n \subseteq B$ (using Lemma 5.16) which obeys $\|x_n - x_m\| > \frac{1}{2}$ for all $n \neq m$.

This sequence is thus not Cauchy and so cannot possibly have a convergent subsequence. It is only possible to construct such a sequence if $X$ is infinite dimensional. ☐

We may thus ask if there exist weaker topologies so that the closed unit ball *is* compact. It turns out that on $X^*$ the weak-star topology is sufficiently weak to allow for this.

**Theorem 5.18** (Banach-Alaoglu)**.** *Let $X^*$ be the dual of some Banach space $X$. Then the closed unit ball in $X^*$*

$$\overline{B_1(0_{X^*})} := \left\{ \lambda \in X^* \,\middle|\, \|\lambda\|_{op} \leq 1 \right\}$$

*is compact in the weak-star topology.*

It should be mentioned that there is something dishonest about presenting Theorem 5.18 as a response to Claim 5.17. Indeed, they are statements about $X$ and $X^*$, *not* the same Banach space. In light of this, one may ask whether for any given Banach space $X$ there is a predual $Y$ such that $X = Y^*$. The answer is in fact no ($L^1([0,1])$ is an example).

*Proof.* For any $x \in X$, define

$$B_x := \overline{B_{\|x\|}(0_{\mathbb{C}})} \equiv \{ z \in \mathbb{C} \mid |z| \leq \|x\| \} .$$

Since this set is compact in $\mathbb{C}$, by Tychonoff,

$$\mathcal{B} := \prod_{x \in X} B_x$$

is compact in the product topology. We may think of $\mathcal{B}$ as a *set of maps,* each of which of the form

$$b : X \to \mathbb{C}$$

obeying $|b(x)| \leq \|x\|$. If $b$ were linear we would say $\|b\|_{\text{op}} = 1$ though this makes no sense. In particular, we may think of $\overline{B_1(0_{X^*})}$ as that subset of $\mathcal{B}$ which consists of maps $b : X \to \mathbb{C}$ *which are also linear,* so

$$\overline{B_1(0_{X^*})} =: \mathcal{B}_{\text{Linear}} .$$

We claim that the subspace topology on $\overline{B_1(0_{X^*})}$ induced by the product topology on $\mathcal{B}$ coincides with the subspace topology on $\overline{B_1(0_{X^*})}$ from $X^*$ taken with the weak-star topology. Indeed, recall that the product topology on $\mathcal{B}$ is defined as the initial topology generated by the set of projections

$$\begin{aligned} p_x : \mathcal{B} &\to B_x \\ b &\mapsto b(x) . \end{aligned}$$

As such it is clear at this point that on linear maps, $p_x = J(x)$ where $J : X \to X^{**}$ is the injection from Theorem 5.3. But this is precisely the definition of the weak-star topology (the initial topology generated by $J(X)$).

But now, since $\mathcal{B}$ is compact, and closed subsets of compact spaces are compact, it is only left to show that $\overline{B_1(0_{X^*})} \in \text{Closed}(\mathcal{B})$. To that end, for any $x, y \in X$, $\lambda \in \mathbb{C}$, define

$$\begin{aligned} \varphi_{xy\lambda} : \mathcal{B} &\to \mathbb{C} \\ b &\mapsto b(x + \lambda y) - b(x) - \lambda b(y) . \end{aligned}$$

We note that $\varphi_{xy\lambda}$ is a continuous map. Indeed, the map $b \mapsto b(x)$ is continuous *by definition of the product topology* and to obtain $\varphi_{xy\lambda}$ we merely have to take this map and compose it with other continuous maps. We now identify

$$\mathcal{B}_{\text{Linear}} = \bigcap_{x,y \in X, \lambda \in \mathbb{C}} \ker(\varphi_{xy\lambda}) .$$

I.e., the kernel forces these maps to be linear which is why they set within $X^*$. Since the kernel of a continuous map is closed, the intersection of closed sets is closed and we are done. ☐

Note: there is also a proof with nets which we made sure to avoid.

# 6 Banach Algebras

Consider $X$ a Banach space, and $\mathcal{B}(X) \equiv \mathcal{B}(X \to X)$ the space of all continuous linear maps $X \to X$. Via Claim 3.20 we know that $\mathcal{B}(X)$ is a Banach space in its own right with norm $\|\cdot\|_{\text{op}}$. On $\mathcal{B}(X)$ we may consider a *multiplication* operation which is merely composition of functions. It is clear that composition of linear maps is linear, and moreover, thanks to Lemma 3.19, any $A, B \in \mathcal{B}(X)$ obey

$$\|AB\|_{\text{op}} \leq \|A\|_{\text{op}} \|B\|_{\text{op}}$$

so that continuous linear maps compose to a continuous linear map. This turns out to be a good model for

**Definition 6.1** (Banach algebra)**.** A Banach space $\mathcal{A}$ (with norm $\|\cdot\|$) is called a *Banach algebra* iff there is a multiplication map defined

$$\cdot : \mathcal{A}^2 \to \mathcal{A}$$

which is associative and distributive w.r.t. scalar multiplication and vector addition, and moreover, such that the norm is *submultiplicative*

$$\|ab\| \leq \|a\| \|b\| \qquad (a, b \in \mathcal{A})$$

and such that $\exists \mathbb{1} \in \mathcal{A}$ such that $a\mathbb{1} = \mathbb{1}a = a$ for all $a \in \mathcal{A}$ and $\|\mathbb{1}\| = 1$.

Clearly $\mathcal{B}(X)$ is a Banach algebra with $\|\cdot\|_{\text{op}}$ and the identity map $\mathbb{1} : X \to X$.

*Claim* 6.2. Multiplication in a Banach algebra is automatically continuous.

*Proof.* Let $U \in \text{Open}(\mathcal{A})$ and consider some $a, b \in \mathcal{A}$ with $ab \in U$. Then there exists some $\varepsilon > 0$ such that

$$B_\varepsilon(ab) \subseteq U\,.$$

Let $\tilde{a} \in B_\delta(a)$ and $\tilde{b} \in B_\delta(b)$. Then

$$\begin{aligned} \left\|\tilde{a}\tilde{b} - ab\right\| &= \left\|\tilde{a}\tilde{b} - \tilde{a}b + \tilde{a}b - ab\right\| \\ &\leq \|\tilde{a}\| \left\|\tilde{b} - b\right\| + \|\tilde{a} - a\| \|b\| \\ &\leq (\|a\| + \|\tilde{a} - a\|) \left\|\tilde{b} - b\right\| + \|b\| \|\tilde{a} - a\| \\ &\leq (1 + \|a\|)\,\delta + \|b\|\delta \end{aligned}$$

so pick $\delta := \min\left(\left\{ 1, \frac{1}{1+\|a\|}\frac{1}{2}\varepsilon, \frac{1}{\|b\|}\frac{1}{2}\varepsilon \right\}\right)$ to get the desired constraint. ☐

**Example 6.3.** Let $C([0,1] \to \mathbb{C})$ be the space of continuous functions. Then if we define multiplication pointwise, this turns out to be a *commutative* Banach algebra. The constant 1 function is the unit element.If we replace $[0,1]$ with a finite set we obtain $\mathbb{C}^n$ with componentwise multiplication, and for $n = 1$ we get the Banach algebra $\mathbb{C}$. If we take $\mathcal{B}(\mathbb{C}^n)$ we get the Banach algebra of $n \times n$ complex matrices, which is *not* commutative.

## 6.1 Invertible elements

**Definition 6.4** (Invertible elements)**.** An element $a \in \mathcal{A}$ is called *left invertible* iff there exists some $x \in \mathcal{A}$ with $xa = \mathbb{1}$ and right invertible iff there exists some $y \in \mathcal{A}$ with $ay = \mathbb{1}$. If $a \in \mathcal{A}$ is both left and right invertible then we call it simply *invertible.* In this case it is clear that the left and right inverses are equal:

$$xa = \mathbb{1} = ay$$

so

$$y = \mathbb{1}y = (xa)\,y = x\,(ay) = x\mathbb{1} = x\,.$$

We then denote this (*unique*) inverse by $a^{-1}$. We denote the space of all invertible elements within $\mathcal{A}$ as $\mathcal{G}_{\mathcal{A}} \equiv \mathcal{G}(\mathcal{A})$.

**Lemma 6.5.** *If for some $x \in \mathcal{A}$ we have $\|x - \mathbb{1}\| < 1$ then $x \in \mathcal{G}_{\mathcal{A}}$,*

$$x^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - x)^n$$

*is a series that converges in operator norm, and*

$$\left\|x^{-1}\right\| \leq \frac{1}{1 - \|\mathbb{1} - x\|} \,. \tag{6.1}$$

*Proof.* Define $y := \mathbb{1} - x$ so that $\|y\| =: r < 1$. By submul., we have $\|y^n\| \leq \|y\|^n = r^n$. We claim the sequence $\{ z_N \}_N$ given by

$$z_N := \sum_{n=0}^{N} y^n$$

is Cauchy. Indeed,

$$\|z_N - z_M\| \leq \sum_{n=N+1}^{M} \|y^n\| \leq \sum_{n=M+1}^{M} r^n = \frac{r^{N+1}}{1-r}\left(1 - r^{M-N}\right) \,.$$

So by completeness $z := \sum_{n=0}^{\infty} y^n$ exists in $\mathcal{A}$. Now,

$$z(\mathbb{1} - y) = \lim_{N\to\infty} \sum_{n=0}^{N} y^n (\mathbb{1} - y) = \lim_{N\to\infty} \left[\sum_{n=0}^{N} \left(y^n - y^{n+1}\right)\right] = \lim_{N\to\infty} \left[\mathbb{1} - y^{N+1}\right] = \mathbb{1}$$

and similarly $(\mathbb{1} - y) z = \mathbb{1}$. We learn that $\mathbb{1} - y = x$ is invertible and $x^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - x)^n$ converges in operator norm. Moreover,

$$\left\|x^{-1}\right\| \leq \sum_{n=0}^{\infty} \|\mathbb{1} - x\|^n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} = \frac{1}{1 - \|1 - x\|} \,.$$

□

*Claim* 6.6. $\mathcal{G}_{\mathcal{A}} \in \text{Open}(\mathcal{A})$ and the ${}^{-1} : \mathcal{G}_{\mathcal{A}} \to \mathcal{G}_{\mathcal{A}}$ mapping $a \mapsto a^{-1}$ is a homeomorphism.

*Proof.* Let $a \in \mathcal{G}$. Then if $\tilde{a} \in B_{\|a^{-1}\|^{-1}}(a)$, we have

$$\begin{array}{rcl} \|\tilde{a} - a\| & < & \left\|a^{-1}\right\|^{-1} \\ & \updownarrow & \\ \|\tilde{a} - a\|\left\|a^{-1}\right\| & < & 1 \\ & \downarrow & \\ \left\|a^{-1}\tilde{a} - \mathbb{1}\right\| & < & 1 \end{array}$$

so by Lemma 6.5 we get that $a^{-1}\tilde{a}$ is invertible. Thus

$$\left(a^{-1}\tilde{a}\right)^{-1} a^{-1}\tilde{a} = \mathbb{1}$$

so that $\left(a^{-1}\tilde{a}\right)^{-1} a^{-1}$ is a left inverse for $\tilde{a}$. Similarly,

$$\begin{array}{rcl} \|\tilde{a} - a\| & < & \left\|a^{-1}\right\|^{-1} \\ & \updownarrow & \\ \|\tilde{a} - a\|\left\|a^{-1}\right\| & < & 1 \\ & \downarrow & \\ \left\|\tilde{a}a^{-1} - \mathbb{1}\right\| & < & 1 \end{array}$$

so that $\tilde{a}a^{-1}$ is invertible, i.e.,

$$\tilde{a}a^{-1}\left(\tilde{a}a^{-1}\right)^{-1}=\mathbb{1}$$

and hence $a^{-1}\left(\tilde{a}a^{-1}\right)^{-1}$ is the right inverse of $\tilde{a}$. We conclude that $\tilde{a}$ is invertible, so that

$$B_{\|a^{-1}\|^{-1}}(a)\subseteq\mathcal{G}_{\mathcal{A}}$$

and $\mathcal{G}_{\mathcal{A}}$ is open. Moreover, we note that the above implies that together with (6.1):

$$\left\|\tilde{a}^{-1}\right\|\leq\left\|a^{-1}\right\|\left\|a\tilde{a}^{-1}\right\|\leq\left\|a^{-1}\right\|\frac{1}{1-\|\tilde{a}a^{-1}-\mathbb{1}\|}\leq\frac{\left\|a^{-1}\right\|}{1-\|a^{-1}\|\|\tilde{a}-a\|}\,. \tag{6.2}$$

To show continuity of $^{-1}:\mathcal{G}_{\mathcal{A}}\to\mathcal{G}_{\mathcal{A}}$, let us observe the resolvent identity

$$a^{-1}-b^{-1}=a^{-1}(b-a)\,b^{-1}\,. \tag{6.3}$$

This implies that

$$\begin{aligned}\left\|a^{-1}-b^{-1}\right\| &\leq \left\|a^{-1}\right\|\|a-b\|\left\|b^{-1}\right\| \\ &\leq \left\|a^{-1}\right\|\|a-b\|\frac{\left\|a^{-1}\right\|}{1-\|a^{-1}\|\|a-b\|} \\ &= \frac{\left\|a^{-1}\right\|^2\|a-b\|}{1-\|a^{-1}\|\|a-b\|}\,.\end{aligned}$$

In the last step we hav penultimate step we have used (6.2) which relies on $\|a-b\|<\left\|a^{-1}\right\|^{-1}$. Hence we see that for any $\varepsilon>0$, if

$$\|a-b\|\leq\frac{1}{2}\left\|a^{-1}\right\|^{-1}\min\left(\left\{1,\frac{1}{\|a^{-1}\|}\varepsilon\right\}\right)$$

then

$$\left\|a^{-1}-b^{-1}\right\|\leq\varepsilon\,.$$

□

**Lemma 6.7.** *In a Banach algebra $\mathcal{A}$, $a,b\in\mathcal{A}$ one has $\mathbb{1}-ab\in\mathcal{G}_{\mathcal{A}}$ iff $\mathbb{1}-ba\in\mathcal{G}_{\mathcal{A}}$.*

*Proof.* Assume that $\mathbb{1}-ab\in\mathcal{G}_{\mathcal{A}}$. Then we claim

$$(\mathbb{1}-ba)^{-1}=\mathbb{1}+b\,(\mathbb{1}-ab)^{-1}\,a\,.$$

Indeed,

$$\begin{aligned}(\mathbb{1}-ba)\left(\mathbb{1}+b\,(\mathbb{1}-ab)^{-1}\,a\right) &= \mathbb{1}-ba+(\mathbb{1}-ba)\,b\,(\mathbb{1}-ab)^{-1}\,a \\ &= \mathbb{1}-ba+b\,(\mathbb{1}-ab)^{-1}\,a-bab\,(\mathbb{1}-ab)^{-1}\,a \\ &=: \star\end{aligned}$$

Now we use

$$\begin{aligned}ab\,(\mathbb{1}-ab)^{-1} &= (ab-\mathbb{1}+\mathbb{1})\,(\mathbb{1}-ab)^{-1} \\ &= -\mathbb{1}+(\mathbb{1}-ab)^{-1}\end{aligned}$$

to get

$$\begin{aligned}
\star &= \mathbb{1} - ba + b(\mathbb{1} - ab)^{-1} a - bab(\mathbb{1} - ab)^{-1} a \\
&= \mathbb{1} - ba + b(\mathbb{1} - ab)^{-1} a - b\left[-\mathbb{1} + (\mathbb{1} - ab)^{-1}\right] a \\
&= \mathbb{1} - ba + b(\mathbb{1} - ab)^{-1} a + ba - b(\mathbb{1} - ab)^{-1} a \\
&= \mathbb{1}
\end{aligned}$$

□

## 6.2 Banach-space-valued analytic functions

In this short section we mainly want to make the point that complex analysis "goes through" if one replaces $\mathbb{C}$ by $X$ for the codomain of functions, $X$ being a Banach space. For more details see Conway pp. 196 or Rudin pp. 82.

**Definition 6.8** ($\mathbb{C}$-Differentiability of a Banach-valued function)**.** Let $X$ be a Banach space and $\Omega \subseteq \mathbb{C}$ an open connected subset. A function

$$f : \Omega \to X$$

is said to be $\mathbb{C}$-differentiable at $z_0 \in \mathbb{C}$ (synonymous with *holomorphic*) iff

$$\lim_{z \to 0} \frac{f(z_0 + z) - f(z_0)}{z}$$

exists. The limit is to be understood with respect to the topology induced by the norm of $X$. If the limit does exist for all $z_0 \in \Omega$, we obtain a new function, the derivative $f' : \Omega \to X$ given by

$$f'(z_0) := \lim_{z \to 0} \frac{f(z_0 + z) - f(z_0)}{z} \qquad (z_0 \in \Omega) \,.$$

Compare this with Frechet differentiability, which would amount to

**Definition 6.9** (Frechet differentiability)**.** Let $X$ be a $\mathbb{C}$-Banach space and $\Omega \subseteq \mathbb{C}$ an open connected subset. A function

$$f : \Omega \to X$$

is said to be $\mathbb{C}$-differentiable at $z_0 \in \mathbb{C}$ iff there exists some $\mathbb{C}$-linear map $L : \mathbb{C} \to X$ such that

$$\lim_{z \in \mathbb{C} : |z| \to 0} \frac{\| f(z_0 + z) - f(z_0) - Lz \|}{|z|} = 0 \,.$$

Since we know a linear map $L : \mathbb{C} \to X$ is represented by some vector $v \in X$, so that $Lz = zv$, we get

$$\lim_{z \in \mathbb{C} : |z| \to 0} \frac{\| f(z_0 + z) - f(z_0) - vz \|}{|z|} = 0 \,.$$

*Claim* 6.10. The two notions are equivalent.

Note that if $X$ is an $\mathbb{R}$-Banach space then these two notions are *not* equivalent.

By analogy we also define $f : \Omega \to X$ to be weakly-$\mathbb{C}$-differentiable iff $\Lambda \circ f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable (in the usual sense) for any $\Lambda \in X^*$. Clearly the norm convergence implies the weak convergence. It turns out that for Banach spaces these two notions are equivalent

*Claim* 6.11. In a Banach space, if $f : \Omega \to X$ is weakly-$\mathbb{C}$-differentiable it is $\mathbb{C}$ differentiable.

but we avoid giving the details here (see Rudin 3.31).

**Definition 6.12.** Given a function $f:[a,b]\to X$ where $a<b\in\mathbb{R}$ and $X$ is a $\mathbb{C}$-Banach space, we define

$$\int_{[a,b]} f$$

using an analogous Darboux-integral definition: for any $n$-partition $P=a=x_1<x_2<\cdots<x_n=b$ of $[a,b]$ we define, for example,

$$S(f,P):=\sum_{j=2}^{n}(x_j-x_{j-1})f(x_j)$$

and we similarly claim that $f$ is Riemann-integrable on $[a,b]$ iff the

$$\omega(f,P):=\sum_{j=2}^{n}(x_j-x_{j-1})\sup(\{\,\|f(s)-f(t)\|_X \mid s,t\in(x_{j-1},x_j)\,\})$$

can be made arbitrarily small: for any $\varepsilon>0$ there is some partition $P$ such that $\omega(f,P)<\varepsilon$, in which case

$$\int_{[a,b]} f := \lim_n S(f,P_n)$$

where $P_n$ is some sequence of partitions with shrinking size (one shows that $\int_{[a,b]} f$ does not depend on the limit).

**Lemma 6.13** (Another characterization of the vector-valued integral)**.** *Let $f:[a,b]\to X$ be continuous. Then for every $\lambda\in X^*$, the function $\lambda\circ f:[a,b]\to\mathbb{C}$ is Riemann-integrable and*

$$\lambda\left(\int_{[a,b]} f\right)=\int_{[a,b]}\lambda\circ f \qquad (\lambda\in X^*)\ .$$

*Conversely, if $\psi\in X$ is such that*

$$\lambda\psi \quad = \quad \int_{[a,b]}\lambda\circ f \qquad (\lambda\in X^*)$$

*then*

$$\psi=\int_{[a,b]} f\,.$$

*Proof.* See Rudin 3.27 for the forward direction. The converse is left as an exercise to the reader. ☐

*Claim* 6.14. If $f:[a,b]\to X$ is Riemann-integrable then $\|f(\cdot)\|_X:[a,b]\to[0,\infty)$ is also, and we have $\left\|\int_{[a,b]} f\right\|_X\leq\int_{[a,b]}\|f(\cdot)\|_X$.

*Remark* 6.15. There is also the *Bochner* integral which is the Lebesgue-integral version of functions with values in a Banach space. For more information, one may read, e.g., Rudin "Vector-valued integration", pp. 77.

Once we have this integral, we may now define contour integration similarly to how it is done in complex analysis:

**Definition 6.16.** Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth contour. We define the contour integral of $f : \mathbb{C} \to X$ *along* $\gamma$ as

$$\int_\gamma f := \int_{[a,b]} (f \circ \gamma)\, \gamma' \tag{6.4}$$

where the right hand side is interpreted as follows: $\gamma' : [a,b] \to \mathbb{C}$ and $f \circ \gamma : [a,b] \to X$. Then thanks to scalar multiplication in $X$, $(f \circ \gamma)\, \gamma' : [a,b] \to X$ and we may ask whether $(f \circ \gamma)\, \gamma'$ is Riemann-integrable in the sense of Definition 6.12. If it is, then we define $\int_\gamma f$ as an element in $X$ via (6.4). The integral does not depend on the parametrization of $\gamma$.

The following collection of statements have proofs which are identical to those found in elementary complex analysis textbooks, and they use nothing other than the fact that the Banach space $X$ has a norm.

**Lemma 6.17** (ML lemma)**.** *If $f : \mathbb{C} \to X$ is continuous and $\gamma : [a,b] \to \mathbb{C}$ then*

$$\left\| \int_\gamma f \right\| \leq \sup_{t \in [a,b]} \| f(\gamma(t)) \|_X L(\gamma)$$

*where $L(\gamma)$ is the length of $\gamma$.*

**Theorem 6.18** (Cauchy's integral formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a simply-connected open set and $f : \Omega \to X$ $\mathbb{C}$-differentiable on it. Then for any $\gamma : [a,b] \to \Omega$ simple closed contour taken in CCW, if $z_0$ is a point in the interior of $\gamma$,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{(z - z_0)^{n+1}} f(z)\, \mathrm{d}z \qquad (n \in \mathbb{N}_{\geq 0}) . \tag{6.5}$$

*In particular, any $\mathbb{C}$-differentiable function is in fact smooth.*

*Proof.* Consider the case $n = 0$. Let $\lambda \in X^*$. Then $\lambda \circ f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable and so we have from complex analysis, that

$$(\lambda \circ f)(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{z - z_0} (\lambda \circ f)(z)\, \mathrm{d}z .$$

But now, since $\lambda$ was arbitrary, we have thanks to Lemma 6.13 that

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{z - z_0} f(z)\, \mathrm{d}z ,$$

i.e., (6.5) with $n = 0$ is correct. The same argument follows for $\lambda \circ f^{(n)} : \Omega \to \mathbb{C}$. ☐

**Lemma 6.19** (Cauchy's intequality)**.** *If $f : \mathbb{C} \to X$ is $\mathbb{C}$-differentiable on $\overline{B_R(z_0)}$ then*

$$\left\| f^{(n)}(z_0) \right\|_X \leq \frac{n!}{R^n} \sup_{z \in \overline{B_R(z_0)}} \| f(z) \|_X .$$

**Theorem 6.20** (Rudin 3.32)**.** *Let $X$ be a Banach space and $f : \mathbb{C} \to X$ weakly-entire with $f(\mathbb{C})$ a weakly-bounded subset of $X$. Then $f$ is constant.*

*Proof.* The assumption is equivalent to $\Lambda \circ f : \mathbb{C} \to \mathbb{C}$ being bounded and entire for any $\Lambda \in X^*$. Apply Liouville now on $\Lambda \circ f$ to learn that it is constant. In particular, $(\Lambda f)(z) = (\Lambda f)(0_\mathbb{C})$ for any $\Lambda \in X^*$. So if $f(z) - f(0_\mathbb{C}) \neq 0_X$, define some $\Lambda_0 \in X^*$ which is *not* zero on $f(z) - f(0_\mathbb{C})$ to find a contradiction, i.e., we find that $f$ is constant itself. ☐

## 6.3 The spectrum

We now come to a main concept for Banach algebras, which is the spectrum. It is mainly in this part of the notes that the distinction between $\mathbb{C}$-Banach algebras and $\mathbb{R}$-Banach algebras becomes important. As was mentioned already, we will focus solely on the complex case.

**Definition 6.21** (The spectrum)**.** Let $\mathcal{A}$ be a Banach algebra. For any element $a \in \mathcal{A}$, the spectrum of $a$, denoted by $\sigma(a)$, is a *subset of* $\mathbb{C}$ which is defined as

$$\sigma(a) := \{ z \in \mathbb{C} \mid (a - z\mathbb{1}) \notin \mathcal{G}_{\mathcal{A}} \} .$$

It is sometimes also convenient to refer to the *resolvent set* $\rho(a)$ which is merely the complement of the spectrum

$$\rho(a) := \mathbb{C} \setminus \sigma(a) .$$

One also speaks of *the spectral radius* $r : \mathcal{A} \to [0, \infty)$ which is

$$r(a) := \sup |\sigma(a)| .$$

**Example 6.22.** If $\mathcal{A} = \mathrm{Mat}_{n\times n}(\mathbb{C})$ then $\sigma(a)$ is precisely the set of $n$ complex eigenvalues of the $n \times n$ matrix $a$.

In principle the spectrum need not be a *finite* set.

**Theorem 6.23.** *For any $a \in \mathcal{A}$, $\sigma(a)$ is a* non-empty compact *subset of* $\mathbb{C}$.

*Proof.* Fix $a \in \mathcal{A}$. Define $\psi : \mathbb{C} \to \mathcal{A}$ by

$$z \mapsto a - z\mathbb{1} .$$

As the composition of continuous functions, $\psi$ is clearly continuous, so that $\psi^{-1}(\mathcal{G}_{\mathcal{A}}) \in \mathrm{Open}(\mathbb{C})$. But

$$\begin{aligned}\psi^{-1}(\mathcal{G}_{\mathcal{A}}) &\equiv \{ z \in \mathbb{C} \mid \psi(z) \in \mathcal{G}_{\mathcal{A}} \} \\ &= \{ z \in \mathbb{C} \mid (a - z\mathbb{1}) \in \mathcal{G}_{\mathcal{A}} \} \\ &\equiv \rho(a)\end{aligned}$$

so the resolvent set is open, i.e., the spectrum is closed. Next, we show the spectrum is bounded. We will show $r(a) \leq \|a\|$. Let $z \in \mathbb{C}$ such that $|z| > \|a\|$. Then

$$1 > \frac{\|a\|}{|z|} = \left\| \frac{a}{z} \right\| = \left\| 1 - \left(1 - \frac{a}{z}\right) \right\| .$$

Hence $1 - \frac{a}{z}$ is invertible, so that $a - z\mathbb{1}$ is invertible. Hence $z \in \rho(a)$ or $z \notin \sigma(a)$.

Define now a map

$$\begin{aligned}\varphi : \rho(a) &\to \mathcal{A} \\ z &\mapsto (a - z\mathbb{1})^{-1}\end{aligned} \tag{6.6}$$

which makes sense since $a - z\mathbb{1}$ is invertible precisely when $z \in \rho(a)$. Moreover, the domain of $\varphi$ is open by the above. We claim that $\varphi$ is $\mathbb{C}$-differentiable for any point $z_0 \in \rho(a)$. Indeed:

$$\begin{aligned}\frac{\varphi(z_0 + z) - \varphi(z_0)}{z} &= \frac{(a - (z_0 - z)\mathbb{1})^{-1} - (a - z_0\mathbb{1})^{-1}}{z} \\ &= \frac{(a - (z_0 - z)\mathbb{1})^{-1}([a - z_0\mathbb{1}] - [a - (z_0 - z)\mathbb{1}])(a - z_0\mathbb{1})^{-1}}{z} \\ &= \frac{(a - (z_0 - z)\mathbb{1})^{-1} z (a - z_0\mathbb{1})^{-1}}{z} \\ &= \varphi(z_0 - z)\varphi(z_0)\end{aligned}$$

We thus find that $\varphi$ is indeed $\mathbb{C}$-differentiable and $\varphi'(z_0) = \varphi(z_0)^2$. This is also a weak statement: for any $\lambda \in X^*$, $\lambda \circ \varphi : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-differentiable with derivative

$$(\lambda \circ \varphi)'(z) = [(\lambda \circ \varphi)(z)]^2 \,.$$

We claim that $\varphi$ is bounded at infinity. Indeed, assuming $|z| > \|a\|$, we have

$$\left\|\frac{a}{z}\right\| < 1$$

so that

$$\left\|\mathbb{1} - \left(\mathbb{1} - \frac{a}{z}\right)\right\| < 1$$

and hence using the estimate (6.1) we have

$$\left\|\left(\mathbb{1} - \frac{a}{z}\right)^{-1}\right\| \leq \frac{1}{1 - \left\|\mathbb{1} - \left(\mathbb{1} - \frac{a}{z}\right)\right\|} = \frac{1}{1 - \left\|\frac{a}{z}\right\|} \,.$$

Hence

$$\begin{aligned} \|\varphi(z)\| &\equiv \left\|(a - z\mathbb{1})^{-1}\right\| \\ &\leq |z|^{-1}\left\|\left(\frac{1}{z}a - \mathbb{1}\right)^{-1}\right\| \\ &\leq |z|^{-1}\frac{1}{1 - \left\|\frac{1}{z}a\right\|} \\ &\overset{|z|\to\infty}{\to} 0\,. \end{aligned}$$

We learn that

$$|\lambda(\varphi(z))| \quad \leq \quad \|\lambda\|_{\text{op}}\|\varphi(z)\|$$

so that $\lambda \circ \varphi$ is also bounded at infinity. Hence, if $\rho(a) = \mathbb{C}$, $\varphi$ emerges as a weakly-entire weakly-bounded function. But then, the analog of Liouville's theorem Theorem 6.20 would imply that $\varphi$ is constant. However, we have calculated $\varphi'(z) = -\varphi(z)^2$ which contradicts that. So $\sigma(a) \neq \varnothing$. □

**Lemma 6.24** (Fekete)**.** *A sequence* $\{ a_n \}_n \subseteq \mathbb{R}$ *is sub-additive iff*

$$a_{n+m} \leq a_n + a_m \qquad (n, m \in \mathbb{N}) \,.$$

*If* $\{ a_n \}_n$ *is sub-additive then* $\lim_{n\to\infty} \frac{1}{n}a_n$ *exists and equals* $\inf \frac{1}{n}a_n$*.*

*Proof.* Let us write $s^* := \inf_n \frac{a_n}{n}$ and let $k \in \mathbb{N}$ with $\frac{a_k}{k} < b$ where $b \in (s^*, \infty)$. For any $n > k$, there are two integers $p_n, q_n$ such that

$$n = p_n k + q_n$$

ad $0 \leq q_n \leq k - 1$. We thus find

$$\begin{aligned} a_n &= a_{p_n k + q_n} \\ &\leq p_n a_k + a_{q_n} \end{aligned}$$

so

$$\frac{a_n}{n} \leq \frac{p_n k}{n}\frac{a_k}{k} + \frac{a_{q_n}}{n} \,.$$

Now, as $n \to \infty$, $\frac{p_n k}{n} \to 1$ and $\frac{a_{q_n}}{n} \to 0$ as the numerator is bounded (by hypothesis, otherwise the lemma is trivial) and Hence we find for all $b > s^*$:

$$s^* \leq \lim_n \frac{a_n}{n} \leq \frac{a_k}{k} < b \,.$$

Now let $b \to s^*$ to get

$$s^* = \inf_n \frac{a_n}{n} = \lim_n \frac{a_n}{n}\,.$$

Then $\liminf_n \frac{1}{n}a_n \geq s^*$, so suffice to show $\limsup_n \frac{1}{n}a_n \leq s^*$. Assume otherwise. Then there exists a sequence $\{ a_{n_k} \}_k$ and $\varepsilon > 0$ such that $\frac{a_{n_k}}{n_k} > s^* + \varepsilon$ for all $k$. Let $a_m$ be such that $\frac{a_m}{m} < s^* + \frac{1}{2}\varepsilon$. □

**Lemma 6.25** (Gelfand's formula)**.** *For any $a \in \mathcal{A}$, the limit $\lim_{n\to\infty}\|a^n\|^{\frac{1}{n}}$ exists and obeys*

$$r(a) = \lim_{n\to\infty}\|a^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}}\|a^n\|^{\frac{1}{n}}\,.$$

*Proof.* First we observe that the sequence

$$b_n \quad := \quad \log(\|a^n\|)$$

is sub-additive. Indeed,

$$\begin{aligned} b_{n+m} &= \log\left(\left\|a^{n+m}\right\|\right) \\ &\leq \log(\|a^n\|\|a^m\|) \\ &\leq b_n + b_m\,. \end{aligned}$$

Hence by Fekete Lemma 6.24 $\lim_{n\to\infty}\frac{1}{n}b_n$ exists so by continuity of log, $\lim_{n\to\infty}\|a^n\|^{\frac{1}{n}}$ exists and equals $\inf_{n\in\mathbb{N}}\|a^n\|^{\frac{1}{n}}$.

Next, if $|z| > \|a\|$, we have, using the same definition (6.6) in the preceding proof:

$$\varphi(z) = -\sum_{n=0}^{\infty} z^{-n-1}a^n$$

which converges uniformly on $\partial B_R(0_{\mathbb{C}})$ for every $R > \|a\|$. Hence we may integrate this term by term (using Theorem 6.18) to get:

$$\begin{aligned} \oint_{z\in\partial B_R(0_{\mathbb{C}})} z^m\varphi(z)\,\mathrm{d}z &= -\oint_{z\in\partial B_R(0_{\mathbb{C}})} z^m\sum_{n=0}^{\infty}z^{-n-1}a^n\mathrm{d}z \\ &= -\sum_{n=0}^{\infty}a^n\oint_{z\in\partial B_R(0_{\mathbb{C}})} z^m z^{-n-1}\mathrm{d}z \\ &= -\sum_{n=0}^{\infty}a^n 2\pi\mathrm{i}\delta_{n,m} \\ &= -2\pi\mathrm{i}a^m \end{aligned}$$

and so

$$a^n = \frac{-1}{2\pi\mathrm{i}}\oint_{z\in\partial B_R(0_{\mathbb{C}})} z^n\varphi(z)\,\mathrm{d}z \qquad (R > \|a\|; n\in\mathbb{N}_{\geq 0})\,.$$

Since $\rho(a) \equiv \mathbb{C}\setminus\sigma(a)$ contains all $z$ with $|z| > r(a)$, we may deform the contour of integration to get

$$a^n = \frac{-1}{2\pi\mathrm{i}}\oint_{z\in\partial B_R(0_{\mathbb{C}})} z^n\varphi(z)\,\mathrm{d}z \qquad (R > r(a)\,; n\in\mathbb{N}_{\geq 0})\,. \tag{6.7}$$

Taking the norm on both sides yields

$$\|a^n\| \leq R^{n+1}\underbrace{\sup_{z\in\partial B_R(0_{\mathbb{C}})}\|\varphi(z)\|}_{\text{bounded}} \qquad (R > r(a))$$

so we get

$$\limsup_{n\to\infty}\|a^n\|^{\frac{1}{n}} \quad\leq\quad r(a)\ .$$

Conversely, if $z\in\sigma(a)$,

$$z^n\mathbb{1}-a^n=(z\mathbb{1}-a)\left(z^{n-1}\mathbb{1}+\cdots+a^{n-1}\right)$$

shows that $z^n\mathbb{1}-a^n$ is *not* invertible. Hence $z^n\in\sigma(a^n)$. This yields $|z^n|\leq\|a^n\|$. Hence

$$r(a)\leq\inf_n\|a^n\|^{\frac{1}{n}}\ .$$

□

**Theorem 6.26** (Gelfand-Mazur)**.** *If $\mathcal{A}$ is a Banach algebra where $\mathcal{A}\setminus\{0\}\subseteq\mathcal{G}_{\mathcal{A}}$ then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

*Proof.* Le $a\in\mathcal{A}$ and $z_1\neq z_2$. Then it can't be that both $a-z_1\mathbb{1}=0$ and $a-z_2\mathbb{1}=0$ and so, by hypothesis, at least one of them is invertible. But $\sigma(a)\neq\varnothing$, so $\sigma(a)$ consists of exactly one point: that one point which makes $a-z\mathbb{1}=0$. Hence for any $a\in\mathcal{A}$, $a=z\mathbb{1}$ for some $z\in\mathbb{C}$. This provides the desired isometric isomorphism. □

**Lemma 6.27.** *Let $\{x_n\}_n\subseteq\mathcal{G}_{\mathcal{A}}$, $x\in\partial\mathcal{G}_{\mathcal{A}}$ and $x_n\to x$. Then $\left\|(x_n)^{-1}\right\|\to\infty$.*

*Proof.* Assume otherwise. Then $\exists M<\infty$ with $\left\|(x_n)^{-1}\right\|<M$ for infinitely many $n$. Since also $x_n\to x$, take one of these $n$ such that also

$$\|x_n-x\|<\frac{1}{M}\ .$$

Then

$$\left\|\mathbb{1}-(x_n)^{-1}x\right\|=\left\|(x_n)^{-1}(x_n-x)\right\|<1\ .$$

Hence $(x_n)^{-1}x$ is invertible. But $x=x_n(x_n)^{-1}x$, so $x\in\mathcal{G}_{\mathcal{A}}$. But $x\in\partial\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\mathcal{A}}\in\text{Open}(\mathcal{A})$, so $x\notin\mathcal{G}_{\mathcal{A}}$! □

**Theorem 6.28** (Continuity of the spectrum)**.** *Let $a\in\mathcal{A}$ and $\sigma(a)\subseteq\Omega\in\text{Open}(\mathbb{C})$. Then if $b$ is sufficiently close to $a$ in norm, e.g.,*

$$b\in B_{\left[\sup_{z\in\Omega^c}\left\|(a-z\mathbb{1})^{-1}\right\|\right]^{-1}}(a)$$

*then $\sigma(b)\subseteq\Omega$ also.*

*Proof.* Recall that we know the map

$$\rho(a)\ni z\mapsto\left\|(a-z\mathbb{1})^{-1}\right\|$$

is continuous so

$$\Omega^c\ni z\mapsto\left\|(a-z\mathbb{1})^{-1}\right\|$$

is bounded, as $\Omega^c$ is closed. Now if $b\in B_{\left[\sup_{z\in\Omega^c}\left\|(a-z\mathbb{1})^{-1}\right\|\right]^{-1}}(a)$, and $z\in\Omega^c$, we rewrite

$$b-z\mathbb{1}=(a-z\mathbb{1})\left((a-z\mathbb{1})^{-1}(b-a)+\mathbb{1}\right)$$

and this is manifestly invertible: the product of two invertibles is invertible, and moreover $(a-z\mathbb{1})^{-1}(b-a)+\mathbb{1}$ is invertible as

$$\begin{aligned}\left\|(a-z\mathbb{1})^{-1}(b-a)\right\| &\leq \left\|(a-z\mathbb{1})^{-1}\right\|\|b-a\|\\ &< 1\end{aligned}$$

by construction. This implies $z \notin \sigma(b)$. ☐

**Corollary 6.29.** *Let $\Omega \subseteq \mathbb{C}$ and $a_n \to a$ in norm. If*

$$\sigma(a_n) \subseteq \Omega \qquad (n \in \mathbb{N})$$

*then*

$$\sigma(a) \subseteq \overline{\Omega}\,.$$

*Proof.* As in the preceding proof, for $z \in \overline{\Omega}^c$, $a_n - z\mathbb{1}$ is invertible by assumption and we may thus write

$$a - z\mathbb{1} = (a_n - z\mathbb{1})\left((a_n - z\mathbb{1})^{-1}(a - a_n) + \mathbb{1}\right)\,.$$

Now this will be invertible if

$$\|a - a_n\|\left\|(a_n - z\mathbb{1})^{-1}\right\| < 1\,.$$

Now since $z \in \overline{\Omega}^c$, $\exists \varepsilon > 0$ such that $B_\varepsilon(z) \subseteq \overline{\Omega}^c$, and so

$$\left\|(a_n - z\mathbb{1})^{-1}\right\| < \frac{1}{\varepsilon}\,. \tag{6.8}$$

Hence for every $z \in \overline{\Omega}^c$ there is a choice of $n$ sufficiently large so that $\|a - a_n\| < \varepsilon$, and hence, invertibility.

Note that it was necessary to take the closure here since $\Omega^c$ might not be an open set, but we need $z$ to be a *strictly positive* distance from $\Omega$. ☐

**Lemma 6.30.** *For any $a, b \in \mathcal{A}$,*

$$\sigma(ab) \setminus \{\,0\,\} = \sigma(ba) \setminus \{\,0\,\}\,.$$

*Proof.* Invoking Lemma 6.7 we get the result. ☐

## 6.4 The polynomial functional calculus

For fixed $a \in \mathcal{A}$ in a Banach algebra, it is clear that we may define a map

$$\chi_a : \mathcal{P} \to \mathcal{A}$$

where $\mathcal{P}$ is the space of all polynomials in one complex variable and $\mathcal{A}$ is a Banach algebra via the mapping

$$\chi_a : p \mapsto p(a)$$

since $p$ involves only operations which are unambiguously defined in $\mathcal{A}$: If

$$p(z) = \sum_{j=1}^{n} c_j z^j$$

with $\{\,c_j\,\}_{j=1}^n \subseteq \mathbb{C}$ the coefficients then it is clear what is meant by

$$p(a) \equiv \sum_{j=1}^{n} c_j a^j \in \mathcal{A}\,.$$

This mapping is called a *functional calculus* and currently since we only know how to apply it to polynomials we have the polynomial functional calculus. Our goal throughout the semester, in some sense, will be to gradually decrease the regularity of functions on which we may do this. The price to pay will be more and more assumptions on the $\mathcal{A}$ and the elements on which functional calculus may be applied, the culmination of which shall be *the spectral theorem of normal operators on Hilbert space.*

For now let us still remain with Banach algebras and go from polynomials to power series.

## 6.5 The holomorphic functional calculus

Consider the entire function

$$\exp : \mathbb{C} \to \mathbb{C} .$$

Its power series expansion

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad (z \in \mathbb{C})$$

converges for every $z$, absolutely, so that for $a \in \mathcal{A}$ we may make sense of $\exp(a)$ as follows. The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} a^n$$

converges in norm. Indeed, it is Cauchy: If $N < M$ then

$$\left\| \sum_{n=0}^{N} \frac{1}{n!} a^n - \sum_{n=0}^{M} \frac{1}{n!} a^n \right\| \leq \sum_{n=N+1}^{M} \frac{1}{n!} \|a\|^n$$

the latter expression goes to zero since $\exp(\|a\|)$ exists as a series. Hence by completeness of $\mathcal{A}$ we have $\exp(a)$.

What if $f$ is not entire? Then we need to be careful about the location of $\sigma(a) \subseteq \mathbb{C}$.

The easiest thing to do is:

Now if $f : \mathbb{C} \to \mathbb{C}$ is merely holomorphic within a ball $B_R(0_{\mathbb{C}})$ for some $R > 0$, we know it may be written as a power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (|z| \leq R) .$$

Let now $a \in \mathcal{A}$ with $r(a) < R$, i.e., all the spectrum of $a$ is contained within $B_R(0_{\mathbb{C}})$. Then we *define*

$$f(a) := \sum_{n=0}^{\infty} c_n a^n .$$

Thanks to the spectral radius formula Lemma 6.25 one may prove that this series is a Cauchy sequence and hence converges, it goes along the lines of:

$$\begin{aligned} \left\| \sum_{n=0}^{N} c_n a^n - \sum_{n=0}^{M} c_n a^n \right\| &\leq \sum_{n=N+1}^{M} |c_n| \|a^n\| \\ &= \sum_{n=N+1}^{M} |c_n| \left( \|a^n\|^{\frac{1}{n}} \right)^n \\ &\sim \sum_{n=N+1}^{M} |c_n| (r(a))^n \\ &\leq \sum_{n=N+1}^{M} |c_n| R^n \end{aligned}$$

which exists thanks to the assumption on $f$.

But we want something better: we don't want to assume that the spectrum of $a$ is contained in *one* disc on which $f$ is holomorphic.

Let us start with rational functions which have poles in various places.

**Lemma 6.31.** *Let $a \in \mathcal{A}$ $\alpha \in \rho(a)$ and $\Omega := \mathbb{C} \setminus \{ \alpha \}$. Assume further that we choose some $\gamma_j : [a,b] \to \Omega$, $j = 1, \dots, m$ a collection of $m$ oriented loops which surround $\sigma(a)$ within $\Omega$, such that*

$$\frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{m} \oint_{\gamma_j} \frac{1}{z - \lambda} \mathrm{d}z = \begin{cases} 1 & \lambda \in \sigma(a) \\ 0 & \lambda \notin \Omega \end{cases} \tag{6.9}$$

*(these are guaranteed to exist thanks to Lemma C.1). Then*

$$\frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{m} \oint_{\gamma_j} (\alpha - z)^n (z\mathbb{1} - a)^{-1} \mathrm{d}z = (\alpha\mathbb{1} - a)^n \qquad (n \in \mathbb{Z}) . \tag{6.10}$$

*Proof.* Let us show first the case $n=0$, namely, we want to show that

$$\frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{m}\oint_{\gamma_j}(z\mathbb{1}-a)^{-1}\,\mathrm{d}z=\mathbb{1}\,.$$

For brevity we replace

$$\sum_{j=1}^{m}\oint_{\gamma_j}\cdot\to\oint_{\Gamma}\cdot\,. \tag{6.11}$$

First we want to replace $\oint_\Gamma$ with just one CCW simple contour $\gamma_R$ whose image is $\partial B_R(0_{\mathbb{C}})$, such that $R>\|a\|$. On $\gamma_R$, we have

$$(z\mathbb{1}-a)^{-1}=\sum_{n=0}^{\infty}z^{-n-1}a^n$$

which converges absolutely and uniformly on $z\in\partial B_R(0)$. Hence we may integrate term by term to obtain

$$\frac{1}{2\pi \mathrm{i}}\oint_{\gamma_R}(z\mathbb{1}-a)^{-1}\,\mathrm{d}z=\mathbb{1}\,.$$

But $\rho(a)\ni z\mapsto(z\mathbb{1}-a)^{-1}$ is holomorphic, so we may deform the contour of integration Theorem 6.18 to replace $\oint_{\gamma_R}$ with $\oint_\Gamma$. To do so, we note that even though $\Omega$ may not be an open disc, on all of its holes, $z\mapsto(z\mathbb{1}-a)^{-1}$ is actually holomorphic since there is no spectrum there. So we may deform the contours of $\Gamma$ around these holes to nothing without changing the integral. Once this has been done, we may deform all outer contours into one big contour around the whole spectrum (note that Rudin doesn't really explain this point).

For the general case: we denote

$$y_n:=\frac{1}{2\pi \mathrm{i}}\oint_\Gamma(\alpha-z)^n(z\mathbb{1}-a)^{-1}\,\mathrm{d}z\qquad(n\in\mathbb{Z})$$

and claim that

$$(\alpha\mathbb{1}-a)\,y_n=y_{n+1}\qquad(n\in\mathbb{Z})\,. \tag{6.12}$$

This would yield the claim since we have just shown $y_0=\mathbb{1}$.

To show (6.12), employ the resolvent identity (6.3) to get that when $z\notin\sigma(a)$,

$$(z\mathbb{1}-a)^{-1}=(\alpha\mathbb{1}-a)^{-1}+(\alpha-z)(\alpha\mathbb{1}-a)^{-1}(z\mathbb{1}-a)^{-1}\,.$$

Plug this into

$$\begin{aligned}
y_n &\equiv \frac{1}{2\pi \mathrm{i}}\oint_\Gamma(\alpha-z)^n(z\mathbb{1}-a)^{-1}\,\mathrm{d}z\\
&= \frac{1}{2\pi \mathrm{i}}\oint_\Gamma(\alpha-z)^n\left((\alpha\mathbb{1}-a)^{-1}+(\alpha-z)(\alpha\mathbb{1}-a)^{-1}(z\mathbb{1}-a)^{-1}\right)\mathrm{d}z\\
&= \frac{1}{2\pi \mathrm{i}}\oint_\Gamma(\alpha-z)^n(\alpha\mathbb{1}-a)^{-1}\,\mathrm{d}z+\frac{1}{2\pi \mathrm{i}}\oint_\Gamma(\alpha-z)^n(\alpha-z)(\alpha\mathbb{1}-a)^{-1}(z\mathbb{1}-a)^{-1}\,\mathrm{d}z\\
&= (\alpha\mathbb{1}-a)^{-1}\underbrace{\frac{1}{2\pi \mathrm{i}}\oint_\Gamma(\alpha-z)^n\,\mathrm{d}z}_{=0\text{ as }\alpha\text{ not encircled by }\Gamma}+(\alpha\mathbb{1}-a)^{-1}\frac{1}{2\pi \mathrm{i}}\oint_\Gamma(\alpha-z)^{n+1}(z\mathbb{1}-a)^{-1}\,\mathrm{d}z\\
&= (\alpha\mathbb{1}-a)^{-1}y_{n+1}\,.
\end{aligned}$$

Indeed, in the penultimate line we have used (6.9). ☐

**Corollary 6.32.** *Let $R:\mathbb{C}\to\mathbb{C}$ be a rational function, i.e.,*

$$R(z)=p(z)+\sum_{k=1}^{n}\sum_{l=1}^{q}c_{k,l}(z-z_k)^{-l} \tag{6.13}$$

*where $p$ is a polynomial, $n\in\mathbb{N}$, and $\{z_k\}_k,\{c_{k,l}\}_{k,l}\subseteq\mathbb{C}$. Let now $a\in\mathcal{A}$ such that $\{z_k\}_{k=1}^n\subseteq\rho(a)$. Assume further that we choose some $\sigma(a)\subseteq\Omega\in\operatorname{Open}(\mathbb{C})$ such that $R$ is holomorphic on $\Omega$, and $\gamma_j:[a,b]\to\Omega$, $j=1,\dots,m$ a collection of $m$ oriented loops which surround $\sigma(a)$ within $\Omega$, such that*

$$\frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{1}{z-\lambda}\mathrm{d}z=\begin{cases}1 & \lambda\in\sigma(a)\\ 0 & \lambda\notin\Omega\end{cases}.$$

*(Here we use the notation (6.11)). Then $R(a)$ obeys the Cauchy integral formula, in the sense that*

$$p(a)+\sum_{k=1}^{n}\sum_{l=1}^{q}c_{k,l}(a-z_k)^{-l}=\frac{1}{2\pi\mathrm{i}}\oint_\Gamma R(z)(z\mathbb{1}-a)^{-1}\mathrm{d}z\,. \tag{6.14}$$

*Proof.* Let us plug in (6.13) into the RHS of (6.14) to get

$$\begin{aligned}\text{RHS} &\equiv \frac{1}{2\pi\mathrm{i}}\oint_\Gamma R(z)(z\mathbb{1}-a)^{-1}\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\oint_\Gamma\left[p(z)+\sum_{k=1}^{n}\sum_{l=1}^{q}c_{k,l}(z-z_k)^{-l}\right](z\mathbb{1}-a)^{-1}\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\oint_\Gamma p(z)(z\mathbb{1}-a)^{-1}\mathrm{d}z+\frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{n}\sum_{l=1}^{q}c_{k,l}\oint_\Gamma(z-z_k)^{-l}(z\mathbb{1}-a)^{-1}\mathrm{d}z\,.\end{aligned}$$

The first term with the polynomial yields $p(a)$. This actually follows from (6.7) (with another change of contour from $\gamma_R$ to $\Gamma$). The second term equals the second term on the LHS via an application of Lemma 6.31. ☐

Hence we see that rational functions do obey the Cauchy formula.

**Theorem 6.33** (Holomorphic functional calculus)**.** *Let $a\in\mathcal{A}$ and $\Omega\subseteq\mathbb{C}$ be an open subset such that $\sigma(a)\subseteq\Omega$. In particular we do* not *assume that $\Omega$ is connected. Let $f:\Omega\to\mathbb{C}$ be holomorphic. Let $\gamma_j:[a,b]\to\Omega$, $j=1,\dots,m$ be a collection of $m$ oriented loops which surround $\sigma(a)$ within $\Omega$ but are disjoint from $\sigma(a)$, such that*

$$\frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{1}{z-\lambda}\mathrm{d}z=\begin{cases}1 & \lambda\in\sigma(a)\\ 0 & \lambda\notin\Omega\end{cases}$$

*with the notation from (6.11). Define*

$$f(a):=\frac{1}{2\pi\mathrm{i}}\oint_\Gamma f(z)(z\mathbb{1}-a)^{-1}\mathrm{d}z\,. \tag{6.15}$$

*Then $f\mapsto f(a)\in\mathcal{A}$ is a continuous unital-algebra monomorphism:*

- *$f(a)g(a)=(fg)(a)$ and $f(a)+g(a)=(f+g)(a)$.*
- *$(z\mapsto1)(a)=\mathbb{1}$ and $(z\mapsto z)(a)=a$.*
- *Continuous in the sense that: if $\{f_n:\Omega\to\mathbb{C}\}_n$ is a sequence of holomorphic functions converging uniformly in compact subsets of $\Omega$, then $f_n(a)\to f(a)$ in norm.*

This definition certainly makes sense as it agrees with the more naive definitions when $f$ is a polynomial, rational function, or entire!

*Proof.* First we want to give meaning to the RHS of (6.15). Since $\rho(a) \ni z \mapsto (z\mathbb{1}-a)^{-1}$ is holomorphic, we may ask whether its contour integral exists, in the sense of Theorem 6.18. Indeed it does: as $z$ stays away from $\sigma(a)$, and inversion is continuous on $\mathcal{A}$, the function is continuous and hence integrable.

Next, it is clear that $f \mapsto f(a)$ is linear, so to show it is injective we ask what is its kernel. If for some $f$ of the above class, $f(a) = 0$, then the integral formula implies that for *that* $f$,

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z)(z\mathbb{1}-a)^{-1} \mathrm{d}z = 0 .$$

If this holds for $a$, then it must also hold for $\lambda\mathbb{1}$ for any $\lambda \in \sigma(a)$. This implies that

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z)(z\mathbb{1}-\mathbb{1}\lambda)^{-1} \mathrm{d}z = 0 \qquad (\lambda \in \sigma(a)) .$$

But this latter formula equals $\frac{1}{2\pi\mathrm{i}}\oint_\Gamma f(z)(z\mathbb{1}-\lambda)^{-1}\mathrm{d}z\mathbb{1}$ so the integral inside must be zero, so that $f = 0$. Hence $f \mapsto f(a)$ is injective.

Continuity follows from the integral representation since

$$z \mapsto \left\|(z\mathbb{1}-a)^{-1}\right\|$$

is bounded on the range of $\Gamma$.

First let us show that $(z \mapsto 1)(a) = \mathbb{1}$: According our *definition*,

$$(z \mapsto 1)(a) \equiv -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (a - z\mathbb{1})^{-1} \mathrm{d}z .$$

Now use

$$(a - z\mathbb{1})(a - z\mathbb{1})^{-1} \equiv \mathbb{1}$$

or $a(a-z\mathbb{1})^{-1} = \mathbb{1} + z(a-z\mathbb{1})^{-1}$. Placing this identity within the integrand we find

$$\begin{aligned} (z \mapsto 1)(a) &= -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{a(a-z\mathbb{1})^{-1} - \mathbb{1}}{z} \mathrm{d}z \\ &= \mathbb{1} - -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{a(a-z\mathbb{1})^{-1}}{z} \mathrm{d}z . \end{aligned}$$

Since this holds for any series of contour $\Gamma$ which encloses the entire spectrum, we may integrate on circles of increasing radii to get the second term zero via $\left\|a(a-z\mathbb{1})^{-1}\right\| \lesssim \frac{\|a\|}{|z|}$ (for large $|z|$) and so it is bounded on these circles.

Next, for $(z \mapsto z)(a) = a$ we have

$$\begin{aligned} (z \mapsto z)(a) &= -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma z(a-z\mathbb{1})^{-1} \mathrm{d}z \\ &= -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \left[a(a-z\mathbb{1})^{-1} - \mathbb{1}\right] \mathrm{d}z \\ &= a\left(-\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (a-z\mathbb{1})^{-1} \mathrm{d}z\right) \\ &= a(z \mapsto 1)(a) \\ &= a\mathbb{1} \\ &= a . \end{aligned}$$

For the multiplicative statement one may use Runge's theorem ([SS03] Theorem 5.7): Any function holomorphic in a neighborhood of $K$ can be approximated uniformly on $K$ by rational functions whose singularities are in $K^c$. It is easy to see the statement holds for rational functions, so one is only left with the limit statement. ☐

We note that the multiplicative property, implies in particular, that $f(a)g(a) = g(a)f(a)$, i.e, functional calculus of an operator commutes!

Some additional properties of the functional calculus are

**Lemma 6.34.** *Let* $a \in \mathcal{A}$ *with* $\sigma(a) \subseteq \Omega \in \operatorname{Open}(\mathbb{C})$ *and* $f : \Omega \to \mathbb{C}$ *holomorphic. Then* $f(a) \in \mathcal{G}_{\mathcal{A}}$ *iff* $0 \notin \operatorname{im}\left(f|_{\sigma(a)}\right)$.

*Proof.* Assume that $0 \notin \operatorname{im}\left(f|_{\sigma(a)}\right)$. Let then $g : \frac{1}{f}$, which is holomorphic on some $\tilde{\Omega} \in \operatorname{Open}(\Omega)$ with $\sigma(a) \subseteq \tilde{\Omega}$. By the holomorphic functional calculus, we have $f(a)\,g(a) = \mathbb{1}$ so that $f(a)$ is invertible.

Conversely, if $0 \in \operatorname{im}\left(f|_{\sigma(a)}\right)$, let $\alpha \in \sigma(a) : f(\alpha) = 0$. Then we have some $h : \Omega \to \mathbb{C}$ holomorphic with

$$f(\lambda) = h(\lambda)(\lambda - \alpha) \qquad (\lambda \in \Omega)$$

so that

$$f(a) = (a - \alpha\mathbb{1})\,h(a) = h(a)(a - \alpha\mathbb{1}) \,.$$

But $a - \alpha\mathbb{1}$ is not invertible (as $\alpha \in \sigma(a)$), so $f(a)$ can't be either. □

**Theorem 6.35** (The spectral mapping theorem)**.** *Let* $a \in \mathcal{A}$ *with* $\sigma(a) \subseteq \Omega \in \operatorname{Open}(\mathbb{C})$ *and* $f : \Omega \to \mathbb{C}$ *holomorphic. Then*

$$\sigma(f(a)) = f(\sigma(a)) \equiv \{\, f(z) \mid z \in \sigma(a) \,\} \,.$$

*Proof.* Let $z \in \mathbb{C}$. We have

$$\begin{aligned} z \in \sigma(f(a)) &\iff (f(a) - z\mathbb{1}) \notin \mathcal{G}_{\mathcal{A}} \\ &\iff 0 \in \operatorname{im}\left((\sigma(a) \ni \lambda \mapsto f(\lambda) - z)|_{\sigma(a)}\right) \\ &\iff z \in \operatorname{im}\left((\sigma(a) \ni \lambda \mapsto f(\lambda))|_{\sigma(a)}\right) \\ &\iff z \in f(\sigma(a)) \,. \end{aligned}$$

□

The spectral mapping theorem allows us to compose holomorphic functions for the functional calculus

**Lemma 6.36.** *Let* $a \in \mathcal{A}$ *and* $\Omega \in \operatorname{Open}(\mathbb{C})$ *with* $\sigma(a) \subseteq \Omega$. *Let* $f : \Omega \to \mathbb{C}$ *be holomorphic. Let* $\tilde{\Omega} \in \operatorname{Open}(\mathbb{C})$ *with* $f(\sigma(a)) \subseteq \tilde{\Omega}$ *and* $g : \tilde{\Omega} \to \mathbb{C}$ *be holomorphic. Then*

$$g(f(a)) = (g \circ f)(a) \,.$$

*Proof.* By the spectral mapping theorem, it is clear that

$$g(f(a))$$

makes sense. Let $\Gamma$ be a collection of contours in $\tilde{\Omega}$ which surrounds $f(\sigma(a))$. Then the functional calculus says

$$g(f(a)) = \frac{\mathrm{i}}{2\pi} \oint_{\Gamma} g(z)\,(f(a) - z\mathbb{1})^{-1}\,\mathrm{d}z \,.$$

We would like to re-express $(f(a) - z\mathbb{1})^{-1}$ as a contour integral for all $z \in \operatorname{im}(\Gamma)$:

$$(f(a) - z\mathbb{1})^{-1} = \frac{\mathrm{i}}{2\pi} \oint_{\tilde{\Gamma}} (f(\zeta) - z)^{-1}\,(a - \zeta\mathbb{1})^{-1}\,\mathrm{d}\zeta \,.$$

For this to hold, $\tilde{\Gamma}$ needs to be a collection of contours in $\Omega$ surrounding $\sigma(a)$ but disjoint from it, and we also need that

$$\Omega \ni \zeta \mapsto (f(\zeta) - z)^{-1}$$

is holomorphic. This is indeed true, since $z$ only ranges within $\tilde{\Omega} \setminus f(\sigma(a))$. We thus find

$$\begin{aligned} g(f(a)) &= \frac{\mathrm{i}}{2\pi} \oint_{\Gamma} g(z) \left[ \frac{\mathrm{i}}{2\pi} \oint_{\tilde{\Gamma}} (f(\zeta) - z)^{-1} (a - \zeta \mathbb{1})^{-1} \mathrm{d}\zeta \right] \mathrm{d}z \\ &= \frac{\mathrm{i}}{2\pi} \oint_{\tilde{\Gamma}} \underbrace{\left[ \frac{\mathrm{i}}{2\pi} \oint_{\Gamma} g(z) (f(\zeta) - z)^{-1} \mathrm{d}z \right]}_{=(g)(f(\zeta)) \equiv (g\circ f)(\zeta)} (a - \zeta \mathbb{1})^{-1} \mathrm{d}\zeta \\ &= \frac{\mathrm{i}}{2\pi} \oint_{\tilde{\Gamma}} (g \circ f)(\zeta) (a - \zeta \mathbb{1})^{-1} \mathrm{d}\zeta \\ &\equiv (g \circ f)(a) \, . \end{aligned}$$

The exchange of integrals is justified thanks to uniform convergence. □

One statement that comes up again and again in applications is

**Theorem 6.37.** *If $a \in \mathcal{A}$ and $\sigma(a)$ does not separate $0$ from $\infty$ then $\log(a) \in \mathcal{A}$ (via the holomorphic functional calculus) and*

$$a = \exp(\log(a)) \, . \tag{6.16}$$

*Proof.* Since $0$ lies in the unbounded connected component of the resolvent set, there is a *holomorphic* function $f : \Omega \to \mathbb{C}$ with $\Omega$ simply connected, open, and containing $\sigma(a)$, such that $\exp \circ f = \mathbb{1}_{\Omega}$, namely, the logarithm with branch cut *outside* of $\Omega$ (note that in this case the branch cut need not be a straight line–this is fine). Hence using the holomorphic functional calculus we may define

$$f(a) \equiv \left[ \frac{\mathrm{i}}{2\pi} \oint f(z) (a - z\mathbb{1})^{-1} \mathrm{d}z \right] \in \mathcal{A}$$

for an appropriate contour within $\Omega$ which encircles $\sigma(a)$. The identity (6.16) holds thanks to Lemma 6.36. □

# 7 Hilbert spaces

In our journey of combining the linear structure of a space and its topological structure, we have become increasingly more and more specific, starting with TVS, specifying to complete *normed* spaces, and now we arrive at our final destination: assuming that the norm satisfies the parallelogram identity, i.e., talking about *complete inner product spaces*. These structures are extremely important for various reasons: they are the mathematical structure of quantum mechanics, and moreover, they are the best infinite-dimensional generalization of Euclidean spaces.

Of course, all of our previous analysis carries over in Hilbert spaces, but as we shall see, the additional assumptions of having an inner product will allow us to do even more. The culmination of that "more" will be the bounded Borel-measurable functional calculus of normal operators on a Hilbert space.

First, the formal definition (which we have already seen in Definition 3.4):

**Definition 7.1** (Inner product space)**.** An inner-product space is a $\mathbb{C}$-vector space $\mathcal{H}$ along with a sesquilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

such that

1. Conjugate symmetry:
$$\langle \psi, \varphi \rangle = \overline{\langle \varphi, \psi \rangle} \qquad (\varphi, \psi \in \mathcal{H}) \, .$$

2. Linearity in second argument:
$$\langle \psi, \alpha\varphi + \tilde{\varphi} \rangle = \alpha \langle \psi, \varphi \rangle + \langle \psi, \tilde{\varphi} \rangle \qquad (\varphi, \tilde{\varphi}, \psi \in \mathcal{H}, \alpha \in \mathbb{C}) \, .$$

3. Positive-definite:
$$\langle \psi, \psi \rangle > 0 \qquad (\psi \in \mathcal{H} \setminus \{ 0 \}) \, .$$

To every inner product one immediately may associate a norm, via

$$\|\psi\| := \sqrt{\langle\psi,\psi\rangle} \qquad (\psi \in \mathcal{H}) \ .$$

The converse, however, hinges on the norm obeying the parallelogram law as we have seen in Claim 3.6.

*Claim* 7.2. Once we have an inner-product, we immediately have the Cauchy-Schwarz inequality

$$|\langle\varphi,\psi\rangle| \leq \|\varphi\|\|\psi\| \qquad (\varphi,\psi \in \mathcal{H}) \ .$$

**Definition 7.3** (Hilbert space)**.** A Hilbert space $\mathcal{H}$ is a inner-product space with $\langle\cdot,\cdot\rangle$ such the induced norm $\|\cdot\|$ from this inner product makes $\mathcal{H}$ into a Banach space (i.e. a complete metric space w.r.t. to the metric induced by $\|\cdot\|$).

Hence we identify a Hilbert space as a Banach space whose norm satisfies the parallelogram identity.

One of the central notions available to us now in Hilbert space, which was not available before, is the notion of *orthogonality* of vectors:

**Definition 7.4** (Orthogonality)**.** Two vectors $\varphi,\psi \in \mathcal{H}$ in a Hilbert space are dubbed *orthogonal* iff

$$\langle\varphi,\psi\rangle_{\mathcal{H}} = 0 \, .$$

A collection $\{ \varphi_i \}_i$ is called *orthonormal* iff

$$\langle\varphi_i,\varphi_j\rangle = \delta_{ij} \ .$$

The following two claims involving orthogonality will be useful. Their proof will be a homework assignment.

*Claim* 7.5. $\varphi \perp \psi$ iff

$$\|\varphi\| \leq \|z\psi + \varphi\| \qquad (\forall z \in \mathbb{C}) \ .$$

*Proof.* We have

$$0 \leq \|z\psi+\varphi\|^2 = |z|^2 \|\psi\|^2 + \|\varphi\|^2 + 2\,\mathbb{R}\mathrm{e}\,\{\langle z\psi,\varphi\rangle\} \ . \tag{7.1}$$

Hence if $\langle\psi,\varphi\rangle = 0$ we have $\|z\psi+\varphi\|^2 \geq \|\varphi\|^2$.

Conversely, if $\psi = 0$ we are finished. Otherwise, let $z := -\frac{\langle\psi,\varphi\rangle}{\|\psi\|^2}$. Plugging this into (7.1) we find

$$\begin{aligned} 0 \leq \|z\psi+\varphi\|^2 &= \|\varphi\|^2 + |z|^2 \|\psi\|^2 + 2\,\mathbb{R}\mathrm{e}\,\{\langle z\psi,\varphi\rangle\} \\ &= \|\varphi\|^2 + \frac{|\langle\psi,\varphi\rangle|^2}{\|\psi\|^2} + 2\,\mathbb{R}\mathrm{e}\left\{\left\langle -\frac{\langle\psi,\varphi\rangle}{\|\psi\|^2}\psi,\varphi\right\rangle\right\} \\ &= \|\varphi\|^2 - \frac{|\langle\psi,\varphi\rangle|^2}{\|\psi\|^2} \end{aligned}$$

which is coincidentally a proof of the Cauchy-Schwarz inequality. But this also shows that

$$\|z\psi+\varphi\|^2 < \|\varphi\|^2$$

for *one* $z$ if $\langle\varphi,\psi\rangle \neq 0$. □

**Theorem 7.6.** *Every nonempty closed convex set* $E \subseteq \mathcal{H}$ *contains a* unique *element of minimal norm.*

*Proof.* Let

$$d := \inf\left(\{ \|x\| \mid x \in E \}\right) \ .$$

Let $\{ x_n \}_n \subseteq E$ so that $\{ \|x_n\| \}_n \to d$. Since $E$ is convex,

$$\frac{1}{2}\left(x_n + x_m\right) \in E$$

and hence

$$\|x_n + x_m\|^2 = 4\left\|\frac{1}{2}(x_n + x_m)\right\|^2 \geq 4d^2 .$$

Then the parallelogram law Claim 3.6

$$\|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2$$

has its right hand side tend to $4d^2$ also, so $\|x_n - x_m\|^2 \to 0$ and hence $\{ x_n \}_n$ is Cauchy, which hence converges to some $x \in E$ (as $E$ is closed) and we have $\|x\| = d$.

For uniqueness, if $y \in E$ with $\|y\| = d$, then

$$\{ x, y, x, y, \dots \}$$

must converge by the above, so $y = x$. □

**Theorem 7.7.** *Let $M$ be a closed subspace of $\mathcal{H}$. Then*

$$M^\perp \equiv \{ \varphi \in \mathcal{H} \mid \varphi \perp \psi \forall \psi \in M \}$$

*is also a closed subspace of $\mathcal{H}$ and $M \cap M^\perp = \{ 0 \}$, and*

$$\mathcal{H} = M \oplus M^\perp .$$

*Proof.* Since $\langle \varphi, \cdot \rangle$ is linear, $M^\perp$ is linear. Also,

$$M^\perp = \bigcap_{\varphi \in M} \langle \varphi, \cdot \rangle^{-1} (\{ 0 \}) \tag{7.2}$$

and $\langle \varphi, \cdot \rangle$ is continuous by the Cauchy-Schwarz inequality, so $M^\perp$ is closed. Next, if $\varphi \in M \cap M^\perp$ then in particular $\langle \varphi, \varphi \rangle = 0$ so $\varphi = 0$.

Finally, let $\varphi \in \mathcal{H}$. The set $\varphi - M$ is a convex, closed subset and hence by Theorem 7.6 we get some $\psi \in M$ such that $\|\varphi - \psi\|$ is minimal. Let

$$\eta := \varphi - \psi .$$

Then

$$\|\eta\| \leq \|\eta + \xi\| \qquad (\xi \in M)$$

by the minimizing property. So by Claim 7.5, $\eta \in M^\perp$. But

$$\varphi = \psi + \eta \in M + M^\perp .$$

□

*Claim* 7.8. For any subspace $W \subseteq \mathcal{H}$, $\left(\overline{W}\right)^\perp = W^\perp$.

*Proof.* $W \subseteq \overline{W}$ and from (7.2) we have $\left(\overline{W}\right)^\perp \subseteq W^\perp$.

Conversely, let $v \in W^\perp$. We want to show that $v \in \left(\overline{W}\right)^\perp$. That is, for all $w \in \overline{W}$, $\langle v, w \rangle = 0$. Let $\{ w_n \}_n \subseteq W$ with $w_n \to w$. Then

$$\begin{aligned} \langle v, w \rangle &= \left\langle v, \lim_n w_n \right\rangle \\ &= \lim_n \langle v, w_n \rangle \\ &= \lim_n 0 \\ &= 0 . \end{aligned}$$

where in the second equality we have used the fact that $\langle v, \cdot\rangle$ is a continuous function on $\mathscr{H}$:

$$\begin{aligned} |\langle v,\varphi\rangle - \langle v,\psi\rangle| &= |\langle v,\varphi-\psi\rangle| \\ &\leq \|v\|\|\varphi-\psi\|\,. \end{aligned}$$

□

*Claim* 7.9. For any subspace $W \subseteq \mathscr{H}$, $\left(W^\perp\right)^\perp = \overline{W}$.

*Proof.* Let $w \in \overline{W}$. Then $\langle w, v\rangle = 0$ for all $v \in \left(\overline{W}\right)^\perp$, which, via Claim 7.8, implies that $\langle w, v\rangle = 0$ for all $v \in W^\perp$, which is equivalent to saying that $w \in \left(W^\perp\right)^\perp$.

Conversely, by Theorem 7.7, for any closed subspace,

$$\begin{aligned} \mathscr{H} &= \overline{W} \oplus \left(\overline{W}\right)^\perp \\ &= \overline{W} \oplus W^\perp\,. && \text{(Via Claim 7.8)} \end{aligned}$$

Now since $W^\perp \in \text{Closed}(\mathscr{H})$ via Theorem 7.7, we may also write

$$\mathscr{H} = W^\perp \oplus \left(W^\perp\right)^\perp\,.$$

Hence we learn that

$$W^\perp \oplus \left(W^\perp\right)^\perp = W^\perp \oplus \overline{W}\,.$$

Now if $\overline{W}$ were a proper subspace of $\left(W^\perp\right)^\perp$, this line would lead to a contradiction. □

## 7.1 Duality in Hilbert spaces and the adjoint

Let us now discuss the duality of a Hilbert space. Our ultimate goal is to define, for every $A \in \mathcal{B}(\mathscr{H})$, an adjoint operator $A^* \in \mathcal{B}(\mathscr{H})$.

**Theorem 7.10.** *There is an anti-$\mathbb{C}$-linear isometric bijection $K : \mathscr{H} \to \mathscr{H}^*$ given by*

$$\varphi \mapsto \langle \varphi, \cdot\rangle\,.$$

*Proof.* Clearly $K$ is anti-$\mathbb{C}$-linear. To show it is isometric, we have by the Cauchy-Schwarz inequality

$$\begin{aligned} \|K(\varphi)\|_{\text{op}} &\equiv \sup\left(\{\, (K(\varphi))(\psi) \mid \|\psi\| = 1\,\}\right) \\ &\equiv \sup\left(\{\, |\langle\varphi,\psi\rangle| \mid \|\psi\| = 1\,\}\right) \\ &\overset{\text{CS}}{\leq} \sup\left(\{\, \|\varphi\|\|\psi\| \mid \|\psi\| = 1\,\}\right) \\ &= \|\varphi\|\,. \end{aligned}$$

But also,

$$\|\varphi\|^2 \equiv \langle\varphi,\varphi\rangle \equiv (K(\varphi))(\varphi) \leq \|K(\varphi)\|\|\varphi\|$$

where the last inequality was Lemma 3.18. Hence $\|K(\varphi)\| = \|\varphi\|$ so $K$ is an isometry. But an isometry is always injective, is it merely remains to show that $K$ is surjective.

Let then $\lambda \in \mathscr{H}^*$. If $\lambda = 0$ then $K(0_{\mathscr{H}}) = 0 = \lambda$. Otherwise, since $\ker(\lambda)$ is a closed linear subspace, the proof above says

$$\mathscr{H} = \ker(\lambda) \oplus \ker(\lambda)^\perp\,.$$

Let therefore $\eta \in \ker(\lambda)^\perp$ and $\eta \neq 0$. Since by linearity we have

$$[(\lambda\psi)\,\eta - (\lambda\eta)\,\psi] \in \ker(\lambda) \qquad (\psi \in \mathscr{H})$$

we have

$$\begin{aligned} 0 &= \langle \eta, [(\lambda\psi)\,\eta - (\lambda\eta)\,\psi] \rangle \\ &= (\lambda\psi)\,\langle \eta, \eta \rangle - (\lambda\eta)\,\langle \eta, \psi \rangle \end{aligned}$$

i.e.,

$$\lambda\psi = \left\langle \frac{\overline{(\lambda\eta)}}{\|\eta\|^2}\eta, \psi \right\rangle$$

or

$$\lambda = \left\langle \frac{\overline{(\lambda\eta)}}{\|\eta\|^2}\eta, \cdot \right\rangle .$$

□

On every Hilbert space, we have the space of all bounded (and hence continuous) linear functions

$$\mathcal{B}(\mathcal{H}) \equiv \{ A : \mathcal{H} \to \mathcal{H} \mid A \text{ linear and } \|A\| < \infty \} .$$

As we have seen in Claim 3.20, this is a Banach algebra (but in general not a Hilbert space in and of itself!). Now, as we shall see, it has additional structure: the adjoint, which maps an operator $A$ to its adjoint $A^*$ (sometimes $A^\dagger$ in physics). It is already clear that given any $A \in \mathcal{B}(\mathcal{H})$, via the above claim, we may map it to an element in $\mathcal{B}(\mathcal{H}^*)$ as follows:

$$A \mapsto KAK^{-1} \equiv (\langle \varphi, \cdot \rangle \mapsto \langle A\varphi, \cdot \rangle) . \tag{7.3}$$

Since $\mathcal{H}^*$ is exhibited as isometrically isomorphic to $\mathcal{H}$ via $K$, we ask how to interpret $KAK^{-1}$ acting directly on $\mathcal{H}$. Let us study this more systematically.

The next two statements show us that, to an extent, operators are determined by their "diagonal matrix elements", if we are allowed to pick arbitrary matrix elements.

*Remark* 7.11 (TODO). Note about dimensionality of $\ker(\lambda)^\perp$ TODO.

**Theorem 7.12.** *If $A \in \mathcal{B}(\mathcal{H})$ and $\langle \varphi, A\varphi \rangle = 0$ for all $\varphi \in \mathcal{H}$ then $A = 0$.*

*Proof.* We also have

$$\langle \varphi + \psi, A(\varphi + \psi) \rangle = 0$$

so

$$\langle \varphi, A\psi \rangle + \langle \psi, A\varphi \rangle = 0 \qquad (\varphi, \psi \in \mathcal{H}) .$$

We may replace $\varphi$ by $\mathrm{i}\varphi$ to get

$$-\mathrm{i}\,\langle \varphi, A\psi \rangle + \mathrm{i}\,\langle \psi, A\varphi \rangle = 0 \qquad (\varphi, \psi \in \mathcal{H}) .$$

If we now multiply the last equation by i and add it to the first, we get

$$\langle \varphi, A\psi \rangle = 0 \qquad (\varphi, \psi \in \mathcal{H}) .$$

Hence taking $\varphi = A\psi$ yields $\|A\psi\|^2 = 0$ so that $A\psi = 0$ for all $\psi \in \mathcal{H}$. □

**Corollary 7.13.** *If $A, B \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle \varphi, A\varphi \rangle = \langle \varphi, B\varphi \rangle \qquad (\varphi \in \mathcal{H})$$

*then $A = B$.*

*Proof.* Apply the preceding theorem to $A - B$. □

Note this would fail if we had an $\mathbb{R}$-Hilbert space, as rotations in $\mathbb{R}^2$ demonstrate (take rotations by 90 degrees CCW or CW)!

**Theorem 7.14.** *If $f : \mathcal{H}^2 \to \mathbb{C}$ is sesquilinear and bounded, i.e.,*

$$S := \sup \left(\left\{\, \left|f\left(\varphi, \psi\right)\right| : \|\varphi\| = \|\psi\| = 1 \,\right\}\right) < \infty$$

*then there is some unique $F \in \mathcal{B}\left(\mathcal{H}\right)$ such that*

$$f\left(\varphi, \psi\right) = \langle F\varphi, \psi\rangle \qquad \left(\varphi, \psi \in \mathcal{H}\right)$$

*and $\|F\| = S$.*

*Proof.* Since

$$\left|f\left(\varphi, \psi\right)\right| \leq S\|\varphi\|\|\psi\|$$

we have for fixed $\varphi$,

$$\mathcal{H} \ni \psi \mapsto f\left(\varphi, \psi\right)$$

a bounded linear functional of norm at most $S\|\varphi\|$. Hence to each such linear functional there corresponds a unique element, which we call $F\varphi$, for which

$$f\left(\varphi, \psi\right) = \langle F\varphi, \psi\rangle$$

and $\|F\varphi\| \leq S\|\varphi\|$. Clearly $F : \mathcal{H} \to \mathcal{H}$ defined in this way is $\mathbb{C}$-linear from the properties of $\langle\cdot,\cdot\rangle$. E.g.,

$$\langle F\left(\alpha\varphi\right), \psi\rangle = f\left(\alpha\varphi, \psi\right) = \overline{\alpha}f\left(\varphi, \psi\right) = \overline{\alpha}\left\langle F\varphi, \psi\right\rangle = \langle \alpha F\varphi, \psi\rangle \; .$$

Hence $F \in \mathcal{B}\left(\mathcal{H}\right)$ with $\|F\| \leq S$. Moreover,

$$\left|f\left(\varphi, \psi\right)\right| = \left|\langle F\varphi, \psi\rangle\right| \leq \|F\varphi\|\|\psi\| \leq \|F\|\|\varphi\|\|\psi\|$$

and hence $S \leq \|F\|$. □

Hence if we have an operator $A \in \mathcal{B}\left(\mathcal{H}\right)$, applying the preceding theorem on the function

$$f\left(\varphi, \psi\right) := \langle \varphi, A\psi\rangle$$

we learn there exists some $F =: A^*$ such that

$$\langle \varphi, A\psi\rangle = \langle A^*\varphi, \psi\rangle \qquad \left(\varphi, \psi \in \mathcal{H}\right)$$

and such that $\|A^*\| = \|A\|$.

We have thus exhibited a map

$$* : \mathcal{B}\left(\mathcal{H}\right) \to \mathcal{B}\left(\mathcal{H}\right)$$

called *the adjoint map* which is a $\mathbb{C}$-anti-linear involution:

$$\begin{aligned}
\left(A+B\right)^* &= A^* + B^* \\
\left(\lambda A\right)^* &= \overline{\lambda}A^* \\
\left(AB\right)^* &= B^*A^* \\
A^{**} &= A \qquad \text{(involution)}
\end{aligned}$$

these can be verified using Corollary 7.13.

*Claim* 7.15 (The C-star identity). We have $\|A\|^2 = \|A^*A\|$.

*Proof.* We have

$$\|A\varphi\|^2 = \langle A\varphi, A\varphi\rangle = \langle \varphi, A^*A\varphi\rangle \leq \|A^*A\|\|\varphi\|^2 \,.$$

But also,

$$\|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2 \,.$$

□

Again we use the properties of $\mathcal{B}(\mathcal{H})$ to make an abstract definition:

**Definition 7.16** (C-star algebra)**.** A C-star algebra is a $\mathbb{C}$-Banach algebra $\mathcal{A}$ together with a $\mathbb{C}$-anti-linear involution

$$* : \mathcal{A} \to \mathcal{A}$$

(i.e. it is a star-algebra) such that

$$\|a^* a\| = \|a\|^2 \qquad (a \in \mathcal{A}) \ .$$

*Remark* 7.17. Note that if we regard $A^*$ as an element of $\mathcal{B}(\mathcal{H}^*)$ (where $\mathcal{H}^*$is the Banach dual of $\mathcal{H}$) rather than as an element of $\mathcal{B}(\mathcal{H})$ then $* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^*)$ may be regarded as a $\mathbb{C}$-linear map. This is how (7.3) is set up.

## 7.2 Kernels and images

**Theorem 7.18.** *If* $A \in \mathcal{B}(\mathcal{H})$ *then*

$$\ker(A^*) = \operatorname{im}(A)^\perp \qquad \ker(A) = \operatorname{im}(A^*)^\perp \ .$$

*Proof.* We have

$$\begin{array}{rcll} A^*\varphi & = & 0 & \\ & \updownarrow & & \\ \langle A^*\varphi, \psi\rangle & = & 0 & (\psi \in \mathcal{H}) \\ & \updownarrow & & \\ \langle \varphi, A\psi\rangle & = & 0 & (\psi \in \mathcal{H}) \\ & \updownarrow & & \\ \varphi & \in & \operatorname{im}(A)^\perp \ . & \end{array}$$

Hence $\ker(A^*) = \operatorname{im}(A)^\perp$. Since $A^{**} = A$, we get the second statement. ☐

**Lemma 7.19.** *We have* $\ker(A) = \ker\left(|A|^2\right)$ *with* $|A|^2 \equiv A^* A$.

*Proof.* We have the chain of implications

$$\begin{aligned} \varphi \in \ker(A) & \iff A\varphi = 0 \\ & \iff A^* A\varphi = |A|^2 \varphi = 0 \\ & \implies \varphi \in \ker\left(|A|^2\right) \ . \end{aligned}$$

Conversely,

$$\begin{aligned} \varphi \in \ker\left(|A|^2\right) & \iff |A|^2 \varphi = 0 \\ & \iff \left\langle |A|^2 \varphi, \psi \right\rangle = 0 \forall \psi \in \mathcal{H} \\ & \iff \langle A\varphi, A\psi\rangle = 0 \forall \psi \in \mathcal{H} \ . \end{aligned}$$

In particular, choose $\psi = \varphi$ to get $\|A\varphi\| = 0$ which implies $A\varphi = 0$ and so $\varphi \in \ker(A)$ as desired. ☐

**Definition 7.20** (Order on self-adjoint operators)**.** We say that $A \in \mathcal{B}(\mathcal{H})$, call it “positive”, denote it by $A \geq 0$, iff

$$\langle \psi, A\psi\rangle \geq 0 \qquad (\psi \in \mathcal{H}) \ .$$

Similarly, for two operators $A, B \in \mathcal{B}(\mathcal{H})$, we say that $A \geq B$ iff $A - B \geq 0$.

**Lemma 7.21.** *For $A \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

1. $\text{im} A \in \text{Closed}(\mathcal{H})$.
2. $0 \notin \sigma\left(|A|^2\right)$ *or zero is an isolated point of* $\sigma\left(|A|^2\right)$.
3. $\exists \varepsilon > 0$ *such that*

$$\|A\varphi\| \quad \geq \quad \varepsilon\|\varphi\| \qquad \left(\varphi \in (\ker A)^\perp\right) .$$

*Proof.* ((1)=>(3)): Assume that $\text{im}(A) \in \text{Closed}(\mathcal{H})$. Consider the bijection $\tilde{A} : \ker(A)^\perp \to \text{im}(A)$. It is clearly bounded since $A$ is bounded. Since $\text{im}(A) \in \text{Closed}(\mathcal{H})$, $\text{im}(A)$ is a Banach space, which implies that $\tilde{A}^{-1} : \text{im}(A) \to \ker(A)^\perp$ is bounded by Corollary 3.33. I.e., $\left\|\tilde{A}^{-1}\right\| < \infty$, which is tantamount to saying that $\exists c < \infty$ such that $\left\|\tilde{A}^{-1}\varphi\right\| < c\|\varphi\|$ for all $\varphi \in \text{im}(A)$. Now if $\psi \in \ker(A)^\perp$, then $A\psi \in \text{im}(A)$, and so $\tilde{A}^{-1}A\psi \equiv \psi$. Hence

$$\begin{aligned} \|\psi\| \quad &\leq \quad c\|A\psi\| \\ &\updownarrow \\ \frac{1}{c}\|\psi\| \quad &\leq \quad \|A\psi\| . \end{aligned}$$

((3)=>(1)): Let $\{\varphi_n\}_n \subseteq \text{im}(A)$ such that $\lim_n \varphi_n = \psi$ for some $\psi \in \mathcal{H}$. Our goal is to show that $\psi \in \text{im}(A)$. Let

$$\eta_n := \tilde{A}^{-1}\varphi_n \in \ker(A)^\perp .$$

Then $A\eta_n = A\tilde{A}^{-1}\varphi_n = \varphi_n$. Now, $\|A\eta_n\| \geq \varepsilon\|\eta_n\|$ by hypothesis. We claim $\{\eta_n\}_n$ is Cauchy. Indeed, $\|\eta_n - \eta_m\| \leq \frac{1}{\varepsilon}\|A(\eta_n - \eta_m)\| = \frac{1}{\varepsilon}\|\varphi_n - \varphi_m\|$. But $\{\varphi_n\}_n$ converges and is hence Cauchy. Hence $\lim_n \eta_n = \xi$ for some $\xi \in \mathcal{H}$ (because regardless of the status of $\text{im}(A)$, $\mathcal{H}$ certainly *is* complete and hence Cauchy sequence converge). Since $A$ is bounded it is continuous, so we find that

$$\begin{aligned} A\xi \quad &= \quad A\lim_n \eta_n \\ &= \quad \lim_n A\eta_n \\ &= \quad \lim_n \varphi_n \\ &= \quad \psi . \end{aligned}$$

We obtain then that $\psi \in \text{im}(A)$ as desired.

((2)<=>(3)): We have $\ker\left(|A|^2\right)^\perp = \ker(A)^\perp$ using Lemma 7.19, and, for fixed $\varepsilon > 0$, we thus have the equivalence

$$\begin{aligned} \|A\varphi\| \quad &\geq \quad \varepsilon\|\varphi\| \qquad \left(\varphi \in (\ker A)^\perp\right) \\ &\updownarrow \\ \left\langle\varphi, |A|^2\varphi\right\rangle \quad &\geq \quad \varepsilon^2\|\varphi\|^2 \qquad \left(\varphi \in \ker\left(|A|^2\right)^\perp\right) . \end{aligned}$$

Now write

$$\mathcal{H} \quad = \quad \ker\left(|A|^2\right) \oplus \ker\left(|A|^2\right)^\perp$$

and consider $|A|^2$ as an operator on the direct sum Hilbert space

$$|A|^2 : \ker\left(|A|^2\right) \oplus \ker\left(|A|^2\right)^\perp \to \ker\left(|A|^2\right) \oplus \ker\left(|A|^2\right)^\perp .$$

Write it in block operator form as

$$|A|^2 \quad = \quad \begin{bmatrix} C_{11} & (C_{21})^* \\ C_{21} & C_{22} \end{bmatrix}$$

where

$$C_{11} : \ker\left(|A|^2\right) \quad \to \quad \ker\left(|A|^2\right)$$

etc. By definition of these spaces, we must have $C_{11} = C_{21} = 0$ since their domain is $\ker\left(|A|^2\right)$. So actually

$$|A|^2 \quad = \quad \begin{bmatrix} 0 & 0 \\ 0 & C_{22} \end{bmatrix}.$$

We note $C_{22}$ need not be an isomorphism since it could fail to be surjective (but it has no kernel by definition). The inequality established above is thus equivalent to

$$\langle \varphi, C_{22}\varphi \rangle \quad \geq \quad \varepsilon^2 \|\varphi\|^2$$

for all vectors on the Hilbert space $\ker\left(|A|^2\right)^\perp$. We now invoke Corollary 8.18 on $C_{22}$ to conclude this estimate is equivalent to that it cannot have spectrum on the interval $\left(0, \varepsilon^2\right)$, which is what we were trying to show. □

## 7.3 Orthogonality and bases

*Claim* 7.22 (Pythagoras). If $\{\varphi_i\}_{i=1}^n$ is an orthonormal set and $\psi \in \mathcal{H}$ then

$$\|\psi\|^2 = \sum_{i=1}^n |\langle \varphi_i, \psi \rangle|^2 + \left\| \psi - \sum_{i=1}^n \langle \varphi_i, \psi \rangle \varphi_i \right\|^2. \tag{7.4}$$

*Proof.* Let us define

$$P := \sum_{i=1}^n \varphi_i \otimes \varphi_i^*.$$

Here we interpret the dual of a vector as a linear functional as follows:

$$\eta^* \quad := \quad \langle \eta, \cdot \rangle \in \mathcal{H}^*.$$

Moreover, $\varphi \otimes \psi^*$ is thus an operator as

$$(\varphi \otimes \psi^*)(\eta) := \langle \psi, \eta \rangle \varphi.$$

We claim that $P = P^* = P^2$. The first equality is obvious. The second one follows via

$$\begin{aligned} P^2 &= \sum_{i,j=1}^n \varphi_i \otimes \varphi_i^* \varphi_j \otimes \varphi_j^* \\ &= \sum_{i,j=1}^n \langle \varphi_i, \varphi_j \rangle \varphi_i \otimes \varphi_j^* \\ &= \sum_{i,j=1}^n \delta_{ij} \varphi_i \otimes \varphi_j^* \\ &= \sum_{i=1}^n \varphi_i \otimes \varphi_i^* \\ &\equiv P. \end{aligned}$$

Hence we have

$$\|\psi\|^2 = \left\|P\psi + P^\perp\psi\right\|^2 = \|P\psi\|^2 + \left\|P^\perp\psi\right\|^2 + 2\,\mathbb{R}\mathrm{e}\left\{\left\langle P\psi, P^\perp\psi\right\rangle\right\} = \|P\psi\|^2 + \left\|P^\perp\psi\right\|^2$$

where in the last step we have used $PP^\perp = 0$. ☐

**Definition 7.23.** Let $\{ E_n \}_{n\in\mathbb{N}}$ be a sequence of *closed* subspaces of $\mathcal{H}$. Then

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} E_n$$

iff $\{ E_n \}_{n\in\mathbb{N}}$ is a mutually orthogonal set (i.e. $\langle u, v\rangle = 0$ if $u \in E_n$ and $v \in E_m$ for $n \neq m$) and $\operatorname{span}\left(\bigcup_{n\in\mathbb{N}} E_n\right)$ (*finite* linear combinations of elements of $E_n$) is dense in $\mathcal{H}$.

Note that thanks to Theorem 7.7, given any closed subspace $M \subseteq \mathcal{H}$, since

$$\mathcal{H} = M \oplus M^\perp$$

we may define an operator

$$P_M : \mathcal{H} \quad\to\quad \mathcal{H}$$

given by

$$\psi \mapsto (\psi_M, 0) \in M \oplus M^\perp\,.$$

This operator is called the projection onto the closed subspace $M$. It is clearly linear and moreover bounded:

$$\begin{aligned} \|P_M\| &\equiv \sup\left(\{\, \|P_M\psi\| \mid \|\psi\| = 1 \,\}\right) \\ &= \sup\left(\{\, \|(\psi_M, 0)\| \mid \|(\psi_M, \psi_{M^\perp})\| = 1 \,\}\right) \\ &\leq 1\,. \end{aligned}$$

In fact, its operator norm is also bounded below by 1 since we may always take a unit vector within $M$, in which case $P_M$ acts trivially on it. In fact, $P_M^2 = P_M^* = P_M$ is a self-adjoint projection.

**Lemma 7.24.** *Let $\{ \varphi_j \}_j \subseteq \mathcal{H}$ be a mutually orthogonal sequence such that*

$$\sum_{j=1}^{\infty}\|\varphi_j\|^2 < \infty\,.$$

*Then*

$$\psi := \lim_{n\to\infty}\sum_{j=1}^{n}\varphi_j$$

*exists and*

$$\|\psi\| = \sqrt{\sum_{j=1}^{\infty}\|\varphi_j\|^2}\,.$$

*Proof.* The sequence $\left\{ \sum_{j=1}^n \varphi_j \right\}_n$ is Cauchy (almost by hypothesis), so the limit exists. But we also have

$$\left\|\sum_{j=1}^{n}\varphi_j\right\|^2 = \sum_{j=1}^{n}\|\varphi_j\|^2$$

by mutual orthogonality. Taking the limit $n \to \infty$ on both sides we obtain the desired equality. ☐

**Theorem 7.25.** *Let* $\{ E_n \}_{n\in\mathbb{N}}$ *be a sequence of* closed *subspaces of* $\mathcal{H}$ *such that*

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} E_n \,.$$

*Then for any* $\psi \in \mathcal{H}$*,*

$$\psi = \lim_{n\to\infty} \sum_{j=1}^{n} P_{E_j}\psi$$

*and*

$$\sum_{j=1}^{\infty} \left\| P_{E_j}\psi \right\|^2 = \|\psi\|^2 \,.$$

*Proof.* For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\psi\|^2 &= \left\| \left( \sum_{j=1}^{n} P_j + \mathbb{1} - \left( \sum_{j=1}^{n} P_j \right) \right) \psi \right\|^2 \\
&= \sum_{j=1}^{n} \left\| P_{E_j}\psi \right\|^2 + \left\| \left( \mathbb{1} - \left( \sum_{j=1}^{n} P_j \right) \right) \psi \right\|^2 && \text{(mutual orthogonality)} \\
&\geq \sum_{j=1}^{n} \left\| P_{E_j}\psi \right\|^2
\end{aligned}$$

so applying the previous lemma with $\varphi_n := P_{E_n}\psi$ we find that

$$\lim_{n\to\infty} \sum_{j=1}^{n} P_{E_j}\psi$$

exists. We claim it equals $\psi$. Indeed,

$$\left( \psi - \sum_{j=1}^{n} P_{E_j}\psi \right) \perp E_m \qquad (m \leq n)$$

and hence in the limit $n \to \infty$,

$$\left( \psi - \lim_{n\to\infty} \sum_{j=1}^{n} P_{E_j}\psi \right) \perp E_m \qquad (m \in \mathbb{N}) \,.$$

Since

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} E_n$$

we have that

$$\left( \psi - \lim_{n\to\infty} \sum_{j=1}^{n} P_{E_j}\psi \right) \perp \mathcal{H}$$

i.e.,

$$\psi = \lim_{n\to\infty} \sum_{j=1}^{n} P_{E_j}\psi \,.$$

□

**Definition 7.26** (orthonormal basis)**.** We now discuss bases for Hilbert spaces. If $S$ is an orthonormal set in $\mathcal{H}$, and no other orthonormal set contains $S$ as a proper subset, then $S$ is called an orthonormal basis (or a *complete orthonormal basis*) for $\mathcal{H}$.

**Theorem 7.27.** *Every Hilbert space $\mathcal{H}$ has an orthonormal basis.*

*Proof.* Let $\mathcal{C}$ be the collection of orthonormal sets in $\mathcal{H}$. Order $\mathcal{C}$ by inclusion: $S_1 < S_2$ iff $S_1 \subseteq S_2$. With this order $<$ on $\mathcal{C}$, $\mathcal{C}$ is partially ordered. It is also non-empty since if $\varphi \in \mathcal{H} \setminus \{ 0 \}$, then $\left\{ \frac{\varphi}{\|\varphi\|} \right\}$ is an orthonormal set. Now if $\{ S_\alpha \}_{\alpha \in A}$ is any *linearly* ordered subset of $\mathcal{C}$, then $\bigcup_{\alpha \in A} S_a$ is an orthonormal set contains each $S_\alpha$ and is hence an upper bound for $\{ S_\alpha \}_{\alpha \in A}$. Since every linearly ordered subset of $\mathcal{C}$ has an upper bound, we can thus apply Zorn's lemma to conclude that $\mathcal{C}$ has a maximal element, that is, an orthonormal set not properly contained in any other orthonormal set. □

**Theorem 7.28.** *Let $\mathcal{H}$ be a Hilbert space and $\{ \varphi_\alpha \}_{\alpha \in A}$ an orthonormal basis. Then*

$$\psi = \sum_{\alpha \in A} \langle \varphi_\alpha, \psi \rangle \varphi_\alpha$$

*and*

$$\|\psi\|^2 = \sum_{\alpha \in A} |\langle \varphi_\alpha, \psi \rangle|^2$$

*for any $\psi \in \mathcal{H}$.*

*Proof.* Apply Theorem 7.25 with $E_n := \mathbb{C}\varphi_n$. □

**Definition 7.29** (Separable metric space)**.** A metric space which has a countable dense subset is called *separable.*

*Remark* 7.30. The notion of isomorphism in the category of Hilbert spaces must preserve also the inner product, i.e.,

$$f : \mathcal{H} \to \tilde{\mathcal{H}}$$

is a Hilbert space isomorphism iff it is a $\mathbb{C}$-linear homeomorphism which obeys

$$\langle f\psi, f\varphi \rangle_{\tilde{\mathcal{H}}} = \langle \psi, \varphi \rangle_{\mathcal{H}} \,.$$

**Theorem 7.31.** *A Hilbert space $\mathcal{H}$ is separable iff it has a countable orthonormal basis $S$. If the basis has $n$ elements, $\mathcal{H}$ is isomorphic to $\mathbb{C}^n$ as a Hilbert space. If $S$ is countable then $\mathcal{H}$ is isomorphic to $\ell^2(\mathbb{N})$ as a Hilbert space.*

*Proof.* First assume that $\mathcal{H}$ is separable as a metric space. Let $\{ \varphi_n \}_n$ be a countable set dense. By omitting some elements of this collection we may make it linearly independent and whose finite linear combination span is the same as the original collection, which is dense. We may then apply a Gram-Schmidt process to this independent collection to a countable orthonormal system.

Conversely, given a countable orthonormal basis $\{ \varphi_n \}_n$, we may approximate any vector by a finite linear combination of elements in this collection (say, with rational coefficients, to make sure it is countable) so that it is dense.

Next, assuming $\mathcal{H}$ is separable, setup the isomorphism

$$U : \mathcal{H} \to \ell^2(\mathbb{N})$$

given by

$$\psi \mapsto (\langle \varphi_1, \psi \rangle, \langle \varphi_2, \psi \rangle, \dots) \,.$$

Theorem 7.28 shows that this map is well-defined and surjective. The calculation

$$\begin{aligned}\langle U\psi, U\varphi\rangle_{\ell^2(\mathbb{N})} &\equiv \sum_{n=1}^{\infty}\overline{(U\psi)_n}\,(U\varphi)_n \\ &= \sum_{n=1}^{\infty}\langle\psi,\varphi_n\rangle\langle\varphi_n,\varphi\rangle \\ &= \sum_{n=1}^{\infty}\langle\psi,\varphi_n\rangle\langle\varphi_n,\varphi\rangle \\ &= \sum_{n=1}^{\infty}\langle\psi, P_{E_n}\varphi\rangle \\ &= \left\langle\psi,\sum_{n=1}^{\infty}P_{E_n}\varphi\right\rangle \\ &= \langle\psi,\varphi\rangle\end{aligned}$$

shows that this map is unitary. □

*Remark* 7.32. Even though all separable Hilbert spaces are isomorphic, it is still convenient and necessary to work with concrete examples. The "abstract", "platonic" separable Hilbert space is usually referred to either as $\ell^2(\mathbb{N})$ or as $\mathbb{C}^\infty$ (the latter more in topology).

While $\ell^\infty(\mathbb{N}\to\mathbb{C})$ is an example of a non-separable Banach space (we may view the power set $\mathscr{P}(\mathbb{N})$ as a subset of $\ell^\infty(\mathbb{N}\to\mathbb{C})$. This set is however uncountable, and moreover, each of its elements is of distance 1 from any other element in it. Form open sets around each of these points). It is not a Hilbert space. If we insist on an example of a non-separable Hilbert space, we have

*Claim* 7.33. Define the space

$$\mathscr{H} := \ell^2(\mathbb{R}) \equiv \left\{ f:\mathbb{R}\to\mathbb{C} \;\middle|\; f^{-1}(\mathbb{C}\setminus\{0\}) \text{ is a countable set and } \sum_{x\in\mathbb{R}}|f(x)|^2<\infty \right\}$$

with the obvious inner product (using the fact that the countable union of countable sets is countable). Then $\mathscr{H}$ is a non-separable Hilbert space.

*Proof.* For each $y\in\mathbb{R}$, define

$$\delta_y(x) := \begin{cases}1 & x=y\\ 0 & x\neq y\end{cases}.$$

The set $\{\delta_y\}_{y\in\mathbb{R}}$ is an uncountable set of elements of distance $\sqrt{2}$ of each other. □

*Remark* 7.34 (Various types of bases). We define a *Hamel* basis (of a Banach space, and hence of a Hilbert space) to be any subset $B$ of a Banach space $X$ such that any vector $\varphi\in X$ may be written as *finite* linear combinations from the collection $B$:

$$\varphi = \sum_{j=1}^{n}\alpha_j b_j$$

with $n\in\mathbb{N}$, $\{\alpha_j\}_{j=1}^n\subseteq\mathbb{C}$ and $\{b_j\}_{j=1}^n\subseteq B$. We have already seen in the homework (HW2 and HW3) that infinite dimensional Banach spaces cannot admit a *countable* Hamel basis, and the same is clearly true for Hilbert spaces. In contrast, a *Schauder basis* allows the linear combinations to be infinite. Hence the above statements imply that a separable Hilbert space has a countable orthonormal Schauder basis. Does every separable Banach space have a Schauder basis? Apparently, according to Enflo, the answer is no.

## 7.4 Direct sums and tensor products

**Definition 7.35.** Given a sequence $\{ \mathcal{H}_n \}_n$ of Hilbert spaces, let

$$\mathcal{H} := \left\{ (x_n)_n \;\middle|\; x_m \in \mathcal{H}_m \wedge \sum_{m=1}^{\infty} \|x_m\|^2 < \infty \right\} .$$

Then $\mathcal{H} =: \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a Hilbert space with the inner product

$$\langle (x_n)_n , (y_n)_n \rangle_{\mathcal{H}} := \sum_{m=1}^{\infty} \langle x_m, y_m \rangle_{\mathcal{H}_m} .$$

*Proof.* We want to prove that the inner product is well-defined:

$$\begin{aligned} \left| \langle (x_n)_n , (y_n)_n \rangle_{\mathcal{H}} \right| &\leq \sum_{m=1}^{\infty} \left| \langle x_m, y_m \rangle_{\mathcal{H}_m} \right| \\ &\leq \sum_{m=1}^{\infty} \|x_m\| \|y_m\| && \text{(Cauchy-Schwarz)} \\ &\leq \left( \sum_{m=1}^{\infty} \|x_m\|^2 \right)^{\frac{1}{2}} \left( \sum_{m=1}^{\infty} \|y_m\|^2 \right)^{\frac{1}{2}} . && \text{(Cauchy-Schwarz)} \end{aligned}$$

The fact it is indeed sesquilinear follow from the linear properties of each inner product. For positive definiteness,

$$\langle (x_n)_n , (x_n)_n \rangle_{\mathcal{H}} \quad \equiv \quad \sum_{m=1}^{\infty} \|x_m\|_{\mathcal{H}_m}^2 < \infty$$

But a series of positive terms can only be zero if each term is zero, which means the whole sequence is zero.

Finally we want to show completeness: assume that $\{ x_m \}_m$ is a Cauchy sequence in $\mathcal{H}$, i.e., that $\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N}$ :

$$\|x_m - x_l\|_{\mathcal{H}} < \varepsilon \qquad (n, m \geq N_\varepsilon) .$$

This last inequality is equivalent to

$$\sum_{n \in \mathbb{N}} \|x_{n,m} - x_{n,l}\|_{\mathcal{H}_n}^2 < \varepsilon^2 \qquad (n, m \geq N_\varepsilon) .$$

which implies that

$$\|x_{n,m} - x_{n,l}\|_{\mathcal{H}_n}^2 < \varepsilon^2 \qquad (n, m \geq N_\varepsilon, n \in \mathbb{N}) .$$

As such, for fixed $n \in \mathbb{N}$, $\{ x_{n,m} \}_m$ is a Cauchy sequence in $\mathcal{H}_n$ which converges to some $y_n$. We want to show that $x_m \to y$ where $y \equiv (y_n)_n$. Hence,

$$\begin{aligned} \|x_m - y\|_{\mathcal{H}}^2 &\equiv \sum_{n \in \mathbb{N}} \|x_{n,m} - y_n\|_{\mathcal{H}_n}^2 \\ &= \sum_{n \in \mathbb{N}} \lim_{l \to \infty} \|x_{n,m} - x_{n,l}\|_{\mathcal{H}_n}^2 \\ &\leq \lim_{l \to \infty} \sum_{n \in \mathbb{N}} \|x_{n,m} - x_{n,l}\|_{\mathcal{H}_n}^2 && \text{(Fatou's lemma)} \\ &= \lim_{l \to \infty} \|x_m - x_l\|_{\mathcal{H}}^2 \\ &\leq \varepsilon^2 . \end{aligned}$$

We still need to show that $y\in\mathcal{H}$:

$$\begin{aligned}\|y\|^2 &\equiv \sum_{n\in\mathbb{N}}\|y_n\|^2_{\mathcal{H}_n}\\ &\leq \sum_{n\in\mathbb{N}}\left(\|y_n-x_{n,m}\|_{\mathcal{H}_n}+\|x_{n,m}\|_{\mathcal{H}_n}\right)^2\\ &< \infty\,.\end{aligned}$$

□

**Lemma 7.36.** *Let $A$ and $B$ be two disjoint sets. Then*

$$\ell^2(A\cup B)\cong\ell^2(A)\oplus\ell^2(B)\ .$$

*Proof.* Setup a basis for $\ell^2(A\cup B)$ given by $\{\ e_\alpha\ \}_{\alpha\in A\cup B}$. Then it is clear how to setup the isomorphism and extend it linearly. □

**Definition 7.37.** Given two Hilbert spaces $\mathcal{H}_1,\mathcal{H}_2$, define their tensor product as follows:

$$V:=\mathcal{H}_1\otimes_{\text{vector space}}\mathcal{H}_2$$

where we mean *the vector space* tensor product (e.g., take the linear span of formal set $\{\ e_j\otimes f_k\ \}$ where $\{\ e_j\ \}_j$ is a basis for $\mathcal{H}_1$ and $\{\ f_k\ \}_k$ is a basis for $\mathcal{H}_2$). On $V$, we define an inner product as

$$\langle\varphi_1\otimes\varphi_2,\psi_1\otimes\psi_2\rangle_V:=\langle\varphi_1,\psi_1\rangle_{\mathcal{H}_1}\langle\varphi_2,\psi_2\rangle_{\mathcal{H}_2}$$

and extend linearly from non-simple vectors.

$$\mathcal{H}_1\otimes_{\text{Hilbert space}}\mathcal{H}_2$$

is then defined as the *completion* of $V$ under this inner product. One way to think about the completion is as a space of equivalence classes of Cauchy sequences. There is also an explicit construction by identifying $\mathcal{H}_1\otimes_{\text{Hilbert space}}\mathcal{H}_2\cong\mathcal{B}(\mathcal{H}_1\to\mathcal{H}_2^*)$ [TODO: expand on this].

*Claim* 7.38. Given two countable sets $A$ and $B$,

$$\ell^2(A\times B)\cong\ell^2(A)\otimes\ell^2(B)\ .$$

*Proof.* Send the basis element

$$e_{(a,b)}\mapsto e_a\otimes e_b$$

and extend linearly. □

**Definition 7.39** (Fock space)**.** Given a Hilbert space $\mathcal{H}$, define

$$\mathcal{F}(\mathcal{H}):=\bigoplus_{n=0}^{\infty}\underbrace{\mathcal{H}\otimes\cdots\otimes\mathcal{H}}_{n\text{ times}}$$

where we use the convention that $\mathcal{H}^{\otimes 0}\equiv\mathbb{C}$. $\mathcal{F}(\mathcal{H})$ is separable if $\mathcal{H}$ is. Its symmetric and anti-symmetric subspaces are called the Bosonic and Fermionic subspaces respectively.

# 8 Some C-star algebra theory

In this section we momentarily leave $\mathcal{B}(\mathcal{H})$ and concentrate on an abstract C-star algebra as in Definition 7.16 $\mathcal{A}$. This is done for the sake of establishing some properties of elements in C-star algebras which only call for this abstract setting.

We shall culminate with the *continuous* functional calculus in this setting, as well as a representation theorem for C-star algebras.

## 8.1 Normals, self-adjoints, unitaries and self-adjoint projections on C-star algebras

**Definition 8.1** (Algebraic properties of elements in a C-star algebra)**.** For $a \in \mathcal{A}$ (with $|a|^2 \equiv a^* a$) we say it is

1. Normal iff $|a|^2 = |a^*|^2$ iff $[a, a^*] = 0$ where
$$|a|^2 \equiv a^* a \tag{8.1}$$
and where
$$[a, b] \equiv ab - ba\,. \tag{8.2}$$
2. *Self-adjoint* iff $a^* = a$.
3. *Positive* iff $a = |b|^2$ for some $b \in \mathcal{A}$.
4. *Unitary* iff $|a|^2 = |a^*|^2 = \mathbb{1}$. Isometry iff $|a|^2 = \mathbb{1}$ and Co-Isometry iff $|a^*|^2 = \mathbb{1}$.
5. *Idempotent* iff $a^2 = a$.
6. S*elf-adjoint projection* iff $a^2 = a^* = a$ (more commonly *orthogonal projection*).
7. *Partial isometry* iff $|a|^2$ is an idempotent (automatically self-adjoint, so we can just as well say that $|a|^2$ is a self-adjoint projection).

*Claim* 8.2. For $a \in \mathcal{A}$, $a = 0$ iff $|a|^2 = 0$.

*Proof.* Clearly if $a = 0$ then $|a|^2 = a^* 0 = 0$. In fact $a^* = 0$ too by $0^* = 0$.

Conversely, if $|a|^2 = 0$ then $\left\| |a|^2 \right\| = 0$. But the C-star identity then says that $\|a\|^2 = 0$, i.e., that $\|a\| = 0$ so that $a = 0$. ☐

**Lemma 8.3.** *$a \in \mathcal{A}$ is a partial isometry iff $a^*$ is a partial isometry, in which case*
$$a \overset{(1)}{=} aa^* a \overset{(2)}{=} aa^* aa^* a \equiv |a^*|^2 a |a|^2 \,.$$

*Proof.* Let us begin by showing (1). Calculate
$$\begin{aligned}
\left|\left(\mathbb{1} - |a^*|^2\right) a\right|^2 &\equiv \left[\left(\mathbb{1} - |a^*|^2\right) a\right]^* \left(\mathbb{1} - |a^*|^2\right) a \\
&= a^* \left(\mathbb{1} - |a^*|^2\right) \left(\mathbb{1} - |a^*|^2\right) a \\
&= a^* \left(\mathbb{1} - 2|a^*|^2 + |a^*|^4\right) a \\
&= |a|^2 - 2a^* aa^* a + a^* aa^* aa^* a \\
&= |a|^2 - 2|a|^2 |a|^2 + |a|^2 |a|^2 |a|^2 \\
&= |a|^2 - 2|a|^2 + |a|^2 && (|a|^2 \text{ is an idempotent}) \\
&= 0\,.
\end{aligned}$$
Now using Claim 8.2 we conclude that $\left(\mathbb{1} - |a^*|^2\right) a = 0$. Hence
$$aa^* a = |a^*|^2 a = \left(|a^*|^2 - \mathbb{1} + \mathbb{1}\right) a = a\,.$$

Next, we want to show that $|a^*|^2$ is an idempotent. This is an immediate application of the first equality:

$$\left(|a^*|^2\right)^2 \equiv (aa^*)^2 \equiv (aa^*a)\, a^* \overset{\star}{=} aa^* \equiv |a^*|^2 \ .$$

Finally, we tend to the second equality:

$$aa^*aa^*a = (aa^*a)\, a^*a \overset{(1)}{=} aa^*a \overset{(1)}{=} a\,.$$

□

This lemma shows us that partial isometries can be considered as "maps" from the range of the SA projection $|a|^2$ to the range of the SA projection $|a^*|^2$.

*Claim* 8.4. If $p \in \mathcal{A}$ is a self-adjoint projection then $\|p\| = 1$.

*Proof.* We have

$$\begin{aligned} \|p\|^2 &= \|p^*p\| \\ &= \left\|p^2\right\| \\ &= \|p\|\,. \end{aligned}$$

□

**Lemma 8.5.** *If $u \in \mathcal{A}$ is a unitary then $\|u\| = 1$ and $\sigma(u) \subseteq \mathbb{S}^1$.*

*Proof.* Again the C-star identity implies

$$\|u\|^2 = \left\||u|^2\right\| = \|\mathbb{1}\| = 1\,.$$

Now let $\lambda \in \sigma(u)$. Of course $\lambda \neq 0$ for otherwise $u$ is not invertible, which is false by $u^*u = \mathbb{1}$. Then by the spectral mapping theorem Theorem 6.35,

$$\lambda^{-1} \in \sigma\left(u^{-1}\right) \overset{u^*=u^{-1}}{=} \sigma(u^*) = \overline{\sigma(u)}\,.$$

As the norm bounds the spectral radius, we have $r(u) \leq \|u\| = 1$, so that $|\lambda| \leq 1$. But the above shows also that $\left|\lambda^{-1}\right| \leq 1$. Hence $|\lambda| = 1$. □

**Theorem 8.6.** *If $a \in \mathcal{A}$ is self-adjoint then the spectral radius $r(a)$ equals the operator norm $\|a\|$.*

*Proof.* By the C-star identity, we have

$$\|a\|^2 = \|a^*a\| = \left\|a^2\right\|$$

from which we conclude that

$$\|a\|^{2n} = \left\|a^{2n}\right\|$$

and hence

$$r(a) \equiv \lim_n \|a^n\|^{\frac{1}{n}} = \lim_n \left\|a^{2^n}\right\|^{2^{-n}} = \|a\|\,.$$

□

**Corollary 8.7.** *There is at most one norm on a star-algebra making it a C-star algebra.*

*Proof.* We have for any $a \in \mathcal{A}$, using the C-star identity and the fact $|a|^2$ is self-adjoint:

$$\|a\|^2 = \|a^* a\| = r\left(|a|^2\right) = \sup_{\lambda \in \sigma\left(|a|^2\right)} |\lambda| \ .$$

But the RHS is independent of the norm. ☐

Actually one only needs the C-star identity as an upper bound:

**Lemma 8.8.** *If $\mathcal{A}$ is a Banach star-algebra such that*

$$\|a\|^2 \leq \|a^* a\|$$

*then $\mathcal{A}$ is a C-star algebra.*

*Proof.* [TODO] ☐

*Claim* 8.9. If $a \in \mathcal{A}$ is self-adjoint then $\sigma(a) \subseteq \mathbb{R}$.

*Proof.* Since $\|a^*\| = \|a\|$, clearly the involution is continuous. Note that $e^{ia}$ (understood via the entire functional calculus) is unitary:

$$\left(e^{ia}\right)^* = \left(\sum_{n=0}^{\infty} \frac{i^n}{n!} a^n\right)^* = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} a^n = e^{-ia} \, .$$

But $e^{ia} e^{-ia} = \mathbb{1}$. Hence $\sigma\left(e^{ia}\right) \subseteq \mathbb{S}^1$.

Now let $\lambda \in \sigma(a)$. Let $b := \sum_{n=1}^{\infty} \frac{i^n}{n!} (a - \lambda \mathbb{1})^{n-1}$. Then

$$e^{ia} - e^{i\lambda}\mathbb{1} = \left(e^{i(a-\lambda\mathbb{1})} - \mathbb{1}\right) e^{i\lambda} = (a - \lambda\mathbb{1})\, b e^{i\lambda} \, .$$

Since $b$ commutes with $a$ and $a - \lambda\mathbb{1}$ is not invertible, $e^{ia} - e^{i\lambda}\mathbb{1}$ is also not invertible. Hence $e^{i\lambda} \in \sigma\left(e^{ia}\right) \subseteq \mathbb{S}^1$ which implies $\lambda \in \mathbb{R}$. ☐

**Lemma 8.10.** *Let $a \in \mathcal{A}$ be such that $a = a^*$. Then:*

1. *If $\sigma(a) \subseteq [0, \infty)$ then $\|a - \|a\|\mathbb{1}\| \leq \|a\|$.*
2. *If for some $t \geq 0$, $\|a - t\mathbb{1}\| \leq t$ then $\sigma(a) \subseteq [0, \infty)$.*

*Proof.* We shall use the spectral mapping theorem, and the fact that the spectral radius equals the norm for self-adjoint elements.

For the first statement, assume $\sigma(a) \subseteq [0, \infty)$. Together with $a$ being self-adjoint we get $\sigma(a) \subseteq [0, \|a\|]$ so that

$$\sigma(a - \|a\|\mathbb{1}) \overset{\text{spec. mapping}}{=} \sigma(a) - \|a\| \subseteq [-\|a\|, 0]$$

and hence $r(a - \|a\|\mathbb{1}) \leq \|a\|$ which is what we need.

For the second statement, assume there is some $t \geq 0$ for which $\|a - t\mathbb{1}\| \leq t$. Then

$$\sigma(a - t\mathbb{1}) \overset{\text{spec. mapping}}{=} \sigma(a) - t \overset{\text{assumption}}{\subseteq} [-t, t] \ .$$

So if there were $\lambda \in \sigma(a)$ with $\lambda < 0$, it would violate this last inclusion. ☐

**Lemma 8.11.** *Let $a,b\in\mathcal{A}$ be such that $a=a^*$ and $b=b^*$, and such that $\sigma(a)\subseteq[0,\infty)$ and $\sigma(b)\subseteq[0,\infty)$. Then*

$$\sigma(a+b)\subseteq[0,\infty)\,.$$

*Proof.* Using the first item in Lemma 8.10 we find that

$$\begin{cases}\|a-\|a\|\mathbb{1}\| & \leq\|a\|\\ \|b-\|b\|\mathbb{1}\| & \leq\|b\|\end{cases}.$$

Combining the two together we get

$$\begin{aligned}\|a+b-\|a\|\mathbb{1}-\|b\|\mathbb{1}\| & \overset{\Delta\neq}{\leq} \|a-\|a\|\mathbb{1}\|+\|b-\|b\|\mathbb{1}\|\\ & \leq \|a\|+\|b\|\,.\end{aligned}$$

Thus, invoking the second item in Lemma 8.10 (and the trivial fact that $(a+b)^*=a^*+b^*$) we get the desired result. ☐

**Lemma 8.12.** *The power series of $z\mapsto\sqrt{1-z}$ about zero converges absolutely for all $z\in\overline{B_1(0_{\mathbb{C}})}$.*

*Proof.* We have

$$\sqrt{1-z}=1-\sum_{n=1}^{\infty}\frac{1}{2n-1}\binom{2n}{n}\left(\frac{z}{4}\right)^n\,.$$

☐

To define the square root of a positive element, we would have liked to use the holomorphic functional calculus. Unfortunately this is not possible if

$$0\in\sigma(a)$$

so we must find an alternative route. The best such route is the continuous functional calculus, which we shall encounter below in Theorem 8.40. Another possibility is the so-called *Gelfand representation*. However, here we choose to proceed with the tools we already have:

**Theorem 8.13** (Square root lemma)**.** *If $a\in\mathcal{A}$ has $a=a^*$ and $\sigma(a)\subseteq[0,\infty)$ then there exists some $b\in\mathcal{A}$ such that: (1) $[b,a]=0$, (2) $b=b^*$, (3) $\sigma(b)\subseteq[0,\infty)$, and (4) $b^2=a$.*

*Proof.* Consider the power series about zero of

$$B_1(0_{\mathbb{C}})\ni z\mapsto\sqrt{1-z}\equiv(1-z)^{\frac{1}{2}}=\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}(-1)^j z^j\,.$$

We have

$$\binom{\frac{1}{2}}{j}\equiv\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-j+1\right)}{j!},\qquad\binom{\frac{1}{2}}{0}\equiv1\,.$$

In fact

$$c_j:=-\binom{\frac{1}{2}}{j}(-1)^j\quad\in\quad(0,1)\qquad(j\in\mathbb{N}_{\geq1})\,.$$

It is well-known that this series actually converges absolutely also on the boundary of the disc, i.e., on $\overline{B_1(0_{\mathbb{C}})}$. As a result we have the *norm* convergent series in $\mathcal{A}$

$$\sqrt{\mathbb{1}-x} := \mathbb{1} - \sum_{j=1}^{\infty} c_j x^j \qquad (x \in \mathcal{A} : \|x\| \leq 1) \ . \tag{8.3}$$

Applying it to $x := \mathbb{1} - \frac{1}{\|a\|} a$ we have

$$\begin{aligned} \sigma(x) &\overset{\text{spec. mapping}}{=} 1 - \frac{1}{\|a\|}\sigma(a) \\ &\subseteq 1 - \frac{1}{\|a\|}[0, \|a\|] \\ &= 1 - [0,1] \\ &= [0,1] \end{aligned}$$

and so since $x = x^*$, $\|x\| = r(x) \leq 1$. We are therefore justified to plug in $x$ into (8.3) in order to get

$$\begin{aligned} b &:= \sqrt{a} \\ &= \sqrt{\|a\| \frac{1}{\|a\|} a} \\ &= \sqrt{\|a\|}\sqrt{\mathbb{1} - \left(\mathbb{1} - \frac{1}{\|a\|} a\right)} \\ &= \sqrt{\|a\|}\left(\mathbb{1} - \sum_{j=1}^{\infty} c_j \left(\mathbb{1} - \frac{1}{\|a\|} a\right)^j\right) \end{aligned} \tag{8.4}$$

as a norm convergent series. Since the convergence is absolute we can square the series and re-arrange the terms to obtain the fact that $b^2 = a$. Since $* : \mathcal{A} \to \mathcal{A}$ is norm continuous and the partial sums in the series for $b$ are self-adjoint, so is $b$ itself. Moreover, $b$ also has positive spectrum. Indeed, since $c_j > 0$ and $\sigma\left(\frac{1}{\|a\|} a\right) \subseteq [0,1]$, we get

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} c_j \left(\mathbb{1} - \frac{1}{\|a\|} a\right)^j \right\| &\leq \sum_{j=1}^{\infty} c_j \left\| \left(\mathbb{1} - \frac{1}{\|a\|} a\right)^j \right\| \\ &\leq \sum_{j=1}^{\infty} c_j \left\| \mathbb{1} - \frac{1}{\|a\|} a \right\|^j \\ &\leq \sum_{j=1}^{\infty} c_j \\ &= 1 \,. \end{aligned}$$

So the spectral mapping theorem implies

$$\begin{aligned} \sigma\left(\mathbb{1} - \sum_{j=1}^{\infty} c_j \left(\mathbb{1} - \frac{1}{\|a\|} a\right)^j\right) &= 1 - \sigma\left(\sum_{j=1}^{\infty} c_j \left(\mathbb{1} - \frac{1}{\|a\|} a\right)^j\right) \\ &\subseteq 1 - [-1, 1] \\ &= [0,2] \\ &\subseteq [0, \infty) \ . \end{aligned}$$

Since the series for $b$ converges absolutely, it commutes with $a$. ☐

**Lemma 8.14** (The positive and negative parts of a self-adjoint element)**.** *Given any $a \in \mathcal{A}$ with $a = a^*$ we know that $\sigma(a) \subseteq \mathbb{R}$. Then by the spectral mapping $\sigma\left(a^2\right) \subseteq [0,\infty)$ and so by Theorem 8.13, $\sqrt{a^2}$ exists as a self-adjoint operator with positive spectrum. We then define the positive and negative parts of a as*

$$a^{\pm} \quad := \quad \frac{1}{2}\left(\sqrt{a^2} \pm a\right) .$$

*Then $a^{\pm}$ are self-adjoint elements with positive spectra such that $a^+ a^- = 0$.*

*Proof.* First note that

$$\begin{aligned} a^+ a^- &= \frac{1}{4}\left(\sqrt{a^2}+a\right)\left(\sqrt{a^2}-a\right) \\ &= \frac{1}{4}\left(a^2 - a^2 - \sqrt{a^2}a + a\sqrt{a^2}\right) \\ &= \frac{1}{4}\left[a, \sqrt{a^2}\right] . \end{aligned}$$

But, from Lemma 8.12 it is clear that $\sqrt{a^2}$ is a norm-convergent series in $a^2$, and hence commutes with $a$.

Let $p_n$ be the polynomial approximation to $\sqrt{a^2}$ up to order $n$:

$$\begin{aligned} p_n &:= \sqrt{\|a^2\|}\left(\mathbb{1} - \sum_{j=1}^{n} c_j \left(\mathbb{1} - \frac{1}{\|a^2\|}a^2\right)^j\right) \\ &= \|a\|\left(\mathbb{1} - \sum_{j=1}^{n} c_j \left(\mathbb{1} - \left(\frac{1}{\|a\|}a\right)^2\right)^j\right) \end{aligned}$$

so that

$$\begin{aligned} 2a^{\pm} &= \lim_n \left(p_n \pm a\right) \\ &= \|a\| \lim_n \left(\mathbb{1} - \sum_{j=1}^{n} c_j \left(\mathbb{1} - \left(\frac{1}{\|a\|}a\right)^2\right)^j \pm \frac{1}{\|a\|}a\right) . \end{aligned}$$

One verifies that for any $n \in \mathbb{N}$ the polynomial

$$(-1,1) \ni \alpha \mapsto 1 - \sum_{j=1}^{n} c_j \left(\mathbb{1} - \alpha^2\right)^j \pm \alpha$$

is positive and hence by Corollary 6.29, the spectrum of $a^{\pm}$ is contained in $[0,\infty)$. □

**Theorem 8.15** (Characterization of positive elements)**.** *$a \in \mathcal{A}$ is positive iff $a = a^*$ and $\sigma(a) \subseteq [0,\infty)$.*

*Proof.* Assume first that $a$ is positive, i.e., that $a = |b|^2$. Then clearly $|b|^2$ is self-adjoint and we are left to show that $\sigma\left(|b|^2\right) \subseteq [0,\infty)$.

To that end, first let us show that if, for some generic $a \in \mathcal{A}$, $\sigma\left(-|a|^2\right) \subseteq [0,\infty)$ then $a = 0$. Indeed, we first remark that using Lemma 6.30 we have

$$\sigma\left(|a|^2\right) \setminus \{ 0 \} = \sigma\left(|a^*|^2\right) \setminus \{ 0 \} .$$

Hence $\sigma\left(-|a|^2\right) \subseteq [0,\infty)$ implies $\sigma\left(-|a^*|^2\right) \subseteq [0,\infty)$. Let us write

$$a = b + \mathrm{i}c$$

where $b=\mathbb{Re}\{a\}\equiv\frac{1}{2}(a+a^*)$ and $c=\mathbb{Im}\{a\}\equiv\frac{1}{2\mathrm{i}}(a-a^*)$. Then

$$\begin{aligned}|a|^2+|a^*|^2 &= (b+\mathrm{i}c)^*(b+\mathrm{i}c)+(b+\mathrm{i}c)(b+\mathrm{i}c)^* \\ &= (b-\mathrm{i}c)(b+\mathrm{i}c)+(b+\mathrm{i}c)(b-\mathrm{i}c) \\ &= b^2-\mathrm{i}cb+\mathrm{i}bc+c^2+b^2-\mathrm{i}bc+\mathrm{i}cb+c^2 \\ &= 2b^2+2c^2\,.\end{aligned}$$

Hence $|a|^2=2b^2+2c^2+\left(-|a^*|^2\right)$. By the spectral mapping theorem Theorem 6.35, $b^2$ and $c^2$ have positive spectra, and the sum of self-adjoint elements with positive spectra has positive spectra Lemma 8.11. Hence $|a|^2$ has positive spectra. But by assumption, $\sigma\left(-|a|^2\right)\subseteq[0,\infty)$, i.e.,

$$\sigma\left(|a|^2\right)\subseteq[0,\infty)\cap(-\infty,0]=\{\,0\,\}$$

so that

$$\|a\|^2=\left\||a|^2\right\|=r\left(|a|^2\right)=0$$

which implies $a=0$ as desired.

Now since $|a|^2=:b$ is Hermitian, so using Lemma 8.14 we may write

$$b=b^+-b^-$$

with $b^\pm$ self-adjoint with positive spectrum, and $b^+b^-=0$. Now with $c:=ab^-$ we have

$$\begin{aligned}-|c|^2 &= -b^-a^*ab^- \\ &= -b^-bb^- \\ &= -b^-\left(b^+-b^-\right)b^- \\ &= \left(b^-\right)^3\,.\end{aligned}$$

By the spectral mapping since $\sigma(b^-)\subseteq[0,\infty)$, the same holds for the third power. Thus

$$\sigma\left(-|c|^2\right)\subseteq[0,\infty)$$

so that by the first part of this proof, $c=0$. Hence, $ab^-=0$ by Claim 8.2 so that $b^-=0$, i.e.,

$$b=\sqrt{b^2}\,.$$

But by Theorem 8.13 $\sqrt{b^2}$ has $\sigma\left(\sqrt{b^2}\right)\subseteq[0,\infty)$ so that

$$\sigma\left(|a|^2\right)\subseteq[0,\infty)\;.$$

Conversely, assuming that $a=a^*$ has $\sigma(a)\subseteq[0,\infty)$, apply Theorem 8.13 to get $\sqrt{a}$ which is self-adjoint and for which

$$\left(\sqrt{a}\right)^*\sqrt{a}=\left(\sqrt{a}\right)\sqrt{a}=a\,.$$

□

Thanks to this characterization of positive elements, we may extend $\sqrt{a^2}$ from self-adjoints to arbitrary $a\in\mathcal{A}$ as $\sqrt{a^*a}\equiv|a|$.

**Lemma 8.16.** *The square root defined in Theorem 8.13 is actually unique: if $a\geq0$, and $c\in\mathcal{A}$ is such that: $c\geq0$, $c^2=a$ then $c=\sqrt{a}$.*

*Proof.* Let $b:=\sqrt{a}$ for convenience. We have

$$ca = c^3=ac$$

so that $c$ commutes with $a$ and hence with $b$. Thus

$$(b-c)\,b\,(b-c)+(b-c)\,c\,(b-c)=(b-c)^2\,(b+c)=(b-c)\left(b^2-c^2\right)=(b-c)\,(a-a)=0\,.$$

But both $(b-c)\,b\,(b-c)$ and $(b-c)\,c\,(b-c)$ are positive: $b-c$ is self-adjoint, and

$$w^*qw$$

is clearly positive iff $q$ is positive, so they are both zero separately (as in the proof of Theorem 8.15, we would have the spectrum of a negative operator in $[0,\infty)$ which would imply that it is zero). Their difference is

$$(b-c)^3 \quad = \quad 0\,.$$

But since $b-c$ is self-adjoint, the C-star identity implies

$$\|(b-c)\|^4 \quad = \quad \left\|(b-c)^4\right\|=0$$

so that $b=c$. □

**Corollary 8.17.** *For any $a=a^*$, $a\leq\|a\|\mathbb{1}$ and if $0\leq a\leq b$ then $\|a\|\leq\|b\|$.*

*Proof.* For the first part, we have by the spectral mapping theorem that

$$\begin{aligned}\sigma\left(\|a\|\mathbb{1}-a\right) &= \|a\|-\sigma\left(a\right)\\ &= \subseteq[0,\infty)\end{aligned}$$

but then, thanks to the above theorem, if $\|a\|\mathbb{1}-a$ is self-adjoint and has positive spectrum, then it is positive.

Next, $a\leq b\leq\|b\|\mathbb{1}$ from which we find $\|b\|\mathbb{1}-a$ having positive spectrum. But

$$\sigma\left(\|b\|\mathbb{1}-a\right)=\|b\|-\sigma\left(a\right)$$

which means that $\sigma\left(a\right)\subseteq[0,\|b\|]$, i.e., $\|a\|\leq\|b\|$. □

**Corollary 8.18.** *If $a=a^*$ obeys $a\geq\varepsilon\mathbb{1}$ for some $\varepsilon>0$ then $a$ is invertible and $\left\|a^{-1}\right\|\leq\frac{1}{\varepsilon}$. This is an if and only if condition.*

*Proof.* By the above, we have

$$\sigma\left(a-\varepsilon\mathbb{1}\right)=\sigma\left(a\right)-\varepsilon$$

positive, i.e., all points $\lambda\in\sigma\left(a\right)$ for which $a-\lambda\mathbb{1}$ is not invertible have $\lambda\geq\varepsilon>0$ so that $a$ is invertible. Moreover,

$$\sigma\left(a^{-1}\right)=\frac{1}{\sigma\left(a\right)}$$

and we have

$$\sup\left|\sigma\left(a^{-1}\right)\right|=\frac{1}{\inf\left|\sigma\left(a\right)\right|}\leq\frac{1}{\varepsilon}\,.$$

But $\left\|a^{-1}\right\|=r\left(a^{-1}\right)\equiv\sup\left|\sigma\left(a^{-1}\right)\right|$. □

**Corollary 8.19.** *If $a=a^*$, $b=b^*$ with $a$ invertible obeys $0\leq a\leq b$ then $0\leq b^{-1}\leq a^{-1}$.*

*Claim* 8.20. $\sigma\left(p\right)\subseteq\{\,0,1\,\}$ for any self-adjoint projection $p\in\mathcal{A}$. Furthermore, if $p$ is normal with $\sigma\left(p\right)\subseteq\{\,0,1\,\}$ then $p$ is a self-adjoint projection.

*Claim* 8.21. If $u$ is normal and $\sigma\left(u\right)\subseteq\mathbb{S}^1$ then $u$ is unitary.

## 8.2 The GNS representation [extra]

Given a Hilbert space $\mathcal{H}$, we have seen that $\mathcal{B}(\mathcal{H})$ serves as a model for a C-star algebra. It turns out that *all* C-star algebras may be realized as sub-algebras of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$, and this is the contents of this section. In this section we mainly follow [Con19].

### 8.2.1 Maximal ideals and homomorphisms

Recall from algebra that an ideal $\mathcal{I}$ in an algebra $\mathcal{A}$ is a subgroup with respect to addition, and moreover, is closed under multiplication:

$$\begin{aligned}\mathcal{A}\mathcal{I} &\subseteq \mathcal{I}\\ \mathcal{I}\mathcal{A} &\subseteq \mathcal{I}\,.\end{aligned}$$

(Strictly speaking what we formulated here is a two-sided ideal. One also speaks of left and right ideals). We may also speak of *maximal* ideals (in the sense of set inclusion): An ideal $\mathcal{I} \subsetneq \mathcal{A}$ is *maximal* iff there is no other ideal $\mathcal{J} \subsetneq \mathcal{A}$ such that

$$\mathcal{I} \subsetneq \mathcal{J} \subsetneq \mathcal{A}\,.$$

According to this convention, maximal ideals are necessarily proper ideals.

*Claim* 8.22. Every proper ideal is contained in a maximal ideal.

*Proof.* Let

$$S := \{\, \mathcal{J} \subsetneq \mathcal{A} \text{ is a proper ideal} \mid \mathcal{I} \subseteq \mathcal{J} \,\}\,.$$

Then $\mathcal{I} \in S$ and $S$ is partially ordered by set inclusion. If $\{\mathcal{J}_i\}_i$ is a totally ordered subset of $S$, then $\mathcal{J} := \bigcup_i \mathcal{J}_i$ is also a proper ideal. So $\mathcal{J}$ is an upper bound on $\{\mathcal{J}_i\}_i$. So Zorn's lemma yields that $S$ has a maximal element. □

*Claim* 8.23. Let $\mathcal{A}$ be an Abelian Banach algebra. Then $a \notin \mathcal{G}_{\mathcal{A}}$ iff $a \in \mathcal{M}$ for some maximal ideal $\mathcal{M}$.

*Proof.* Let $\mathcal{M}$ be a maximal ideal and let $a \in \mathcal{M}$. If $a \in \mathcal{G}_{\mathcal{A}}$ then $\exists a^{-1} \in \mathcal{A}$. Then $a^{-1}a \in \mathcal{M}$. But that implies $\mathbb{1} \in \mathcal{M}$, which means that $\mathcal{M} = \mathcal{A}$, in contradiction with $\mathcal{M}$ being maximal and hence proper. Hence if $a \in \mathcal{M}$ for $\mathcal{M}$ maximal then $a \notin \mathcal{G}_{\mathcal{A}}$. Conversely, if $a \notin \mathcal{G}_{\mathcal{A}}$ then consider the set

$$\mathcal{I} := \{\, ab \mid b \in \mathcal{A} \,\}\,.$$

This is a vector subspace which is also an ideal thanks to $\mathcal{A}$ being Abelian. It is also a proper ideal since $a$ is not invertible, so it can't contain $\mathbb{1}$. But every proper ideal is contained in a maximal ideal, see Claim 8.22. □

*Claim* 8.24. In a commutative Banach algebra $\mathcal{A}$, every maximal ideal is closed.

*Proof.* The closure $\overline{\mathcal{M}}$ of an ideal $\mathcal{M}$ is also an ideal. But as $\mathcal{M}$ is maximal, we must have either $\mathcal{M} = \overline{\mathcal{M}}$ or $\mathcal{A} = \overline{\mathcal{M}}$. But it can't be the case that $\overline{\mathcal{M}} = \mathcal{A}$. Indeed, we know that $\mathcal{M} \cap \mathcal{G}_{\mathcal{A}} = \varnothing$ via the proof in Claim 8.23. Hence $\mathcal{M} \subseteq \mathcal{G}_{\mathcal{A}}^c$. But we know that $\mathcal{G}_{\mathcal{A}}^c$ is closed (see Claim 6.6) so that $\overline{\mathcal{M}} \subseteq \mathcal{G}_{\mathcal{A}}^c$ so that it can't be that $\overline{\mathcal{M}} = \mathcal{A}$. □

**Proposition 8.25.** *Let $\mathcal{A}$ be an Abelian Banach algebra. Then there is a bijection between the space of non-zero homomorphisms and maximal ideals.*

*Proof.* Let $\mathcal{M}$ be a maximal ideal. Then since it is closed, $\mathcal{A}/\mathcal{M}$ is a Banach algebra with identity. Let

$$\pi : \mathcal{A} \to \mathcal{A}/\mathcal{M}$$

be the quotient map. We will show that $\mathcal{A}/\mathcal{M} \cong \mathbb{C}$. Let $a \in \mathcal{A}$. If $\pi(a) \notin \mathcal{G}_{\mathcal{A}/\mathcal{M}}$, then

$$\pi(\mathcal{A}a) = \pi(a)(\mathcal{A}/\mathcal{M})$$

is a proper ideal of $\mathcal{A}/\mathcal{M}$. Define

$$\mathcal{I} := \{ b \in \mathcal{A} : \pi(b) \in \pi(\mathcal{A}a) \} = \pi^{-1}(\pi(\mathcal{A}a)) .$$

Then $\mathcal{I}$ is a proper ideal of $\mathcal{A}$ and $\mathcal{M} \subseteq \mathcal{I}$. But $\mathcal{M}$ is maximal, so $\mathcal{M} = \mathcal{I}$. Hence,

$$\pi(a\mathcal{A}) \subseteq \pi(\mathcal{I}) = \pi(\mathcal{M}) = (0)$$

so that $\pi(a) = (0)$. I.e., $\mathcal{A}/\mathcal{M}$ is a field. Thus Theorem 6.26 implies that

$$\mathcal{A}/\mathcal{M} = \{ z + \mathcal{M} \mid z \in \mathbb{C} \} \cong \mathbb{C} .$$

Let

$$\begin{aligned} \tilde{\varphi} : \mathcal{A}/\mathcal{M} &\to \mathbb{C} \\ z + \mathcal{M} &\mapsto z . \end{aligned}$$

Then $\varphi : \mathcal{A} \to \mathbb{C}$ is defined via $\varphi := \tilde{\varphi} \circ \pi$. Hence we find that $\varphi$ is a homomorphism and $\ker(\varphi) = \mathcal{M}$.

Conversely, if $\varphi : \mathcal{A} \to \mathbb{C}$ is a non-zero homomorphism, then $\ker(\varphi)$ is a proper ideal and $\mathcal{A}/\ker(\varphi) \cong \mathbb{C}$ again by Theorem 6.26. So $\ker(\varphi)$ is maximal.

Lastly, if $\varphi, \tilde{\varphi}$ are two non-zero homomorphisms and $\ker(\varphi) = \ker(\tilde{\varphi})$, then there is some $z \in \mathbb{C}$ such that $\varphi = \alpha\tilde{\varphi}$. But

$$1 = \varphi(\mathbb{1}) = \alpha\tilde{\varphi}(\mathbb{1}) = \alpha$$

so $\varphi = \tilde{\varphi}$. □

**Definition 8.26** (maximal ideal space for $\mathcal{A}$)**.** Let $\mathcal{A}$ be an Abelian Banach algebra and define

$$\Sigma := \{ \varphi : \mathcal{A} \to \mathbb{C} \mid \varphi \neq 0 \land \varphi \text{ is an algebra homomorphism} \}$$

with the weak-star topology as in Definition 5.14 (as a subset of $\mathcal{A}^*$). $\Sigma$ is called *the maximal ideal space of* $\mathcal{A}$.

**Theorem 8.27.** *If $\mathcal{A}$ is an Abelian Banach algebra then its maximal ideal space $\Sigma$ is a compact Hausdorff space and*

$$\sigma(a) = \{ h(a) \mid h \in \Sigma \} \qquad (a \in \mathcal{A}) .$$

*Proof.* For the first part the proof would follow very closely along the lines of Theorem 5.18 and so we omit it [see [Con19] pp. 219 for details].

Let now $\varphi \in \Sigma$ and $\lambda = \varphi(a)$. Then by linearity, $\varphi(a - \lambda\mathbb{1}) = 0$, i.e., $a - \lambda\mathbb{1} \in \ker(\varphi)$ so that by the homomorphism property $a - \lambda\mathbb{1}$ cannot be invertible, i.e., $\lambda \in \sigma(a)$. Conversely, let $\lambda \in \sigma(a)$. Hence $a - \lambda\mathbb{1}$ is not invertible, and so, $(a - \lambda)\mathcal{A}$ is a proper ideal. Let $\mathcal{M}$ be the maximal ideal in $\mathcal{A}$ containing $(a - \lambda\mathbb{1})\mathcal{A}$, according to Claim 8.22. Let $\varphi \in \Sigma$ such that $\mathcal{M} = \ker(\varphi)$, according to Proposition 8.25. Then

$$0 \quad = \quad \varphi(a - \lambda\mathbb{1}) = \varphi(a) - \lambda$$

so that $\sigma(a) \subseteq \{ h(a) \mid h \in \Sigma \}$. □

**Definition 8.28** (Gelfand transform)**.** Let $\mathcal{A}$ be an Abelian Banach algebra with maximal ideal space $\Sigma$. For any $a \in \mathcal{A}$, define the Gelfand transform of $a$ as the function $\hat{a} : \Sigma \to \mathbb{C}$ given by

$$\hat{a}(\varphi) := \varphi(a) \qquad (\varphi \in \Sigma) .$$

**Theorem 8.29.** *Let $\mathcal{A}$ be an Abelian Banach algebra with maximal ideal space $\Sigma$. Then the Gelfand transform maps $\mathcal{A}$ into $C(\Sigma)$ continuously as a homomorphism with norm 1, and kernel*

$$\bigcap \{ \mathcal{M} \subseteq \mathcal{A} \mid \mathcal{M} \text{ is a maximal ideal of} \mathcal{A} \} .$$

*Proof.* See [Con19] Theorem 8.9. ☐

### 8.2.2 Representations of C-star algebras

**Definition 8.30** (Representation)**.** Given a C-star algebra $\mathcal{A}$, a representation of it is a pair $(\pi, \mathcal{H})$ where $\mathcal{H}$ is a Hilbert space and

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

is a star-homomorphism, with $\pi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}(\mathcal{H})}$.

**Definition 8.31.** A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C-star algebra is *cyclic* iff there exists a vector $\psi$ in $\mathcal{H}$ such that

$$\overline{\pi(\mathcal{A})\psi} = \mathcal{H}.$$

Two representations are said to be equivalent iff there exists a unitary

$$U : \mathcal{H}_1 \to \mathcal{H}_2$$

with

$$U\pi_1(a)U^{-1} = \pi_2(a) \qquad (a \in \mathcal{A}).$$

**Theorem 8.32.** *If $\pi$ is a representation of a C-star algebra $\mathcal{A}$ then there is a family of* cyclic *representations $\{\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)\}_i$ such that $\pi$ and $\bigoplus_i \pi_i$ are equivalent.*

*Proof.* See [Con19] Theorem VIII.5.9. ☐

### 8.2.3 States on C-star algebras

**Definition 8.33.** A *state* $\rho$ on a C-star algebra $\mathcal{A}$ is a linear functional $\rho : \mathcal{A} \to \mathbb{C}$ which is positive (i.e. $\rho(a) \geq 0$ whenever $a \geq 0$) and has $\|\rho\|_{\text{op}} = 1$.

### 8.2.4 The Gelfand-Naimark-Segal construction

**Theorem 8.34.** *Let $\mathcal{A}$ be a C-star algebra. Then:*

1. *If $\rho : \mathcal{A} \to \mathbb{C}$ is a positive linear function on $\mathcal{A}$, then there exists a cyclic representation $(\pi_\rho, \mathcal{H}_\rho)$ of $\mathcal{A}$ with cyclic vector $\psi$ such that*
$$\rho(a) = \langle \psi, \pi_\psi(a)\psi \rangle \qquad (a \in \mathcal{A}).$$
2. *If $(\pi, \mathcal{H})$ is a cyclic representation of $\mathcal{A}$ with cyclic vector $\psi$ and*
$$\rho(a) := \langle \psi, \pi(a)\psi \rangle \qquad (a \in \mathcal{A})$$
*and if $(\pi_\rho, \mathcal{H}_\rho)$ is constructed as in the first item, then $\pi$ and $\pi_\rho$ are unitarily equivalent.*

**Theorem 8.35.** *If $\mathcal{A}$ is a C-star algebra then there exists a representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ such that $\pi$ is an isometry. If $\mathcal{A}$ is separable, then $\mathcal{H}$ may be chosen as separable.*

## 8.3 The continuous functional calculus [extra]

**Theorem 8.36.** *Let $\mathscr{A}$ be an Abelian C-star algebra and $\Sigma$ its maximal ideal space. Then the Gelfand transform $\gamma : \mathscr{A} \to C(\Sigma)$ is an isometric star-isomoprhims of $\mathscr{A}$ onto $C(\Sigma)$.*

**Proposition 8.37.** *Let $\mathscr{A}$ be an Abelian C-star algebra with maximal ideal space $\Sigma$ and $a \in \mathscr{A}$ such that $\mathscr{A} = C^*(a)$, i.e., $\mathscr{A}$ is the C-star algebra generated by $a$. Then the map*

$$\tau : \Sigma \quad \to \quad \sigma(a)$$

*defined by*

$$\tau(\varphi) := \varphi(a)$$

*is a homeomoprhism, and if $p(z, \overline{z})$ is a polynomial and $\gamma : \mathscr{A} \to C(\Sigma)$ is the Gelfand transform, then*

$$\gamma(p(a, a^*)) \quad = \quad p \circ \tau \,.$$

**Definition 8.38.** Let $\mathscr{A}$ be a C-star algebra and $a \in \mathscr{A}$ be normal (so that $C^*(a)$ is an Abelian C-star algebra). Define

$$\rho : C(\sigma(a)) \to C^*(a)$$

as follows. Any $f \in C(\sigma(a))$ is the limit of polynomials $p_n(z, \overline{z})$. We thus define

$$f \quad \mapsto \quad \lim_n p_n(a, a^*) \ .$$

We thus identify $\rho(f)$ as $f(a)$, *the functional calculus of $a$*.

**Theorem 8.39.** *If $\mathscr{A}$ is a C-star algebra and $a$ is a normal element of $\mathscr{A}$ then the functional calculus has the following properties:*

1. *$f \mapsto f(a)$ is a star-monomorphism.*
2. *$\|f(a)\| = \|f\|_\infty$.*
3. *$f \mapsto f(a)$ is an extension of the holomorphism functional calculus.*

**Theorem 8.40.** *For any normal $a \in \mathscr{A}$ in a C-star algebra, there is a unique $*$-algebra morphism*

$$C(\sigma(a) \to \mathbb{C}) \to C^*(\mathbb{1}_{\mathscr{A}}, a)$$

*which maps $z \mapsto z$ into $a$. It agrees with the polynomial functional calculus, and maps $z \mapsto \overline{z}$ into $a^*$.*

# 9 Bounded operators on Hilbert space

We begin with yet another characterization of the operator norm in Hilbert spaces:

*Claim* 9.1. We have for any $A \in \mathscr{B}(\mathscr{H})$ that

$$\|A\|_{\text{op}} = \sup(\{\, |\langle\varphi, A\psi\rangle| \mid \|\varphi\| = \|\psi\| = 1 \,\}) \ .$$

## 9.1 Normals, unitaries, projections and positive elements in Hilbert space

This section should be contrasted with the corresponding one above for C-star algebras. Here we concentrate on further information or characterizations that are enabled by the structure of the Hilbert space.

*Claim* 9.2. On a Hilbert space, $A \in \mathscr{B}(\mathscr{H})$ is normal iff

$$\|A\varphi\| = \|A^*\varphi\| \qquad (\varphi \in \mathscr{H}) \ .$$

*Proof.* We have

$$\|A\varphi\|^2 = \langle A\varphi, A\varphi\rangle = \left\langle \varphi, |A|^2 \varphi \right\rangle .$$

Now since $A$ is normal, $|A|^2 = |A^*|^2$, so this equals

$$\left\langle \varphi, |A^*|^2 \varphi \right\rangle = \langle A^*\varphi, A^*\varphi\rangle = \|A^*\varphi\|^2 .$$

These relations are equivalences thanks to Corollary 7.13. ☐

**Corollary 9.3.** *If $A \in \mathcal{B}(\mathcal{H})$ is normal then*

1. $\ker(A) = \ker(A^*)$.
2. $\operatorname{im}(A)$ *is dense in* $\mathcal{H}$ *iff* $A$ *is injective.*
3. $A$ *is invertible iff* $\exists \delta > 0$ *such that* $\|A\psi\| \geq \delta\|\psi\|$ *for all* $\psi \in \mathcal{H}$.
4. *If* $A\psi = \lambda\psi$ *for some* $\psi \in \mathcal{H}$, $\lambda \in \mathbb{C}$ *then* $A^*\psi = \overline{\lambda}\psi$.
5. *If* $\lambda, \mu$ *are two distinct eigenvalues of* $A$ *then the corresponding eigenspaces are orthogonal to each other.*

*Proof.* The first statement is an immediate corollary of the preceding claim. We also have $\operatorname{im}(A)^\perp = \ker(A^*)$ which implies the second statement. By [TODO: cite correct lemma], the third item implies $\operatorname{im}(A)$ is closed, hence the second item implies $\operatorname{im}(A) = \mathcal{H}$ so that $A$ is invertible; the converse is the open mapping theorem. For the fourth item, consider $\ker(A - \lambda\mathbb{1})$ and the first item. Finally, if $\psi, \varphi$ are two eigenvectors of $\lambda, \mu$ respectively, then

$$\overline{\lambda}\langle\psi,\varphi\rangle = \langle\lambda\psi,\varphi\rangle = \langle A\psi,\varphi\rangle = \langle\psi, A^*\varphi\rangle = \langle\psi,\overline{\mu}\varphi\rangle = \overline{\mu}\langle\psi,\varphi\rangle$$

But $\lambda \neq \mu$, so $\langle\psi,\varphi\rangle = 0$. ☐

*Claim* 9.4. If $U \in \mathcal{B}(\mathcal{H})$ then the following are equivalent:

1. $U$ is unitary.
2. $\operatorname{im}(U) = \mathcal{H}$ and $\langle U\psi, U\varphi\rangle = \langle\psi,\varphi\rangle$.
3. $\operatorname{im}(U) = \mathcal{H}$ and $\|U\psi\| = \|\psi\|$.

Each of the following properties implies the others for an idempotent $P \in \mathcal{B}(\mathcal{H})$:

1. $P = P^*$.
2. $|P|^2 = |P^*|^2$.
3. $\operatorname{im}(P) = \ker(P)^\perp$.
4. $\langle\psi, P\psi\rangle = \|P\psi\|^2$.

Moreover, self-adjoint projections $P, Q$ have $\operatorname{im}(P) \perp \operatorname{im}(Q)$ iff $PQ = 0$.

We have already seen above in the chapter about C-star algebras various characterizations of an operator $A \in \mathcal{B}(\mathcal{H})$ being positive $A \geq 0$ (which we use to mean $A = |B|^2$ for some $B \in \mathcal{B}(\mathcal{H})$).

**Lemma 9.5.** *For $A \in \mathcal{B}(\mathcal{H})$ the following two are equivalent: (1) $A \geq 0$ and (2) $A = A^*$ and*

$$\langle\psi, A\psi\rangle \geq 0 \qquad (\psi \in \mathcal{H}) .$$

*Proof.* Clearly one direction is trivial:

$$\langle \psi, A\psi \rangle = \left\langle \psi, |B|^2 \psi \right\rangle = \|B\psi\|^2 \geq 0 \,.$$

Conversely, assume $A = A^*$ and $\langle \psi, A\psi \rangle \geq 0$ for all vectors $\psi$. Let $\lambda \in \sigma(A)$. Since $A$ is self-adjoint, using the Weyl criterion [TODO: cite] we know that there is a sequence $\{ \varphi_n \}_n$ such that $\|\varphi_n\| = 1$ and $\|(A - \lambda \mathbb{1}) \varphi_n\| \to 0$. Then for any $\varepsilon > 0$ there is some $N_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N_\varepsilon}$ we have

$$\begin{aligned} \langle \varphi_n, (A - \lambda \mathbb{1}) \varphi_n \rangle &\leq |\langle \varphi_n, (A - \lambda \mathbb{1}) \varphi_n \rangle| \\ &\quad \text{(Cauchy-Schwarz)} \\ &\leq \|\varphi_n\| \|(A - \lambda \mathbb{1}) \varphi_n\| \\ &\quad \text{(Assumption on } \{\varphi_n\}_n) \\ &< \varepsilon \end{aligned}$$

Hence for such $n \in \mathbb{N}_{\geq N_\varepsilon}$ we find that

$$\begin{aligned} \lambda &= \lambda \|\varphi_n\|^2 \\ &= \langle \varphi_n, \lambda \varphi_n \rangle \\ &> \langle \varphi_n, A\varphi_n \rangle - \varepsilon \end{aligned}$$

Note that the first term on the RHS is always $\geq 0$ by our hypothesis. Now as we send $\varepsilon \to 0$ we learn that $\lambda \geq 0$, which is what we wanted to prove. We then conclude by Theorem 8.15. □

## 9.2 The weak and strong operator topologies on $\mathcal{B}(\mathcal{H})$

In addition to the operator norm topology on $\mathcal{B}(\mathcal{H})$ we have

**Definition 9.6** (Strong operator topology)**.** $\text{Open}_{\text{strong}}(\mathcal{B}(\mathcal{H}))$ is the initial topology w.r.t. all the maps

$$\begin{aligned} E_\psi : \mathcal{B}(\mathcal{H}) &\to \mathcal{H} \\ A &\mapsto A\psi \end{aligned}$$

as $\psi$ ranges in $\mathcal{H}$. A sequence $\{ A_n \}_n$ converges to some $A$ iff

$$A_n \psi \to A\psi$$

in $\mathcal{H}$ for any $\psi \in \mathcal{H}$ (i.e., it is *not* uniform in $\psi$).

**Definition 9.7** (Weak operator topology)**.** $\text{Open}_{\text{strong}}(\mathcal{B}(\mathcal{H}))$ is the initial topology w.r.t. all the maps

$$\begin{aligned} E_{\psi,\varphi} : \mathcal{B}(\mathcal{H}) &\to \mathbb{C} \\ A &\mapsto \langle \psi, A\varphi \rangle \end{aligned}$$

as $\psi, \varphi$ range in $\mathcal{H}$. A sequence $\{ A_n \}_n$ converges to some $A$ iff

$$\langle \psi, A_n \varphi \rangle \to \langle \psi, A\varphi \rangle$$

for all $\psi, \varphi \in \mathcal{H}$ iff $A_n \varphi \to A\varphi$ in the weak topology on $\mathcal{H}$ (considered as a Banach space, as defined in in Definition 5.7).

*Remark* 9.8. Note that not *all* bounded linear functionals on $\mathcal{B}(\mathcal{H})$ are of the form

$$\langle \psi, \cdot \varphi \rangle$$

for some $\psi, \varphi$ and hence the weak topology on $\mathcal{B}(\mathcal{H})$ (considered as a Banach space as in Definition 5.7) is not as weak as the weak operator topology. Since $\langle \psi, \cdot \varphi \rangle \in (\mathcal{B}(\mathcal{H}))^*$ it is clear that weak convergence implies weak operator convergence, but not vice versa. Using the identification

$$\mathcal{B}(\mathcal{H}) \quad \cong \quad \mathcal{H} \otimes \mathcal{H}^*$$

we may identify

$$(\mathcal{B}(\mathcal{H}))^* \cong \mathcal{H}^* \otimes \mathcal{H}$$

and hence $\langle \psi, \cdot \varphi \rangle$ as $\psi^* \otimes \varphi$. In this regard the distinction between the weak operator topology on $\mathcal{B}(\mathcal{H})$ and the weak topology on $\mathcal{B}(\mathcal{H})$ (when considered as a Banach space, as in Definition 5.7) is that the weak operator topology only considers the simple tensors within $\mathcal{H}^* \otimes \mathcal{H}$.

*Claim* 9.9. Norm convergence implies strong convergence implies weak convergence. All of these not vice versa.

*Proof.* If we have $A_n \to A$ in norm, then

$$\|(A_n - A)\psi\| \leq \|A_n - A\| \|\psi\|$$

so that we have strong convergence. Similarly, if we have $A_n \to A$ strongly then

$$|\langle \varphi, (A_n - A)\psi \rangle| \leq \|\varphi\| \|(A_n - A)\psi\|$$

so that $A_n \to A$ weakly. □

**Example 9.10** (Strong does not imply norm)**.** On $\mathcal{H} = \ell^2(\mathbb{N})$, let $P_j := e_j \otimes e_j^*$ where $\{ e_j \}_{j=1}^{\infty}$ is the standard orthonormal basis (in physics, the "position" basis). Then $\|P_j\| = 1$ for any $j$, and yet,

$$\|P_j \psi\| \equiv |\psi_j| \to 0$$

since $\psi \in \ell^2$. So $\{ P_j \}_j$ converges strongly to zero but does not converge in norm.

**Example 9.11** (weak does not imply strong)**.** Again on $\mathcal{H} := \ell^2(\mathbb{N})$, let $R$ be the unilateral right shift operator:

$$R(\psi_1, \psi_2, \dots) \equiv (0, \psi_1, \psi_2, \dots) \qquad \left(\psi \in \ell^2\right).$$

Then $A_n := R^n$ converges *weakly* to zero:

$$\begin{aligned} |\langle \varphi, A_n \psi \rangle| &\equiv |\langle \varphi, R^n \psi \rangle| \\ &= \left| \sum_{j=n+1}^{\infty} \overline{\varphi_j} \psi_{j-n} \right| \\ &\leq \sqrt{\sum_{j=n+1}^{\infty} \|\varphi_j\|^2} \|\psi\| \qquad \text{(Cauchy-Schwarz)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

On the other hand, this sequence cannot converge strongly to zero since

$$\|A_n \psi\|^2 = \sum_{j=1}^{\infty} \left|(A_n \psi)_j\right|^2 = \sum_{j=n+1}^{\infty} |\psi_{j-n}|^2 = \|\psi\|^2.$$

**Example 9.12** (An example of something that actually converges in norm)**.** On any Hilbert space, consider $A_n := \frac{1}{n}\mathbb{1}$ which converges in norm to 0 as $n \to \infty$.

**Lemma 9.13.** *When the Hilbert space* $\mathcal{H}$ *has an orthonormal basis* $\{ e_n \}_{n=1}^{\infty}$ *then for each* $A \in \mathcal{B}(\mathcal{H})$*, the sequence*

$$\sum_{n,m=1}^{N} \langle e_n, Ae_m \rangle \, e_n \otimes e_m^*$$

*converges* strongly *as* $N \to \infty$*.*

*Proof.* Let $\psi \in \mathcal{H}$ be given. Then

$$\begin{aligned}
\left\| \left( A - \sum_{n,m=1}^{N} \langle e_n, Ae_m \rangle \, e_n \otimes e_m^* \right) \psi \right\|^2 &= \left\| A\psi - \sum_{n,m=1}^{N} \langle e_m, \psi \rangle \langle e_n, Ae_m \rangle \, e_n \right\|^2 \\
&= \sum_{l=1}^{\infty} \left| \left\langle e_l, \left[ A\psi - \sum_{n,m=1}^{N} \langle e_m, \psi \rangle \langle e_n, Ae_m \rangle \, e_n \right] \right\rangle \right|^2 \\
&= \sum_{l=1}^{\infty} \left| \langle e_l, A\psi \rangle - \sum_{m=1}^{N} \langle e_m, \psi \rangle \langle e_l, Ae_m \rangle \, \chi_{\{ 1,\dots,N \}}(l) \right|^2 \\
&= \sum_{l=1}^{\infty} \left| \sum_{m=1}^{\infty} \langle e_l, Ae_m \rangle \langle e_m, \psi \rangle - \sum_{m=1}^{N} \langle e_m, \psi \rangle \langle e_l, Ae_m \rangle \, \chi_{\{ 1,\dots,N \}}(l) \right|^2 \\
&= \sum_{l=1}^{N} \left| \sum_{m=1}^{\infty} \langle e_l, Ae_m \rangle \langle e_m, \psi \rangle - \sum_{m=1}^{N} \langle e_m, \psi \rangle \langle e_l, Ae_m \rangle \right|^2 + \\
&\quad + \sum_{l=N+1}^{\infty} \left| \sum_{m=1}^{\infty} \langle e_l, Ae_m \rangle \langle e_m, \psi \rangle \right|^2 \\
&= \sum_{l=1}^{N} \left| \sum_{m=N+1}^{\infty} \langle e_l, Ae_m \rangle \langle e_m, \psi \rangle \right|^2 + \sum_{l=N+1}^{\infty} \left| \sum_{m=1}^{\infty} \langle e_l, Ae_m \rangle \langle e_m, \psi \rangle \right|^2 \\
&= \sum_{l=1}^{N} \left| \sum_{m=N+1}^{\infty} \langle e_l, Ae_m \rangle \langle e_m, \psi \rangle \right|^2 + \sum_{l=N+1}^{\infty} \left| \langle e_l, A\psi \rangle \right|^2 .
\end{aligned}$$

Now, the second term converges to zero as $N \to \infty$ since

$$\sum_{l=1}^{\infty} \left| \langle e_l, A\psi \rangle \right|^2 \equiv \|A\psi\|^2 \le \|A\| \|\psi\| < \infty \, .$$

The first term converges to zero as $N \to \infty$ since

$$\sum_{l=1}^{\infty} \left| \sum_{m=1}^{\infty} \langle e_l, Ae_m \rangle \langle e_m, \psi \rangle \right|^2 = \sum_{l=1}^{\infty} \left| \langle e_l, A\psi \rangle \right|^2 = \|A\psi\| \le \|A\| \|\psi\| < \infty \, .$$

□

## 9.3 The spectrum

We have already discussed the spectrum of elements of a Banach algebra, and $\mathcal{B}(\mathcal{H})$ is an example of a Banach algebra. Here we want to refine the discussion with the new structure coming from the Hilbert space.

**Definition 9.14** (Point spectrum)**.** The point spectrum of $A \in \mathcal{B}(\mathcal{H})$ is defined as

$$\sigma_p(A) \equiv \{ \lambda \in \mathbb{C} \mid \ker(A - \lambda \mathbb{1}) \neq \{0\} \} .$$

In particular, this is the set of numbers for which $A - \lambda \mathbb{1}$ is not *injective*, i.e., for which there exists some $\psi \in \mathcal{H} \setminus \{0\}$ for which

$$A\psi = \lambda \psi .$$

In this context, $\psi$ is called an eigenvector of $A$ and $\lambda$ the corresponding eigenvalue.

Contrast this with the general definition of a point $\lambda \in \sigma(A)$ which required that $A - \lambda \mathbb{1}$ is not invertible. Hence we are asking for less now so that

$$\sigma_p(A) \subseteq \sigma(A)$$

clearly. Also, the points $\lambda \in \mathbb{C}$ at which $A - \lambda \mathbb{1}$ is not invertible but *is* injective are those for which $A - \lambda \mathbb{1}$ is not surjective: $\operatorname{im}(A - \lambda \mathbb{1}) \neq \mathcal{H}$. We further decompose this set as follows:

**Definition 9.15** (Continuous spectrum)**.** The continuous spectrum of $A \in \mathcal{B}(\mathcal{H})$ is given by

$$\sigma_c(A) \equiv \left\{ \lambda \in \sigma_p(A)^c \,\middle|\, \operatorname{im}(A - \lambda \mathbb{1}) \neq \mathcal{H} \wedge \overline{\operatorname{im}(A - \lambda \mathbb{1})} = \mathcal{H} \right\} .$$

**Definition 9.16** (Residual spectrum)**.** The residual spectrum of $A \in \mathcal{B}(\mathcal{H})$ is given by

$$\sigma_r(A) \equiv \sigma(A) \setminus \left( \sigma_p(A) \bigcup \sigma_c(A) \right) .$$

The residual spectrum may be characterized as those points $\lambda \in \mathbb{C}$ for which $A - \lambda \mathbb{1}$ *is* injective, but for which $\operatorname{im}(A - \lambda \mathbb{1})$ is a proper but not dense subspace of $\mathcal{H}$.

*Remark* 9.17. Clearly if $\mathcal{H} \cong \mathbb{C}^n$ then $\sigma(A) = \sigma_p(A)$ for any $A \in \mathcal{B}(\mathcal{H})$.

**Example 9.18.** Let $A$ on $\ell^2(\mathbb{N})$ be given by

$$(A\psi)_n = \frac{1}{n}\psi_n \qquad \left( \psi \in \ell^2; n \geq 1 \right) .$$

Then

$$\sigma_p(A) = \left\{ \frac{1}{n} \,\middle|\, n \geq 1 \right\} .$$

Indeed, the eigenvectors of $A$ are the elements of the orthonormal basis $\{ e_n \}_{n \in \mathbb{N}}$ with eigenvalues $\frac{1}{n}$. However, since the spectrum is closed we must also have $0 \in \sigma(A)$. But zero is not an eigenvalue: indeed, $A$ had a zero eigenvector $\psi$ then

$$\frac{1}{n}\psi_n = 0 \qquad (n \geq 1)$$

which implies $\psi = 0$. Let us see that $A$ is self-adjoint: For any $\varphi, \psi \in \mathcal{H}$,

$$\langle \varphi, A^*\psi \rangle \equiv \langle A\varphi, \psi \rangle \equiv \sum_{n \in \mathbb{N}} \overline{(A\varphi)_n}\psi_n = \sum_{n \in \mathbb{N}} \overline{\frac{1}{n}\varphi_n}\psi_n = \sum_{n \in \mathbb{N}} \frac{1}{n}\overline{\varphi_n}\psi_n = \sum_{n \in \mathbb{N}} \overline{\varphi_n}(A\psi)_n \equiv \langle \varphi, A\psi \rangle .$$

Later we will see that self-adjoint operators have no residual spectrum. Hence $0 \in \sigma_c(A)$.

**Example 9.19.** Let $X$ on $L^2([0,1] \to \mathbb{C})$ be given by

$$(X\psi)(x) = x\psi(x) .$$

One has to verify this operator is well-defined and bounded. Then

$$\sigma(X) = \sigma_c(X) .$$

One should think of the eigenvectors as the Dirac delta functions at each point in $[0,1]$. These are clearly not $L^2$ functions but rather distributions and hence they cannot contribute to the point spectrum of $X$, which is empty.

*Claim* 9.20. $\overline{\lambda} \in \sigma_r(A^*)$ implies that $\lambda \in \sigma_p(A)$. Conversely, $\lambda \in \sigma_p(A)$ implies $\overline{\lambda} \in \sigma_r(A^*) \cup \sigma_p(A^*)$.

*Proof.* First assume that $\overline{\lambda} \in \sigma_r(A^*)$. Then $\overline{\operatorname{im}(A^* - \overline{\lambda}\mathbb{1})}$ is a proper closed subspace of $\mathcal{H}$, so its complement is not zero. But by Claim 7.8 and Theorem 7.18 we have

$$\left(\overline{\operatorname{im}(A^* - \overline{\lambda}\mathbb{1})}\right)^{\perp} = \left(\operatorname{im}(A^* - \overline{\lambda}\mathbb{1})\right)^{\perp} = \ker(A - \lambda\mathbb{1})$$

which is, as we just said, is non-empty, so $\lambda \in \sigma_p(A)$.

Conversely, if $\lambda \in \sigma_p(A)$ then as we just saw,

$$\left(\overline{\operatorname{im}(A^* - \overline{\lambda}\mathbb{1})}\right)^{\perp} \neq \{0\}.$$

This could either mean that $\overline{\lambda} \in \sigma_r(A^*)$, unless, it somehow happened that *also* $\ker(A^* - \overline{\lambda}\mathbb{1}) \neq \{0\}$, in which case, $\overline{\lambda} \in \sigma_p(A^*)$. □

**Theorem 9.21.** *If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint then $\sigma_r(A) = \varnothing$ and eigenvectors corresponding to distinct eigenvalues of $A$ are orthogonal.*

*Proof.* Thanks to Claim 8.9 and Claim 9.20 we have that if $\lambda \in \sigma_r(A)$, then $\lambda \in \sigma_p(A)$. But by definition,

$$\sigma_r(A) \cap \sigma_p(A) = \varnothing!$$

Next, assume that $A\psi = \lambda\psi$ and $A\varphi = \mu\varphi$. We want to show that if $\lambda \neq \mu$ then $\psi \perp \varphi$. To that end, assume that $\lambda \neq 0$ (WLOG; it can't be that both are zero!). Then

$$\langle \psi, \varphi \rangle = \frac{1}{\lambda}\overline{\lambda}\langle \psi, \varphi \rangle = \frac{1}{\lambda}\langle \lambda\psi, \varphi \rangle = \frac{1}{\lambda}\langle A\psi, \varphi \rangle = \frac{1}{\lambda}\langle \psi, A\varphi \rangle = \frac{\mu}{\lambda}\langle \psi, \varphi \rangle .$$

But via Claim 8.9 we know that $\overline{\lambda} = \lambda$ and so either $\frac{\mu}{\lambda} = 1$ or $\langle \psi, \varphi \rangle = 0$ as desired. □

### 9.3.1 The Weyl criterion for the spectrum

**Theorem 9.22** (Weyl's criterion)**.** *For any $A = A^* \in \mathcal{B}(\mathcal{H})$, $\lambda \in \sigma(A)$ iff there exists some $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\|\varphi_n\| = 1$ such that*

$$\lim_n \|(A - \lambda\mathbb{1})\varphi_n\| = 0 .$$

*The sequence $\{\varphi_n\}_n$ is called a Weyl sequence.*

*Proof.* Assume that $\lambda \notin \sigma(A)$. Then $0 \notin \sigma(A - \lambda\mathbb{1})$, or by spectral mapping $0 \notin \sigma\left((A - \lambda\mathbb{1})^2\right)$. But thanks to self-adjointness, this means $0 \notin \sigma\left(|A - \lambda\mathbb{1}|^2\right)$. Since $|A - \lambda\mathbb{1}|^2 \geq 0$, it being invertible is tantamount to $|A - \lambda\mathbb{1}|^2 \geq \varepsilon\mathbb{1}$ for some $\varepsilon > 0$, which means that

$$\|(A - \lambda\mathbb{1})\varphi_n\|^2 = \left\langle \varphi_n, |A - \lambda\mathbb{1}|^2 \varphi_n \right\rangle \geq \varepsilon\|\varphi_n\|^2 = \varepsilon .$$

Hence no normalized sequence $\{\varphi_n\}_n$ may converge to zero.

Conversely, if for all $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\|\varphi_n\| = 1$,

$$\lim_n \|(A - \lambda\mathbb{1})\varphi_n\| \neq 0$$

then there exists some $\varepsilon > 0$ such that

$$\|(A-\lambda\mathbb{1})\,\psi\|^2 \geq \varepsilon\|\psi\|^2 \qquad (\psi\in\mathcal{H})\ .$$

This implies that $A-\lambda\mathbb{1}$ is injective. As we have seen, this condition also implies the closed range of $A-\lambda\mathbb{1}$ Lemma 7.21. Since $A$ has no residual spectrum Theorem 9.21, this means that $A-\lambda\mathbb{1}$ is surjective, i.e., it can't be that $\lambda\in\sigma(A)$. □

## 9.4 The polar decomposition

One should have basic knowledge of the polar decomposition for $n\times n$ complex matrices: For any $A\in\mathrm{Mat}_{n\times n}(\mathbb{C})$ there is a partial isometry $U$ such that

$$A = U\,|A|$$

where $|A|\equiv\sqrt{A^*A}\geq 0$. For matrices $\sqrt{A^*A}$ is understood as follows: $A^*A$ is normal, so we may diagonalize to define its square root. The proof of the existence of the polar decomposition for matrices follows via the singular-value decomposition: there exist to partial isometries $W,V$ such that

$$A = W\Sigma V^* = \underbrace{(WV^*)}_{=:U}\underbrace{(V\Sigma V^*)}_{=:|A|}\ .$$

It turns out that this is also true for operators.

*Remark* 9.23. Note that for matrices $U$ may always be extended from a partial isometry to a unitary, since whatever kernel $A$ has may be captured by $|A|$. Not so for operators on infinite dimensional Hilbert space. If $U$ is a partial isometry, it is isometric on $\ker(U)^\perp$, so its image is closed, and we may write

$$U:\ker(U)\oplus\ker(U)^\perp\to\mathrm{im}(U)^\perp\oplus\mathrm{im}(U)\ .$$

Then it is clear that $U$ may be extended to a unitary iff $\dim\ker(U)=\dim\mathrm{im}(U)^\perp$. For those who know: this happens when $U$ is Fredholm of index zero, or, if it's not Fredholm as both of these spaces are infinite dimensional (hence they are isomorphic). We shall see this more systematically later below.

We have already encountered the notion of partial isometries in Definition 8.1 in the context of C-star algebras. Here we give another characterization in Hilbert space:

**Lemma 9.24.** *$U\in\mathcal{B}(\mathcal{H})$ is a partial isometry iff $U$ is an isometry on $\ker(U)^\perp$, i.e., $\|U\psi\|=\|\psi\|$ for all $\psi\in\ker(U)^\perp$.*

*Proof.* Let $U\in\mathcal{B}(\mathcal{H})$ be a partial isometry. Then $|U|^2$ is a self-adjoint projection and so is $|U^*|^2$. We have $\ker(U)=\ker\left(|U|^2\right)$, so that

$$\psi\in\ker(U)^\perp=\ker\left(|U|^2\right)^\perp=\mathrm{im}\left(|U|^2\right)$$

implies $\psi$ is in the range of the projection $|U|^2$. But for any projection $P$, if $\psi\in\mathrm{im}(P)$, $P\psi=\psi$. Hence on $\mathrm{im}\left(|U|^2\right)$, $|U|^2$ acts as the identity: $|U|^2\,\psi=\psi$. Thus,

$$\|U\psi\|^2=\left\langle\psi,|U|^2\,\psi\right\rangle=\|\psi\|^2$$

as desired. Conversely, if $U$ is an isometry on $\ker(U)^\perp$, then by definition $\psi\in\ker(U)^\perp$we have $|U|^2\,\psi=\psi$. Using the decomposition

$$\mathcal{H} \quad = \quad \ker(U)\oplus\ker(U)^\perp$$

we then have for general $\psi = \psi_1 + \psi_2$,

$$\begin{aligned}\left(|U|^4 - |U|^2\right)\psi &= \left(|U|^4 - |U|^2\right)(\psi_1 + \psi_2) \\ &= \left(|U|^4 - |U|^2\right)\psi_2 \\ &= \psi_2 - \psi_2 \\ &= 0\end{aligned}$$

so $|U|^2$ is indeed an idempotent. □

Hence, for a partial isometry $U$, since the range of an isometry is always closed, we may write

$$U : \ker(U) \oplus \ker(U)^\perp \to \operatorname{im}(U) \oplus \operatorname{im}(U)^\perp$$

and consider

$$\begin{aligned}\tilde{U} : \ker(U)^\perp &\to \operatorname{im}(U) \\ \psi &\mapsto U\psi\end{aligned}$$

as an isometric isomorphism. $\ker(U)^\perp$ is called the *initial subspace* and $\operatorname{im}(U)$ the *final subspace*. Similarly,

$$\tilde{U}^* : \operatorname{im}(U) \to \ker(U)^\perp$$

is also an isometric isomorphism which is the inverse of $\tilde{U}$. We have that $|U|^2$ as a self-adjoint projection onto the initial and $|U^*|^2$ a self-adjoint projection onto the final subspace.

**Theorem 9.25.** *Let $A \in \mathcal{B}(\mathcal{H})$. Then there is a partial isometry $U \in \mathcal{B}(\mathcal{H})$ such that*

$$A = U|A|$$

*where $|A| = \sqrt{A^*A}$. $U$ is uniquely determined by constraining $\ker(U) = \ker(A)$. Moreover, we have $\operatorname{im}(U) = \overline{\operatorname{im}(A)}$. In particular, we have uniqueness in the sense that if*

$$A = WP$$

*for some $W, P \in \mathcal{B}(\mathcal{H})$ with $W$ a partial isometry with $\ker(W) = \ker(P)$ and $P \geq 0$ then $P = |A|$ and $W = U$.*

Note that if $A$ is invertible then $|A|$ is also invertible and positive (using the *holomorphic* functional calculus). Then we may define

$$U := A|A|^{-1}$$

and verify it is unitary. Indeed, $U$ is easily an isometry:

$$\begin{aligned}|U|^2 &\equiv \left(A|A|^{-1}\right)^* A|A|^{-1} \\ &= |A|^{-1}|A|^2|A|^{-1} \\ &= \mathbb{1}\,.\end{aligned}$$

Whence we learn it is a partial isometry. But an invertible partial isometry is a unitary.

**Example 9.26.** The right shift $R$ on $\ell^2(\mathbb{N})$ has

$$|R|^2 \equiv R^*R = \mathbb{1}$$

but

$$|R^*|^2 = RR^* = \mathbb{1} - e_1 \otimes e_1^*$$

so that $R$ is not unitary, but is an isometry and a partial isometry. Hence $|R| = \mathbb{1}$ and so the polar decomposition is

$$R = U\mathbb{1}$$

with $U := R$, a *partial isometry*.

*Proof of Theorem 9.25.* For the general case (not assuming $A$ is invertible), define

$$U : \mathrm{im}\,(|A|) \to \mathrm{im}\,(A)$$

by

$$U\,|A|\,\psi := A\psi\,.$$

$U$ is well-defined: if $|A|\,\psi = |A|\,\varphi$ then

$$\|A\psi - A\varphi\| = \|A(\psi-\varphi)\| \overset{\star}{=} \||A|\,(\psi-\varphi)\| = 0$$

where in $\star$ we use

$$\|A\psi\|^2 = \langle A\psi, A\psi\rangle = \left\langle \psi, |A|^2\,\psi\right\rangle = \langle |A|\,\psi, |A|\,\psi\rangle = \||A|\,\psi\|^2\,.$$

I.e.,

$$\|U\,|A|\,\psi\| \quad\equiv\quad \|A\psi\| = \||A|\,\psi\|$$

so that $U$ is an isometry. Let us extend $U$ to an isometry

$$\tilde{U} : \overline{\mathrm{im}\,(|A|)} \to \overline{\mathrm{im}\,(A)} \tag{9.1}$$

as follows. Let $\psi \in \overline{\mathrm{im}\,(|A|)}$. Then there is some $\{\,\varphi_n\,\} \subseteq \mathcal{H}$ so that $|A|\,\varphi_n \to \psi$. Define

$$\tilde{U}\psi \quad:=\quad \lim_n A\varphi_n\,.$$

This limit exists because

$$\|A\varphi_n - A\varphi_m\| \quad=\quad \|A(\varphi_n-\varphi_m)\| = \||A|\,(\varphi_n-\varphi_m)\|$$

and the latter expression is arbitrarily small as $|A|\,\varphi_n$ converges. Thus $A\varphi_n$ converges to some element in $\overline{\mathrm{im}\,(A)}$. Next, $\tilde{U}$ may further be extended to an operator $\mathcal{H} \to \mathcal{H}$ by extending to zero on $\left(\overline{\mathrm{im}\,(|A|)}\right)^\perp$. Since $|A|$ is self-adjoint, we have via Theorem 7.18 that $\overline{\mathrm{im}\,(|A|)}^\perp = \ker(|A|)$. So

$$\ker(U) = \ker(|A|) = \ker(A)\,.$$

To prove uniqueness, assume that

$$A \quad=\quad UP$$

for some partial isometry $U$ and positive $P$. Then

$$|A|^2 \quad=\quad PU^*UP = P\,|U|^2\,P$$

and we assume that $|U|^2$ is a projection onto the image of $P$ (since $\ker\left(|U|^2\right) = \ker(U) = \ker(P)$). So

$$|A|^2 \quad=\quad P^2\,.$$

By uniqueness of the positive square root this implies $P = |A|$. So

$$U\,|A| \quad=\quad W\,|A|$$

so that $U$ and $W$ agree on $\mathrm{im}\,(|A|)$ which is their initial space, so $U = W$. □

*Claim* 9.27. $U$ is the strong-limit of polynomials in $A$ and $A^*$.

*Proof.* [TODO, once we have the functional calculus] □

**Theorem 9.28** (SVD of operators)**.** *Given the polar decomposition, we may form a singular-value decomposition for operators: Given any $A \in \mathcal{B}(\mathcal{H})$, there are two partial isometries $W, V$ and a positive operator $D \geq 0$ such that*

$$A = WDV^* .$$

*The eigenvalues of $D$ are called the singular values of the operator.*

*Proof.* [TODO, but requires spectral theorem so perhaps move downwards] □

## 9.5 Compact operators

In infinite dimensional Hilbert spaces, compact operators may be considered as those operators which mostly resemble finite dimensional matrices embedded in $\mathcal{B}(\mathcal{H})$, or as limits of such.

**Definition 9.29** (finite-rank operator)**.** An operator $A \in \mathcal{B}(\mathcal{H})$ is called of *finite rank* iff $\operatorname{im}(A)$ is a finite-dimensional subspace of $\mathcal{H}$.

*Claim* 9.30. $A \in \mathcal{B}(\mathcal{H})$ is of finite rank iff

$$A = \sum_{n=1}^{N} \alpha_n \varphi_n \otimes \psi_n^*$$

where $N \in \mathbb{N}$ (the rank of the operator), $\{ \alpha_n \}_{n=1}^{N} \subseteq [0, \infty)$ are the singular values of $A$, and $\{ \varphi_n \}_{n\in\mathbb{N}}$, $\{ \psi_n \}_{n\in\mathbb{N}}$ are two orthonormal bases of $\mathcal{H}$.

*Proof.* Since $\operatorname{im}(A)$ is finite-dimensional of dimension $N$, it is closed. So we may write

$$\mathcal{H} = \operatorname{im}(A) \oplus (\operatorname{im}(A))^{\perp} .$$

Furthermore, we always have

$$\mathcal{H} = \ker(A)^{\perp} \oplus \ker(A) .$$

In this decomposition,

$$A : \ker(A)^{\perp} \oplus \ker(A) \to \operatorname{im}(A) \oplus (\operatorname{im}(A))^{\perp}$$

and

$$A = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix}$$

with $\tilde{A} : \ker(A)^{\perp} \to \operatorname{im}(A)$ being an isomorphism, which means that $\ker(A)^{\perp}$ is finite dimensional of dimension $N$ as well and $\tilde{A}$ is really an invertible matrix. Then let us write an SVD of

$$\tilde{A} = WDV^*$$

with $D \geq 0$ diagonal and $V, W : \operatorname{im}(A) \to \ker(A)^{\perp}$ unitary. Then $\Lambda$ is the diagonal given by $\alpha_1, \dots, a_N$, which are the singular values and let $\{ \psi_n \}_{n=1}^{N}$ be the columns of $V$, $\{ \varphi_n \}_{n=1}^{N}$ be the columns of $W$. Complete these two orthonormal sets into bases of $\mathcal{H}$ to get the result. □

**Example 9.31.** The operator $u \otimes v^*$ for any two $u, v \in \mathcal{H}$ is of finite rank. Indeed, it is of rank 1. The operator $\mathbb{1}$ is *not* of finite rank, and neither is, e.g., $\exp(-X^2)$ on $\ell^2(\mathbb{N})$.

**Definition 9.32** (Compact operator)**.** An operator $A \in \mathcal{B}(\mathcal{H})$ is *compact* iff it is the *norm* limit of finite-rank operators. I.e., if it may be written as

$$A = \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n \varphi_n \otimes \psi_n^* \tag{9.2}$$

with $\{ \alpha_n \}_{n=1}^{N} \subseteq [0,\infty)$ and $\{ \varphi_n \}_{n\in\mathbb{N}}$, $\{ \psi_n \}_{n\in\mathbb{N}}$ are two orthonormal bases of $\mathcal{H}$.

*Remark* 9.33. Clearly $A$ is compact if it is finite rank.

We also want a characterization of compact operators which holds valid in Banach spaces:

**Lemma 9.34** (Another characterization of compact operators)**.** *The following are equivalent for a given $A \in \mathcal{B}(\mathcal{H})$:*

1. $A$ is compact.
2. For any bounded sequence $\{ \psi_n \}_n \subseteq \mathcal{H}$, $\{ A\psi_n \}_n$ has a subsequence which converges.
3. For any bounded $B \subseteq \mathcal{H}$, $\overline{A(B)}$ is a *compact set* in $\mathcal{H}$.

*Proof.* [TODO: fix this] Assume that $A \in \mathcal{B}(X \to Y)$ is compact and let $B \subseteq X$ be a bounded set. We want to show $\overline{A(B)}$ is compact. We use the sequential criterion of compactness, so let $\{ \varphi_n \} \subseteq \overline{A(B)}$. Hence each $\varphi_n$ has some $\psi_n \in B$ such that

$$\|A\psi_n - \varphi_n\| \leq 2^{-n}\,.$$

Since $B$ is bounded, $\{ \psi_n \}_n$ is a bounded sequence, and so $\{ A\psi_n \}_n$ has a subsequence which converges. But then the above inequality implies $\varphi_n$ must converge to the same limit as that subsequence.

Conversely, let $\{ \psi_n \}_n \subseteq X$ be a bounded sequence. Then its range is a bounded set so that the closure of its image is compact, admitting a convergent subsequence.

Finally, assume $A$ is compact, i.e., $A = \lim_n A_n$ in norm and $A_n$ are finite rank. Clearly finite rank operators obey the property that the closure of the image of a bounded set is compact: Let $A$ be finite rank and $B$ be bounded. Then if $\psi \in B$,

$$\sum_{j=1}^{N} \alpha_n \varphi_n \otimes \psi_n^* \psi = \sum_{j=1}^{N} \alpha_n \langle \psi_n, \psi \rangle \varphi_n$$

and hence

$$\|A\psi\| \leq \sum_{j=1}^{N} \alpha_n |\langle \psi_n, \psi \rangle| \|\varphi_n\| \leq \left( \sup_{\psi\in B} \|\psi\| \right) \sum_{j=1}^{N} \alpha_n$$

$$\sup_{\psi\in B : \|\psi\|=1} \|A\psi\|$$

so $\overline{A(B)}$ is bounded and finite dimensional, so by the Heine-Borel property for finite-dimensional Euclidean spaces, $\overline{A(B)}$ is compact indeed. Now using Lemma 9.35 below, we get condition (3).

Conversely, assume condition (3). Since $\overline{A(B)}$ is compact, it is separable. Hence, $\overline{\operatorname{im}(A)}$ is a separable subspace of $\mathcal{H}$. Let $\{ e_j \}_j$ be a basis for $\overline{\operatorname{im}(A)}$ and let $P_n$ be the projection onto $\{ e_j \}_{j=1}^{n}$. Set $A_n := P_n A$. Then each $A_n$ has finite rank and we claim that $A_n \to A$ in norm. First, strongly:

$$\begin{aligned} (A_n - A)\psi &= P_n^\perp A\psi \\ &= P_n^\perp \sum_{j=1}^{\infty} \langle e_j, A\psi \rangle e_j \\ &= \sum_{j=n+1}^{\infty} \langle e_j, A\psi \rangle e_j \end{aligned}$$

and hence

$$\|(A_n - A)\,\psi\|^2 \quad = \quad \sum_{j=n+1}^{\infty} |\langle e_j, A\psi\rangle|^2 \;.$$

So, $\|(A_n - A)\,\psi\| \to 0$ as this latter sum is finite for $n = 0$ by $\|A\psi\| < \infty$. Next, for the norm convergence, if $\varepsilon > 0$, there are vectors $\psi_1, \dots, \psi_m$ in the unit ball $B$ of $\mathcal{H}$ such that

$$A\,(B) \subseteq \bigcup_{j=1}^{m} B_{\varepsilon/3}\,(A\psi_j) \;.$$

Now if $\psi \in B$, let $j$ such that $\|A\psi - A\psi_j\| < \frac{1}{3}\varepsilon$. Hence for any $n$,

$$\begin{aligned} \|A\psi - A_n\psi\| \quad &\leq \quad \|A\psi - A\psi_j\| + \|A\psi_j - A_n\psi_j\| + \|P_n\,(A\psi_j - A\psi)\| \\ &\leq \quad 2\|A\psi - A\psi_j\| + \|A\psi_j - A_n\psi_j\| \\ &\leq \quad \frac{2}{3}\varepsilon + \|A\psi_j - A_n\psi_j\| \,. \end{aligned}$$

□

Using the strong convergence, we may find some $n_0 \in \mathbb{N}$ with $\|A\psi_j - A_n\psi_j\| < \frac{1}{3}\varepsilon$ for all $j = 1, \dots, m$ and $n \geq n_0$. Hence $\|A\psi - A_n\psi\| < \varepsilon$ uniformly in $\psi \in B$. Thus $A \to A_n$ in norm.

**Lemma 9.35.** *If $A_n$ is a sequence of operators such that for any bounded $B \subseteq \mathcal{H}$, $\overline{A_n\,(B)}$ is a* compact set *in $\mathcal{H}$, and $A_n \to A$, then also $\overline{A\,(B)}$ is a compact subset.*

*Proof.* [TODO: fix this] Let $\varepsilon > 0$ such that

$$\|A - A_n\| < \frac{1}{3}\varepsilon$$

for all $n$ sufficiently large. Since $A_n$ is compact, for any $\varepsilon > 0$ there are vectors $\psi_1, \dots, \psi_m \subseteq B$ such that

$$A_n\,(B) \subseteq \bigcup_{j=1}^{m} B_{\varepsilon/3}\,(A_n\psi_j) \;.$$

Then given $\psi \in B$, there is some $j$ so that

$$\|A_n\psi_j - A_n\psi\| \leq \frac{1}{3}\varepsilon \,.$$

Thus

$$\begin{aligned} \|A\psi_j - A\psi\| \quad &\leq \quad \|A\psi_j - A_n\psi_j\| + \|A_n\psi_j - A_n\psi\| + \|A_n\psi - A\psi\| \\ &\leq \quad 2\|A - A_n\| + \frac{1}{3}\varepsilon \\ &\leq \quad \varepsilon \,. \end{aligned}$$

Thus $A\,(B) \subseteq \bigcup_{j=1}^{m} B_{\varepsilon}\,(A\psi_j)$. □

**Corollary 9.36.** *Using the above two results in conjunction, we learn that the norm limit of a compact sequence is compact.*

**Theorem 9.37.** *The compact operators $\mathcal{K}\,(\mathcal{H})$ form a* norm-closed *two-sided-star-ideal: if $A, K \in \mathcal{B}\,(\mathcal{H})$ with $K$ compact then $AK, KA$ are both compact, and $K^*$ is compact.*

*Proof.* This is obvious from the fact $*: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is norm continuous and the canonical representation (9.2).

Next, for the two-sided ideal statement, clearly if $K = \lim_n F_n$, then $AK = A \lim_n F_n = \lim_n AF_n$. But each $AF_n$ is a finite rank operator and hence their limit is compact. ☐

**Example 9.38.** An example of a compact operator which is *not* finite rank: $\frac{1}{X}$ on $\ell^2(\mathbb{N})$.

In fact,

*Claim* 9.39. If $A$ is a *multiplication operator* w.r.t. some ONB $\{ e_n \}_n$ of $\mathcal{H}$ then $A$ is compact iff

$$\langle e_n, Ae_n \rangle \equiv A_{nn} \to 0 \qquad (n \to \infty) .$$

*Proof.* First assume that $A_{nn} \to 0$. Then as above, define

$$A^N := P_N A$$

(projection onto the first $N$ coordinates.) $A^N$ is a bounded finite rank operator, with

$$\left\| A - A^N \right\| \leq \sup_{n > N} |A_{nn}| \to 0 .$$

But finite rank operators are compact, and limits of compact are compact, we have exhibited $A$ as the limit of compact operators.

Conversely, if $A_{nn}$ does not tend to zero, then there is some subsequence for which

$$|A_{n_k}| \geq \varepsilon .$$

If $\{ e_j \}_j$ is an ONB (a bounded sequence) then $Ae_{n_k} = A_{n_k} e_{n_k}$ should have a convergent subsequence. But $e_n \to 0$ weakly, which implies $Ae_n \to 0$ weakly. Hence there is a subsequence $Ae_{n_k} \to 0$ in norm. This is a contradiction since $|A_{n_k}| \geq \varepsilon$. ☐

**Example 9.40** (Strong limit of compacts is not compact)**.** We have seen in Lemma 9.13 that if we have an ONB $\{ e_j \}_{j=1}^{\infty}$, then

$$A = \sum_{j,k=1}^{\infty} A_{jk} e_j \otimes e_k^*$$

converges strongly. In fact, each operator $\sum_{j,k=1}^{N} A_{jk} e_j \otimes e_k^*$ is of *finite rank* and hence compact. But since this convergence is merely strongly, it does *not* make the full operator $A$ compact (of course not, it was generic!).

**Theorem 9.41.** *A compact operator maps weakly convergent sequences into norm convergent sequences.*

*Proof.* Let $\psi_n \to \psi$ weakly in $\mathcal{H}$. By the uniform boundedness principle Theorem 3.28, $\{ \|\psi_n\| \}_n$ is a bounded sequence. Let $A \in \mathcal{B}(\mathcal{H})$ be a compact operator. Our goal is to show that $A\psi_n \to A\psi$ in norm. Define

$$\varphi_n \quad := \quad A\psi_n \qquad \varphi := A\psi .$$

For any $\lambda \in \mathcal{H}^*$,

$$\lambda(\varphi_n) - \lambda(\varphi) \quad = \quad \lambda(\varphi_n - \varphi) = \lambda(A(\psi_n - \psi)) = (A^* \lambda)(\psi_n - \psi) .$$

Hence, $\varphi_n \to \varphi$ weakly as $\|A^* \lambda\|$ is bounded. Assume $\varphi_n$ does not converge to $\varphi$ in norm though. Then there must be a subsequence $\{ \varphi_{n_j} \}_j$ which maintains an $\varepsilon > 0$ distance from $\varphi$. But $\psi_{n_j}$ is still a bounded sequence, so $A\psi_{n_j}$ contains a norm convergent subsequence, which apparently does not converge to $A\psi$. This however contradicts $\varphi_n \to \varphi$ weakly. ☐

**Theorem 9.42** (Riesz-Schauder theorem)**.** *Let* $A \in \mathcal{K}(\mathcal{H})$*. Then* $0 \in \sigma(A)$*,* $\sigma(A)$ *is a discrete set where the only possible limit point is zero, and*

$$\left|B_\varepsilon(0_{\mathbb{C}})^c \cap \sigma(A)\right| < \infty \qquad (\varepsilon > 0)\ .$$

*Moreover,*

$$\sigma(A) \setminus \{\,0\,\} \subseteq \sigma_p(A)$$

*and* $\dim(\ker(A - \lambda\mathbb{1})) < \infty$ *for all* $\lambda \neq 0$.

It will turn out much more convenient to prove this theorem using the machinery of Fredholm operators (which we anyway want to introduce), so we postpone proving this until later.

## 9.6 Fredholm operators [mostly extra]

In many regards, Fredholm operators are the *opposite* of compact operators: they explore the full infinity of the Hilbert space; as we shall see, they are precisely the invertible operators *up to compact defects*.

**Definition 9.43** (Fredholm operator)**.** An operator $A \in \mathcal{B}(\mathcal{H})$ is called *Fredholm* iff $\dim\ker A, \dim\operatorname{coker} A$ are both finite dimensional.

Here

$$\operatorname{coker} A \equiv \mathcal{H}/\operatorname{im}(A)$$

in the sense of quotient vector spaces Definition 2.31. Recall that the quotient vector space is defined as:

$$\varphi \sim \psi \quad \Longleftrightarrow \quad \varphi - \psi \in \operatorname{im} A$$

and

$$\begin{aligned} [\varphi] &:= \{\, \psi \in \mathcal{H} \mid \varphi \sim \psi \,\} \\ \operatorname{coker} A &= \{\, [\varphi] \mid \varphi \in \mathcal{H} \,\}\ . \end{aligned}$$

**Definition 9.44** (Fredholm index)**.** For every Fredholm operator $A$, we define the index, an integer associated with that operator, as

$$\operatorname{index}(A) \quad := \quad \dim\ker A - \dim\operatorname{coker} A \in \mathbb{Z}\ .$$

We denote the space of all Fredholm operators as $\mathcal{F}(\mathcal{H})$, so that

$$\operatorname{index} : \mathcal{F} \to \mathbb{Z}\ .$$

Since an operator is injective iff its kernel is trivial, and surjective iff its image is the whole space (in which case the cokernel is trivial), we see that Fredholm operators are defined to be precisely those operators which fail to be invertible up to a finite "problem".

It is convenient to replace the cokernel with the kernel of the adjoint. To that end, we use the following

**Proposition 9.45.** *For any* $A \in \mathcal{B}(\mathcal{H})$*,* $\operatorname{coker} A$ *finite dimensional iff both (1)* $\ker A^*$ *is finite dimensional and (2)* $\operatorname{im}(A) \in \operatorname{Closed}(\mathcal{H})$.

*Proof.* Assume that $\operatorname{im} A \in \operatorname{Closed}(\mathcal{H})$. Then via Theorem 7.18,

$$\begin{aligned} (\ker(A^*))^\perp &= \left((\operatorname{im} A)^\perp\right)^\perp \\ &= \overline{\operatorname{im} A} && \text{(Via Claim 7.9)} \\ &= \operatorname{im} A\,. && \text{(By hypothesis)} \end{aligned}$$

Now, we always have

$$\begin{aligned}\mathcal{H} &= (\ker A) \oplus \left((\ker A)^{\perp}\right) = (\ker A^*) \oplus \left((\ker A^*)^{\perp}\right) \\ &= (\ker A^*) \oplus \mathrm{im} A\,.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{coker} A &\equiv \mathcal{H}/\mathrm{im} A \\ &\cong (\mathrm{im} A)^{\perp} \\ &= \ker A^*\,.\end{aligned}$$

Hence if $\dim\ker A^*$ is finite, so is $\dim\mathrm{coker} A$.

Conversely, assume that $\dim\mathrm{coker} A$ is finite. We want to show that $\mathrm{im} A \in \mathrm{Closed}(\mathcal{H})$.

Define a map

$$\begin{aligned}\eta : (\mathcal{H}/\ker A) \oplus (\mathrm{im} A)^{\perp} &\to \mathcal{H} \\ ([\varphi], \psi) &\mapsto A\varphi + \psi\,.\end{aligned}$$

It is easy to verify that $\eta$ is a bounded linear bijection (it is in verifying that $\eta$ is bounded that we used the fact $\mathrm{coker} A \cong (\mathrm{im} A)^{\perp}$ is finite dimensional). Hence

$$\mathrm{im} A \cong \eta\left((\mathcal{H}/\ker A) \oplus \{\,0\,\}\right) \in \mathrm{Closed}(\mathcal{H})$$

where the last statement is due to the open mapping Theorem 3.32 which says that the inverse of $\eta$ is also continuous, i.e., $\eta$ is a closed map and hence maps closed sets to closed sets. □

**Corollary 9.46.** *As a result of the above, we say that* $A \in \mathcal{F}$ *iff* $\dim\ker A, \dim\ker A^*$ *are both finite dimensional* and $\mathrm{im} A$ *is closed. Then, the index equals*

$$\mathrm{index}(A) = \dim\ker A - \dim\ker A^*\,.$$

Of course, to characterize whether the image of an operator is closed or not it would be convenient to employ Lemma 7.21 above.

When dealing with the space of Fredholm operators, it is customary to endow $\mathcal{F}(\mathcal{H})$ with the *subspace topology* from the topology induced by the operator norm on $\mathcal{B}(\mathcal{H})$. Hence by definition $\mathcal{S} \in \mathrm{Open}(\mathcal{F}(\mathcal{H}))$ iff $\mathcal{S} = \mathcal{T} \cap \mathcal{F}(\mathcal{H})$ for some $\mathcal{T} \in \mathrm{Open}(\mathcal{B}(\mathcal{H}))$, where $\mathrm{Open}(\mathcal{B}(\mathcal{H}))$ is defined in the usual metric since using open balls.

The Fredholm index is continuous, as well shall see, but taking only half of it does not yield a continuous map.

**Lemma 9.47.** $\dim\ker : \mathcal{F}(\mathcal{H}) \to \mathbb{N}_{\geq 0}$ *is upper semicontinuous.*

*Proof.* Decompose $\mathcal{H} = \ker(A) \oplus \ker(A)^{\perp} \cong \ker(A^*) \oplus \ker(A^*)^{\perp}$. Since $\mathrm{im}(A) \cong \ker(A^*)^{\perp}$, we have

$$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

for some isomorphism $a : \ker(A)^{\perp} \to \mathrm{im}(A)$. Taking any norm perturbation $B$ of size at most $\left\|a^{-1}\right\|^{-1}$ will mean that $A + B$ is injective on $\ker(A)^{\perp}$ and hence $\dim\ker A + B \leq \dim\ker A$. □

**Example 9.48.** Here are a few trivial examples for the notion of a Fredholm operator:

1. The identity operator $\mathbb{1}$ is Fredholm and its index is zero. In fact this is the case for any invertible operator.

2. The zero operator $\mathcal{H} \ni v \mapsto 0$ is *not* Fredholm.

3. Recall the position operator $X$ on $\ell^2(\mathbb{N})$, which is not even bounded. Its inverse $A := X^{-1}$ is however bounded: $\|A\| \leq 1$. It is however *not* Fredholm, even though it is self-adjoint and has an empty kernel. To see that $\operatorname{im} A \notin \operatorname{Closed}\left(\ell^2(\mathbb{N})\right)$, use the second characterization of Lemma 7.21 and note that while zero is not in $\sigma\left(|A|^2\right)$, it is an accumulation point and hence not isolated in $\sigma\left(|A|^2\right)$. What is the cokernel of $A$? Is it finite dimensional?

4. The right-shift operator $R$ on $\ell^2(\mathbb{N})$ is Fredholm. Indeed, one checks that

$$|R|^2 \quad = \quad R^* R = \mathbb{1}$$

and hence by the second characterization of Lemma 7.21 it has closed image. Its kernel is empty and the kernel of its adjoint, the left shift operator, is spanned by $\delta_1$ and is hence one dimensional.

$$\operatorname{index}(R) \quad = \quad -1\,.$$

Note that, considered on $\ell^2(\mathbb{Z})$, $R$ is also Fredholm, but it is now invertible and hence has zero index. The right shift operator is the most important example of a Fredholm operator, and in a sense, all other non-zero index operators may be connected to a power of the right shift, as we shall see.

*Claim* 9.49. If $A \in \mathcal{B}(\mathcal{H}_1 \to \mathcal{H}_2)$ with $\mathcal{H}_1, \mathcal{H}_2$ finite dimensional, then $A$ is Fredholm and its index equals

$$\operatorname{index}(A) \quad = \quad \dim(\mathcal{H}_1) - \dim(\mathcal{H}_2)\ .$$

*Proof.* The rank-nullity theorem [KK07] states that

$$\dim(\mathcal{H}_1) \quad = \quad \dim(\ker(A)) + \dim(\operatorname{im}(A))\ .$$

Furthermore, since $\operatorname{coker}(A) \equiv \mathcal{H}_2/\operatorname{im}(A)$, we have

$$\dim(\operatorname{coker}(A)) \quad = \quad \dim(\mathcal{H}_2) - \dim(\operatorname{im}(A))\ .$$

Thus, we have

$$\begin{aligned}\operatorname{index}(A) &\equiv \dim(\ker(A)) - \dim(\operatorname{coker}(A))\\ &= \dim(\mathcal{H}_1) - \dim(\operatorname{im}(A)) - \dim(\mathcal{H}_2) + \dim(\operatorname{im}(A))\\ &= \dim(\mathcal{H}_1) - \dim(\mathcal{H}_2)\ .\end{aligned}$$

as desired. ☐

In particular, any square matrix is Fredholm with index zero: finite dimensions are not very interesting for Fredholm theory. Be that as it may some mechanical models in physics have been studied of finite *non*-square matrices, which have a non-zero index.

**Lemma 9.50.** *(Riesz)* $\mathbb{1} - K \in \mathcal{F}(\mathcal{H})$ *for all* $K \in \mathcal{K}(\mathcal{H})$ *and* $\operatorname{index}(\mathbb{1} - K) = 0$.

*Proof.* Write $K = \lim_n F_n$ (in operator norm) where $F_n$ is finite rank. Hence $\mathbb{1} - K + F_n$ is invertible for $n$ sufficiently large since $\|K - F_n\|$ may be made arbitrarily small. Then,

$$\mathbb{1} - K \quad = \quad (\mathbb{1} - K + F_n)\left(\mathbb{1} - (\mathbb{1} - K + F_n)^{-1} F_n\right)$$

so that

$$\mathbb{1} - K \quad = \quad G(\mathbb{1} - F)$$

with $G$ invertible and $F$ finite rank. Hence $\ker(\mathbb{1} - K) = \ker(\mathbb{1} - F)$. Now, $v \in \ker(\mathbb{1} - F)$ iff $v = Fv$ which implies that $v$ is an eigenvector of $F$ with eigenvalue 1. But this if $F$ is finite rank its eigenspaces are finite dimensional. Same for $\mathbb{1} - F^*$. The two kernels are of the same dimension since $F$ is of finite rank.

TODO: show that $\mathbb{1}-K$ has closed image or that it has finite cokernel. $\square$

**Theorem 9.51** (Atkinson)**.** *$A\in\mathcal{F}(\mathcal{H})$ iff $A$ is invertible up to compacts, i.e., iff there is some operator $B\in\mathcal{B}(\mathcal{H})$, called* the parametrix of $A$, *such that,*

$$\mathbb{1}-AB,\mathbb{1}-BA\in\mathcal{K}(\mathcal{H})\ .$$

*We note that we may have $\mathbb{1}-AB\neq\mathbb{1}-BA$ indeed. Furthermore,* $\operatorname{index}(B)=-\operatorname{index}(A)$.

*Proof.* If $\mathbb{1}-AB,\mathbb{1}-BA\in\mathcal{K}(\mathcal{H})$ then $BA=\mathbb{1}-K$ for some compact $K$ and using Lemma 9.50 we have that $BA$ is Fredholm of index zero. Hence $\ker(BA)$ is finite dimensional. But $\ker(A)\subseteq\ker(BA)$ so that $\ker(A)$ is finite dimensional. Moreover, $\operatorname{im}(AB)\subseteq\operatorname{im}(A)$ (and $AB$ is also Fredholm) so $\operatorname{coker}(A)\subseteq\operatorname{coker}(AB)$. Thus $A$ is Fredholm. Now, using the logarithmic law Theorem 9.57 further below, since $AB=\mathbb{1}-K$, $0=\operatorname{index}(AB)=\operatorname{index}(A)+\operatorname{index}(B)$.

Conversely, assume $A\in\mathcal{F}(\mathcal{H})$. Want to construct two partial inverses: let $P,Q$ be the orthogonal projections onto $\ker(A)$ and $\ker(A^*)$ resp. We claim that $|A|^2+P$ and $|A^*|^2+Q$ are bijections. Indeed, $\ker(A)=\ker\left(|A|^2\right)$ so if $\mathcal{H}\cong\ker\left(|A|^2\right)^\perp\oplus\ker\left(|A|^2\right)$, $|A|^2+P\cong|A|^2\Big|_{\operatorname{im}(P)^\perp}\oplus\mathbb{1}$ and similarly for the other operator. Hence $B:=|A|^2+P$ is invertible, and

$$\mathbb{1}\quad=\quad B^{-1}A^*A+B^{-1}P\,.$$

But now, $B^{-1}P$ is of finite rank and $C:=B^{-1}A^*$ is the sought-after parametrix. $\square$

**Definition 9.52** (Essential spectrum)**.** The *essential spectrum* $\sigma_{\text{ess}}(A)$ of an operator $A\in\mathcal{B}(\mathcal{H})$ is the set of all points $z\in\mathbb{C}$ such that $A-z\mathbb{1}$ is *not* Fredholm. TODO: consolidate this with the definition below.

*Claim* 9.53. If $A\in\mathcal{F}(\mathcal{H})$ and $\operatorname{index}(A)=0$ then $A=G+K$ for some $G$ invertible and $K$ compact.

*Proof.* Since $\operatorname{index}(A)=0$, $\dim\ker A=\dim\ker A^*$. Thus,

$$\mathcal{H}\quad\cong\quad\ker(A)\oplus\ker(A)^\perp\cong\ker(A^*)\oplus\ker(A^*)^\perp\ .$$

But we know that since $\dim\ker A=\dim\ker A^*$, there is a natural linear isomorphism $\eta:\ker(A)\to\ker(A^*)$. We also know that $\operatorname{im}(A)\cong\ker(A^*)^\perp$. Hence, $A|_{\ker(A)^\perp}$ is just an isomorphism

$$\ker(A)^\perp\quad\to\quad\operatorname{im}(A)\ .$$

Hence, the map

$$G:=\eta\oplus A|_{\ker(A)^\perp}:\mathcal{H}\quad\to\quad\mathcal{H}$$

is an isomorphism and $\mathcal{H}\ni(v_1,v_2)\overset{K}{\mapsto}(\eta(v_1),0)\in\mathcal{H}$ is compact and hence the result. $\square$

**Theorem 9.54.** *We have the inclusion*

$$\mathcal{F}(\mathcal{H})+\mathcal{K}(\mathcal{H})\quad\subseteq\quad\mathcal{F}(\mathcal{H})$$

*and the Fredholm index is stable under compact perturbations.*

*Proof.* If $A\in\mathcal{F}(\mathcal{H})$ and $K\in\mathcal{K}(\mathcal{H})$, then by Atkinson Theorem 9.51, there is some parametrix $B$ such that $AB-\mathbb{1},BA-\mathbb{1}$ is compact. But $B$ will be a parametrix of $A+K$ too:

$$(A+K)B-\mathbb{1}\quad=\quad AB+KB-\mathbb{1}$$

which is compact since $KB$ is compact (as the compacts form an ideal). Hence $A+K$ is Fredholm. We postpone the proof that the index remains stable until the next theorem. □

We see from Atkinson's theorem that the essential spectrum is stable under compact perturbations.

**Theorem 9.55.** *(Dieudonne)* $\text{index}:\mathcal{F}(\mathcal{H})\to\mathbb{Z}$ *is operator-norm-continuous and if* $B\in\mathcal{F}(\mathcal{H})$ *is any parametrix of* $A\in\mathcal{F}(\mathcal{H})$ *then*

$$B_{\|B\|^{-1}}(A) \quad\subseteq\quad \mathcal{F}(\mathcal{H}) \,.$$

*In particular,* $\mathcal{F}(\mathcal{H})\in\text{Open}(\mathcal{B}(\mathcal{H}))$.

*Proof.* Let $A\in\mathcal{F}(\mathcal{H})$ and $B$ be any parametrix of it. Take any $\tilde{A}\in B_{\|B\|^{-1}}(A)$. We have

$$\left\|B\left(A-\tilde{A}\right)\right\|\leq\|B\|\left\|A-\tilde{A}\right\|<1$$

by assumption, so that $\mathbb{1}-B\left(A-\tilde{A}\right)$ is invertible. We claim that $\left(\mathbb{1}-B\left(A-\tilde{A}\right)\right)^{-1}B$ is a parametrix for $\tilde{A}$. Now,

$$\begin{aligned}0 &= \text{index}(\mathbb{1})\\ &= \text{index}(\mathbb{1}-K)\\ &= \text{index}\left(\left(\mathbb{1}-B\left(A-\tilde{A}\right)\right)^{-1}B\tilde{A}\right)\\ &= \text{index}\left(B\tilde{A}\right)\end{aligned}$$

Now, via Theorem 9.57 further below we have $\text{index}(B)+\text{index}\left(\tilde{A}\right)$ and since $\text{index}(B)=-\text{index}(A)$ we obtain the result. □

Finally, we finish the proof of Theorem 9.54: if $A\in\mathcal{F}(\mathcal{H})$ and $K\in\mathcal{K}(\mathcal{H})$, the homotopy $[0,1]\ni t\mapsto A+tK\in\mathcal{F}(\mathcal{H})$ interpolates in a norm continuous way between $A$ and $A+K$ and thus the index is constant along this path.

*Claim* 9.56. $A$ is invertible up to compacts iff it is invertible up to finite ranks.

*Proof.* Since finite rank operators are compact one direction is trivial. Now, assume that there is some $B\in\mathcal{F}(\mathcal{H})$ with which $\mathbb{1}-AB,\mathbb{1}-BA\in\mathcal{K}(\mathcal{H})$. Let $\{F_n\}_n$ be a sequence of finite rank operators which converges to $K:=\mathbb{1}-BA$ in operator norm. Then $\|F_n-K\|$ can be made arbitrarily small and hence $W_n:=\mathbb{1}-K+F_n$ is invertible for $n$ sufficiently large. Then,

$$\begin{aligned}BA &= \mathbb{1}-K\\ &= W_n\left(\mathbb{1}-W^{-1}F_n\right)\end{aligned}$$

and hence

$$\mathbb{1}-W_n^{-1}BA \quad=\quad W^{-1}F_n\,.$$

Since finite rank operators form an ideal, $W_n^{-1}F_n$ is finite rank too. This same logic shows that

$$\mathbb{1}-AB\tilde{W}_n^{-1} \quad=\quad \tilde{F}_n\tilde{W}_n^{-1}$$

where now $\tilde{F}_n\tilde{W}_n^{-1}$ is finite rank. So Now, $W_n^{-1}B$ is a partial left inverse and $B\tilde{W}_n^{-1}$is a partial right inverse. Then

$$\begin{aligned} W_n^{-1}BA &= \mathbb{1}-W^{-1}F_n \\ W_n^{-1}BAB\tilde{W}_n^{-1} &= B\tilde{W}_n^{-1}-W^{-1}F_nB\tilde{W}_n^{-1} \\ W_n^{-1}B\left(\mathbb{1}-\tilde{F}_n\tilde{W}_n^{-1}\right) &= B\tilde{W}_n^{-1}-W^{-1}F_nB\tilde{W}_n^{-1} \\ W_n^{-1}B-B\tilde{W}_n^{-1} &= W_n^{-1}B\tilde{F}_n\tilde{W}_n^{-1}-W^{-1}F_nB\tilde{W}_n^{-1} \end{aligned}$$

and since the finite rank operators form an ideal within $\mathcal{B}(\mathcal{H})$, we find that $W_n^{-1}B$ equals $B\tilde{W}_n^{-1}$ up to finite rank operators and hence there is just one parametrix, say, $W_n^{-1}B$. ☐

**Theorem 9.57.** *(Logarithmic law) If $A, B \in \mathcal{F}(\mathcal{H})$ then*

$$\begin{aligned} \text{index}(AB) &= \text{index}(A)+\text{index}(B) \\ \text{index}(A\oplus B) &= \text{index}(A)+\text{index}(B)\,. \end{aligned}$$

*Proof.* The easiest proof is via Fedosov Theorem 9.84: If $\tilde{A}$ is a parametrix for $A$ and $\tilde{B}$ is a parametrix for $B$ then $\tilde{B}\tilde{A}$ is a parametrix for $AB$. If we let $F := B\tilde{B}-\mathbb{1}$ be finite rank, then

$$\begin{aligned} \text{index}(AB) &= \text{tr}\left(AB\tilde{B}\tilde{A}-\tilde{B}\tilde{A}AB\right) \\ &= \text{tr}\left(A(\mathbb{1}+F)\tilde{A}-\tilde{A}A+\tilde{A}A-\tilde{B}\tilde{A}AB\right) \\ &= \text{tr}\left(A\tilde{A}-\tilde{A}A\right)+\text{tr}\left(AF\tilde{A}\right)+\text{tr}\left(\tilde{A}A-\tilde{B}\tilde{A}AB\right) \\ &= \text{index}(A)+\text{tr}\left(\tilde{A}AF\right)+\text{tr}\left(\tilde{A}A-\tilde{B}\tilde{A}AB\right) \\ &= \text{index}(A)+\text{tr}\left(\tilde{A}A(\mathbb{1}+F)-\tilde{B}\tilde{A}AB\right) \\ &= \text{index}(A)+\text{tr}\left(\tilde{A}AB\tilde{B}-\tilde{B}\tilde{A}AB\right)\,. \end{aligned}$$

But now, $\tilde{B}$ is a partial inverse of $\tilde{A}AB$, so the last trace equals $\text{index}\left(\tilde{A}AB\right)$. Let us write $\tilde{A}A=\mathbb{1}+Y$ for some finite rank $Y$. so

$$\begin{aligned} \text{index}\left(\tilde{A}AB\right) &= \text{index}((\mathbb{1}+Y)B) \\ &= \text{index}(B)\,. \end{aligned}$$

The statement about the direct sum is trivial. ☐

**Theorem 9.58.** *(Atiyah-Jähnich) We have $\pi_0(\mathcal{F}(\mathcal{H}))\cong\mathbb{Z}$.*

*Proof.* We already know that $\text{index}:\mathcal{F}(\mathcal{H})\to\mathbb{Z}$ is continuous and is constant on the path-connected components of $\mathcal{F}(\mathcal{H})$. Thus index lifts to a well-defined map on $\pi_0(\mathcal{F}(\mathcal{H}))$. To see that it's surjective it suffices to consider powers of the right-shift operator. So we only need to show it is injective.

First we claim that if $\text{index}(A)=0$ then there is a path from $A$ to $\mathbb{1}$: Via Claim 9.53 we have $A=G+K$ for some invertible $G$ and compact $K$. By Theorem 10.27, there is a path $\gamma$ from $(A-K)^{-1}$ to $\mathbb{1}$, and $\gamma A$ is a path from $\mathbb{1}-\tilde{K}$ to $A$. From there we can define a further homotopy to reduce $\tilde{K}$ to zero.

Next, we need that if $\text{index}(A)=\text{index}(B)$ then there is a path between them. To that end, let $\tilde{B}$ be the parametrix of $B$. Then $\text{index}\left(A\tilde{B}\right)=0$, whence by the above there is a path $\gamma: A\tilde{B}\mapsto\mathbb{1}$. The path $\tilde{\gamma}:=\gamma B$ interpolates between $A-K$ and $B$. Again, a further homotopy brings us to $A$. ☐

### 9.6.1 Back to the Riesz-Schauder theorem

We now want to present the

*Proof of Theorem 9.42.* Let $A \in \mathcal{K}(\mathcal{H})$. Then clearly $A$ cannot be invertible, as it cannot be Fredholm. Indeed, if $A$ were Fredholm, then via Theorem 9.51 $\mathbb{1}$ would be compact, which is false as soon as $\dim \mathcal{H} = \infty$. Thus, since $A$ is not invertible, $0 \in \sigma(A)$.

Next, we want to show that if $\lambda \in \sigma(A) \setminus \{0\}$, then $\lambda$ is an *isolated* point of the *point* spectrum with $\dim(\ker(A - \lambda\mathbb{1})) < \infty$. Clearly, if $\lambda \neq 0$, then $-\lambda\mathbb{1}$ is invertible and hence a compact perturbation from $-\lambda\mathbb{1}$ into $A - \lambda\mathbb{1}$ does not change that fact (via Theorem 9.51). Thus

$$\dim(\ker(A - \lambda\mathbb{1})) < \infty .$$

Now assume that $\lambda \notin \sigma_p(A)$. That necessarily means that $\ker(A - \lambda\mathbb{1}) = \{0\}$. But $\lambda \in \sigma(A)$, so $\operatorname{im}(A - \lambda\mathbb{1}) \neq \mathcal{H}$. Since

$$\operatorname{index}(A - \lambda\mathbb{1}) = \operatorname{index}(-\lambda\mathbb{1}) = \operatorname{index}(\mathbb{1}) = 0$$

we have

$$\ker(A - \lambda\mathbb{1}) \cong \ker\left(A^* - \overline{\lambda}\mathbb{1}\right)$$

so we learn that $\ker\left(A^* - \overline{\lambda}\mathbb{1}\right) = \{0\}$ too. But

$$\left(\overline{\operatorname{im}(A - \lambda\mathbb{1})}\right)^{\perp} = (\operatorname{im}(A - \lambda\mathbb{1}))^{\perp} = \ker\left(A^* - \overline{\lambda}\mathbb{1}\right) = \{0\}$$

which implies that $\overline{\operatorname{im}(A - \lambda\mathbb{1})} \cong \mathcal{H}$. Since $A - \lambda\mathbb{1}$ is Fredholm, it has closed range, so we learn that $\operatorname{im}(A - \lambda\mathbb{1}) \cong \mathcal{H}$, which is in contradiction to $\lambda \notin \sigma_p(A)$.

We have shown thus far that if $\lambda \in \sigma(A) \setminus \{0\}$ then $\lambda \in \sigma_p(A)$ and $\dim\ker(A - \lambda\mathbb{1}) < \infty$. We are left to show that $\lambda$ is an *isolated* element of the point spectrum. To this end, let $\{\lambda_n\}_n$ be a convergent sequence of distinct eigenvalues of $A$. We will show that $\lim_n \lambda_n = 0$. Let $\psi_n$ be the corresponding eigenvectors and

$$\mathcal{M}_n := \operatorname{span}(\{\psi_1, \dots, \psi_n\}) \qquad (n \in \mathbb{N}) .$$

Then

$$\mathcal{M}_1 \subsetneq \mathcal{M}_2 \subsetneq \mathcal{M}_3 \subsetneq \cdots$$

since eigenvectors of distinct eigenvalues are linearly independent. Let $\{\varphi_n\}_n$ be a sequence of unit ectors chosen so that $\varphi_n \in \mathcal{M}_n$ and $\varphi_n \perp \mathcal{M}_{n-1}$. If Hence we have

$$\eta = \sum_{n=1}^{\infty} \langle \varphi_n, \eta \rangle \varphi_n + \varphi_0$$

where $\varphi_0$ is some vector orthogonal to all of $\{\varphi_n\}_n$. Since

$$\|\eta\|^2 = \sum_{n=1}^{\infty} |\langle \varphi_n, \eta \rangle|^2 + \|\varphi_0\|^2$$

it follows that $\lim_{n\to\infty} \langle \varphi_n, \eta \rangle = 0$. Hence $\varphi_n$ converges weakly to zero, so that $\{A\varphi_n\}_n$ converges to 0 in norm via Theorem 9.41. Now, for any $n \in \mathbb{N}$ there exist scalars $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{C}$ such that

$$\varphi_n = \sum_{i=1}^{n} \alpha_i \psi_i$$

so that

$$A\varphi_n = \sum_{i=1}^{n} \alpha_i A\psi_i = \sum_{i=1}^{n} \alpha_i \lambda_i \psi_i = \lambda_n \underbrace{\sum_{i=1}^{n} \alpha_i \psi_i}_{\varphi_n} + \underbrace{\sum_{i=1}^{n-1} \alpha_i (\lambda_i - \lambda_n) \psi_i}_{\in \mathcal{M}_{n-1}} .$$

As such, we find

$$\lim_{n\to\infty} |\lambda_n|^2 \quad \leq \quad \lim_{n\to\infty} \left( |\lambda_n|^2 \|\varphi_n\|^2 + \left\| \sum_{i=1}^{n-1} \alpha_i (\lambda_i - \lambda_n) \psi_i \right\| \right) = \lim_{n\to\infty} \|A\varphi_n\| = 0\,.$$

□

We have an inverse of this theorem for self-adjoint operators using the notion of essential spectrum.

**Definition 9.59** (Essential spectrum)**.** For any $A \in \mathcal{B}(\mathcal{H})$, we define the *essential spectrum* of an operator as

$$\sigma_{\text{ess}}(A) := \{ \lambda \in \mathbb{C} \mid (A - \lambda \mathbb{1}) \notin \mathcal{F}(\mathcal{H}) \}\,.$$

In other words, if the usual spectrum is defined as that set of $\lambda \in \mathbb{C}$ for which the operator $A - \lambda\mathbb{1}$ is not invertible, using Theorem 9.51, we identify the essential spectrum as those $\lambda$ for which $A - \lambda\mathbb{1}$ is *essentially* not invertible, since to be Fredholm is to be invertible up to compact. We also define the discrete spectrum as anything else

$$\sigma_{\text{disc}}(A) := \sigma(A) \setminus \sigma_{\text{ess}}(A)\,.$$

Clearly, $\sigma_{\text{disc}}(A)$ is comprised of eigenvalues of $A$ of finite multiplicity, which is somewhat smaller than the point spectrum (which could contain eigenvalues of infinite multiplicity). Since $\mathcal{F}(\mathcal{H})$ is open, $\sigma_{\text{ess}}(A)$ is closed.

We then have

**Theorem 9.60.** *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, $A = A^*$. Then $A \in \mathcal{K}(\mathcal{H})$ iff*

$$\sigma_{ess}(A) = \{ 0 \}\,.$$

*Proof.* Clearly if $A \in \mathcal{K}(\mathcal{H})$ then $\sigma_{\text{ess}}(A) = \{ 0 \}$ holds, by the very definition of the essential spectrum. We leave the other direction as an exercise to the reader. □

**Theorem 9.61.** *Let $A \in \mathcal{B}(\mathcal{H})$. We have $\lambda \in \sigma_{ess}(A)$ if and only if at least one of the following holds:*

1. *$\lambda \in \sigma_{cont}(A)$.*
2. *$\lambda$ is a limit point of $\sigma_p(A)$.*
3. *$\lambda$ is an eigenvalue of infinite multiplicity.*

**Theorem 9.62** (Weyl's criterion for the essential spectrum)**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, $A = A^*$. Then $\lambda \in \sigma_{ess}(A)$ iff there exists an orthonormal set $\{ \psi_n \}_n$ such that*

$$\lim_{n\to\infty} \|(A - \lambda\mathbb{1})\psi_n\| = 0\,.$$

*Compare this with Theorem 9.22, where the criterion is for $\lambda \in \sigma(A)$ and the assumption that $\psi_n \perp \psi_m$ for $n \neq m$ is dropped.*

## 9.7 Trace-class and Schatten ideals

To generalize the notion of a trace to the infinite dimensional setting, we present the trace-class operators.

**Definition 9.63.** An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be of trace-class iff

$$\mathrm{tr}_{\{\varphi_n\}_{n=1}^\infty}(|A|) < \infty$$

where $|A| \equiv \sqrt{A^* A}$ (as in Theorem 8.13) and $\mathrm{tr}_{\{\varphi_n\}_{n=1}^\infty}$ is the trace calculated in *any* ONB $\{\varphi_n\}_{n=1}^\infty$:

$$\mathrm{tr}_{\{\varphi_n\}_{n=1}^\infty}(|A|) \equiv \sum_{n=1}^\infty \langle \varphi_n, A\varphi_n \rangle \, .$$

We denote the space of all trace-class operators as $\mathcal{J}_1 \equiv \mathcal{J}_1(\mathcal{H})$.

*Claim* 9.64. If $\mathrm{tr}_{\{\varphi_n\}_{n=1}^\infty}(|A|) < \infty$ in one ONB $\{\varphi_n\}_{n=1}^\infty$ then it is finite in any other ONB.

*Proof.* Let $\{\psi_n\}_n$ be any other ONB. Then

$$\begin{aligned}
\mathrm{tr}_{\{\psi_n\}_{n=1}^\infty}(|A|) &= \sum_n \langle \psi_n, |A|\, \psi_n \rangle \\
&= \sum_n \left\| |A|^{\frac{1}{2}} \psi_n \right\|^2 \\
&= \sum_n \left( \sum_m \left| \left\langle \varphi_m, |A|^{\frac{1}{2}} \psi_n \right\rangle \right|^2 \right) \\
&= \sum_m \sum_n \left| \left\langle |A|^{\frac{1}{2}} \varphi_m, \psi_n \right\rangle \right|^2 \\
&= \sum_m \left\| |A|^{\frac{1}{2}} \varphi_m \right\| \\
&= \mathrm{tr}_{\{\varphi_n\}_{n=1}^\infty}(|A|) \, .
\end{aligned}$$

□

Note that it is *not* enough to have

$$\sum_n |\langle \varphi_n, A\varphi_n \rangle| < \infty$$

for some ONB $\{\varphi_n\}_n$ to be trace-class!

**Example 9.65.** The right shift operator is *not* trace-class though $\sum_n |\langle \varphi_n, R\varphi_n \rangle| < \infty$ for it in the position basis.

*Remark* 9.66. It is, however, enough to ask that

$$\sum_n |\langle \varphi_n, A\varphi_n \rangle| < \infty \qquad \forall \text{ONB } \{\varphi_n\}_n \, .$$

This notion is equivalent to trace-class.

**Example 9.67.** Any finite rank operator is trace-class.

*Proof.* First we note that if $F$ is finite rank then so is $|F|$. Indeed, using Claim 9.30 we have

$$F = \sum_{n=1}^N \alpha_n \varphi_n \otimes \psi_n^*$$

so

$$\begin{aligned} |F|^2 &= F^*F \\ &= \sum_{n=1}^N \alpha_n \psi_n \otimes \varphi_n^* \sum_{n=1}^N \alpha_n \varphi_n \otimes \psi_n^* \\ &= \sum_{n=1}^N \alpha_n^2 \psi_n \otimes \psi_n^* \,. \end{aligned}$$

Since this operator is diagonal we can easily take its square root:

$$|F| = \sum_{n=1}^N \alpha_n \psi_n \otimes \psi_n^* \,.$$

Now taking the trace in the $\psi_n$ basis we get

$$\operatorname{tr}(|F|) = \sum_{n=1}^N \alpha_n < \infty \,.$$

□

*Claim* 9.68. Any trace-class operator is compact.

*Proof.* First assume that $\operatorname{tr}\left(|A|^2\right) < \infty$ and let $\varepsilon > 0$. Then we claim there do no exist infinitely many linearly independent $\varphi \in \mathcal{H}$ such that

$$\| |A| \varphi \| \geq \varepsilon \,.$$

Indeed, assume otherwise. Let $\mathcal{E}$ be an infinite orthonormal set such that

$$\| |A| \varphi \| \geq \varepsilon \qquad (\varphi \in \mathcal{E}) \,.$$

Let $\tilde{\mathcal{E}}$ be an orthonormal basis of $\mathcal{H}$ extending $\mathcal{E}$. Then

$$\begin{aligned} \infty &> \operatorname{tr}\left(|A|^2\right) \\ &= \sum_{\varphi \in \tilde{\mathcal{E}}} \left\langle \varphi, |A|^2 \varphi \right\rangle \\ &\geq \sum_{\varphi \in \mathcal{E}} \left\langle \varphi, |A|^2 \varphi \right\rangle \\ &= \sum_{\varphi \in \mathcal{E}} \|A\varphi\|^2 \\ &= \sum_{\varphi \in \mathcal{E}} \| |A| \varphi \|^2 \\ &\geq \sum_{\varphi \in \mathcal{E}} \varepsilon^2 \\ &= \infty \end{aligned}$$

which is a contradiction.

In conclusion, for any $\varepsilon > 0$ we may choose a finite orthonormal set $\{ \varphi_n \}_{n=1}^N$ such that if for some $\psi \in \mathcal{H}$,

$$\| |A| \psi \| \geq \varepsilon$$

then $\psi \in \operatorname{span}\left(\{ \varphi_n \}_{n=1}^N\right)$. Define $F \in \mathcal{B}(\mathcal{H})$ via

$$F\psi \quad := \quad \begin{cases} |A|\,\psi & \psi \in \operatorname{span}\left(\{ \varphi_n \}_{n=1}^N\right) \\ 0 & \text{else} \end{cases} .$$

Then, for any $\psi \in \mathcal{H}$, write $\psi = \psi_1 + \psi_2$ with $\psi_1 \in \operatorname{span}\left(\{ \varphi_n \}_{n=1}^N\right)$ and $\psi_2$ in the complement. We thus get

$$\begin{aligned} (|A| - F)(\psi_1 + \psi_2) &= (|A| - F)\psi_1 + (|A| - F)\psi_2 \\ &= (|A| - F)\psi_2 \\ &= |A|\,\psi_2 \end{aligned}$$

so that

$$\|(|A| - F)\psi\| \quad = \quad \||A|\,\psi_2\| \leq \varepsilon .$$

As a result, $F$ approximates $|A|$ to $\varepsilon$ accuracy and thus $|A|$ is compact.

Let now $A$ be trace-class, i.e., $\operatorname{tr}(|A|) < \infty$ which implies that $|A|^{\frac{1}{2}}$ is compact. Write, using the polar decomposition,

$$A \quad = \quad U\,|A| .$$

Then

$$A \quad = \quad U\,|A|^{\frac{1}{2}}\,|A|^{\frac{1}{2}} .$$

Since $\mathcal{K}(\mathcal{H})$ is a two-sided ideal, we are finished. ◻

**Theorem 9.69.** *If $A \in \mathcal{J}_1(\mathcal{H})$ then*

$$\operatorname{tr}_{\{ \varphi_n \}_{n=1}^\infty}(A)$$

*converges absolutely and is independent of $\{ \varphi_n \}_{n=1}^\infty$.*

**Definition 9.70.** It is thus justified to define, for any $A \in \mathcal{J}_1(\mathcal{H})$,

$$\operatorname{tr}(A) := \operatorname{tr}_{\{ \varphi_n \}_{n=1}^\infty}(A) .$$

*Proof of Theorem 9.69.* Using the above claim, we see that $|A|$ is compact. As a self-adjoint operator with only finite-multiplicitly non-zero eigenvalues, we have using Theorem 9.42 that there exists some $\{ \lambda_n \}_n \subseteq [0, \infty)$ and $\{ \varphi_n \}_n$ orthonormal with

$$|A|\,\varphi_n \quad = \quad \lambda_n \varphi_n .$$

This set forms a basis of $\mathcal{H}$. Indeed, one may appeal to the holomorphic functional calculus for that: we know that

$$\varphi_n \otimes \varphi_n^* = \frac{\mathrm{i}}{2\pi} \oint (|A| - z\mathbb{1})^{-1}\,\mathrm{d}z$$

for any contour which encircles $\lambda_n$ (assuming it has multiplicity one, otherwise it will be a sum of such projections). If we now use the additivity of contour integrals we get

$$\sum_n \varphi_n \otimes \varphi_n^* \quad = \quad \frac{\mathrm{i}}{2\pi} \oint (|A| - z\mathbb{1})^{-1}\,\mathrm{d}z$$

where the contour on the RHS is a contour that contains the entire spectrum of $|A|$. However, we then recognize that contour integral as $\mathbb{1}$ by the holomorphic functional calculus, and as such, $\{ \varphi_n \}$ is an ONB of $\mathcal{H}$.

Then with the polar decomposition $A = U|A|$ we have

$$\begin{aligned}|\mathrm{tr}(A)| &= \left|\sum_{n=1}^{\infty}\langle\varphi_n, U|A|\varphi_n\rangle\right| \\ &= \left|\sum_{n=1}^{\infty}\langle\varphi_n, U\lambda_n\varphi_n\rangle\right| \\ &\leq \sum_{n=1}^{\infty}|\lambda_n|\,\|\varphi_n\|\|U\varphi_n\| \\ &\leq \sum_{n=1}^{\infty}|\lambda_n| \\ &< \infty\,.\end{aligned}$$

I.e., $\sum_{n=1}^{\infty}\lambda_n\langle\varphi_n, U\varphi_n\rangle$ converges absolutely. Then given any other ONB $\{\psi_n\}_n$, we have

$$\begin{aligned}\sum_n \lambda_n\langle\varphi_n, U\varphi_n\rangle &= \sum_n\lambda_n\left\langle\sum_m\psi_m\otimes\psi_m^*\varphi_n, U\varphi_n\right\rangle \\ &= \sum_n\sum_m\lambda_n\overline{\langle\psi_m,\varphi_n\rangle}\langle\psi_m, U\varphi_n\rangle \\ &\overset{\text{abs. conv.}}{=} \sum_m\sum_n\lambda_n\langle\psi_m, U\varphi_n\rangle\langle\varphi_n,\psi_m\rangle \\ &\overset{\lambda_n\varphi_n=|A|\varphi_n}{=} \sum_m\sum_n\langle\psi_m, A\varphi_n\rangle\langle\varphi_n,\psi_m\rangle \\ &= \sum_m\sum_n\langle\psi_m, A\varphi_n\otimes\varphi_n^*\psi_m\rangle \\ &= \sum_m\langle\psi_m, A\psi_m\rangle\,.\end{aligned}$$

Thus $\mathrm{tr}(A)$ does not depend on the basis chosen to calculate the trace. □

**Theorem 9.71.** *The trace is a $\mathbb{C}$-linear map $\mathcal{J}_1(\mathcal{H})\to\mathbb{C}$. It is unitarily invariant: for any $W$ unitary,* $\mathrm{tr}(W^*AW) = \mathrm{tr}(A)$*, and if* $0\leq A\leq B$ *then* $\mathrm{tr}(A)\leq\mathrm{tr}(B)$.

*Proof.* First we show that for any unitary $W$, if $A\in\mathcal{J}_1(\mathcal{H})$ then $W^*AW\in\mathcal{J}_1(\mathcal{H})$. Indeed, this follows from

$$\begin{aligned}|W^*AW|^2 &= (W^*AW)^*W^*AW \\ &= W^*A^*WW^*AW \\ &= W^*|A|^2W\end{aligned}$$

so that, using the fact the square root obeys unitary conjugation (see the definition of the square root Theorem 8.13 as a power series),

$$|W^*AW| = W^*|A|W$$

and since $\mathrm{tr}_{\{\varphi_n\}_n}(|\cdot|)$ does not depend on the basis $\{\varphi_n\}_n$, we find that

$$\mathrm{tr}_{\{\varphi_n\}_n}(|W^*AW|) = \mathrm{tr}_{\{W\varphi_n\}_n}(|A|)\,.$$

Thus we know that $W^*AW\in\mathcal{J}_1(\mathcal{H})$ too. Now thanks to invariance under the choice of ONB (Theorem 9.69) it is clear that $\mathrm{tr}(W^*AW) = \mathrm{tr}(A)$. Finally, if

$$0\leq A\leq B$$

then for any $\psi \in \mathcal{H}$, $\langle \psi, A\psi \rangle \leq \langle \psi, B\psi \rangle$. Hence,

$$\operatorname{tr}(A) = \sum_n \langle \varphi_n, A\varphi_n \rangle \leq \sum_n \langle \varphi_n, B\varphi_n \rangle = \operatorname{tr}(B) \ .$$

□

**Lemma 9.72** (The two-sided star-ideal property)**.** *If $A \in \mathcal{J}_1(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ then $AB, BA, A^* \in \mathcal{J}_1(\mathcal{H})$ and*

$$\begin{aligned} \operatorname{tr}(AB) &= \operatorname{tr}(BA) \ , \\ \operatorname{tr}(A^*) &= \overline{\operatorname{tr}(A)} \, . \end{aligned}$$

*Proof.* First note that any bounded linear operator may be written as the sum of four unitaries. Indeed, Let $B \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} B &= \frac{1}{2}(B + B^*) + \mathrm{i}\frac{1}{2\mathrm{i}}(B - B^*) \\ &=: B_R + \mathrm{i}B_I \end{aligned}$$

so any operator may be written as the sum of two self-adjoints. Moreover, any bounded self-adjoint $H$ may be written as the sum of two unitaries:

$$\begin{aligned} H &= \|H\| \\ &= \frac{1}{2}\|H\|\left(\frac{1}{\|H\|}H + \mathrm{i}\sqrt{\mathbb{1} - \frac{1}{\|H\|^2}H^2}\right) + \frac{1}{2}\|H\|\left(\frac{1}{\|H\|}H - \mathrm{i}\sqrt{\mathbb{1} - \frac{1}{\|H\|^2}H^2}\right) \end{aligned}$$

and one readily verifies that

$$\frac{1}{\|H\|}H \pm \mathrm{i}\sqrt{\mathbb{1} - \frac{1}{\|H\|^2}H^2}$$

are unitary. Next, note that

$$|A| = |UA|$$

for any unitary $U$, so $UA$ is actually trace-class too whenever $A$ is. Also

$$|AU| = U^{-1}|A|U$$

so also $AU$ is trace class. Then

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_n \langle \varphi_n, AB\varphi_n \rangle \\ &\overset{\psi_n := B\varphi_n}{=} \sum_n \langle B^*\psi_n, A\psi_n \rangle \\ &= \sum_n \langle \psi_n, BA\psi_n \rangle \\ &= \operatorname{tr}(BA) \ . \end{aligned}$$

Lastly,

$$\operatorname{tr}(A^*) = \sum_n \langle \varphi_n, A^*\varphi_n \rangle = \sum_n \langle A\varphi_n, \varphi_n \rangle = \sum_n \overline{\langle \varphi_n, A\varphi_n \rangle} = \overline{\operatorname{tr}(A)} \, .$$

□

If we define on $\mathcal{J}_1(\mathcal{H})$ the norm

$$\|A\|_1 := \operatorname{tr}(|A|)$$

then

**Theorem 9.73.** *$\mathcal{J}_1(\mathcal{H})$ is a Banach space with the norm $\|\cdot\|_1$ and*

$$\|A\|_{op} \leq \|A\|_1 \, .$$

*Proof.* We have $\|A\|_{\mathrm{op}} = \||A|\|_{\mathrm{op}}$, but the latter number is equal to $r(|A|)$, i.e., the maximal magnitude of the spectrum of $|A|$. On the other hand $\|A\|_1$ is the sum of *all* elements in the spectrum of $|A|$, which is a trivial upper bound on the maximum. □

**Theorem 9.74.** *We have*

$$\mathcal{J}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$$

*and $A \in \mathcal{K}(\mathcal{H})$ is in $\mathcal{J}_1(\mathcal{H})$ iff $\mathrm{tr}(|A|) < \infty$.*

*Proof.* The first part has been proven in Claim 9.68. □

**Theorem 9.75.** *The trace-class ideal $\mathcal{J}_1(\mathcal{H})$ is the $\|\cdot\|_1$-closure of finite rank operators.*

*Proof.* Given any $A \in \mathcal{J}_1(\mathcal{H})$, we must show that

$$A = \lim_{n\to\infty} F_n$$

where each $F_n$ is finite rank and the limit is in the sense of $\|\cdot\|_1$, i.e.,

$$\|A - F_n\|_1 \overset{n\to\infty}{\to} 0 \, .$$

As we have seen, since $A \in \mathcal{J}_1(\mathcal{H})$, $A$ is compact, so it has the operator-norm convergent approximation

$$A = \sum_{n=1}^{\infty} \alpha_n \varphi_n \otimes \psi_n^* \, .$$

Here $\alpha_n$ are the singular values of $A$ and by definition we have

$$\mathrm{tr}(|A|) = \sum_{n=1}^{\infty} \alpha_n < \infty \, .$$

We then want to show that

$$\left\| \sum_{n=N}^{\infty} \alpha_n \varphi_n \otimes \psi_n^* \right\|_1 \overset{N\to\infty}{\to} 0 \, .$$

We have

$$\begin{aligned} \left\| \sum_{n=N}^{\infty} \alpha_n \varphi_n \otimes \psi_n^* \right\|_1 &= \mathrm{tr}\left( \left| \sum_{n=N}^{\infty} \alpha_n \varphi_n \otimes \psi_n^* \right| \right) \\ &= \mathrm{tr}\left( \sum_{n=N}^{\infty} \alpha_n \psi_n \otimes \psi_n^* \right) \, . \end{aligned}$$

Calculating this in the $\psi_n$ basis we find the result. □

### 9.7.1 Schatten ideals

**Definition 9.76** (Schatten class operators)**.** We define now, for any $p \in [1,\infty)$, the operator $A \in \mathcal{B}(\mathcal{H})$ to be of $p$-th Schatten class, denoted by $A \in \mathcal{J}_p(\mathcal{H})$ iff

$$\operatorname{tr}(|A|^p) < \infty\,.$$

This also yields the p-Schatten norm,

$$\|A\|_p := (\operatorname{tr}(|A|^p))^{\frac{1}{p}}$$

which makes $\left(\mathcal{J}_p(\mathcal{H}), \|\cdot\|_p\right)$ into a Banach space. It is to be contrasted with the usual Lebesgue $L^p(X,\mu)$ spaces, with $X$ a measure space and $\mu$ a Borel measure. In this sense these Schatten classes could be considered the non-commutative analogs of these $L^p$ spaces (function multiplication is Abelian of course). We give $\mathcal{J}_2(\mathcal{H})$ the special name of *Hilbert-Schmidt operators* and it is the only one which is actually itself a Hilbert space under the inner product

$$\langle A, B\rangle_{\text{Hilbert-Schmidt}} := \operatorname{tr}(A^* B)$$

in a similar way to how $L^2$ is a Hilbert space. In fact, we have the Cauchy-Schwarz inequality:

$$\begin{aligned}
\langle A, B\rangle_{\text{HS}} &\equiv \operatorname{tr}(A^*B) \\
&= \sum_n \langle \varphi_n, A^*B\varphi_n\rangle \\
&= \sum_n \left\langle \varphi_n, A^* \sum_m \varphi_m \otimes \varphi_m^* B\varphi_n \right\rangle \\
&= \sum_n \sum_m \langle \varphi_n, A^*\varphi_m\rangle \langle \varphi_m, B\varphi_n\rangle \\
&\overset{\text{CS for }\mathbb{C}}{\leq} \sqrt{\sum_n\sum_m |\langle \varphi_n, A^*\varphi_m\rangle|^2}\sqrt{\sum_n\sum_m |\langle \varphi_m, B\varphi_n\rangle|^2} \\
&= \sqrt{\sum_n\sum_m \langle A^*\varphi_m, \varphi_n\rangle\langle \varphi_n, A^*\varphi_m\rangle}\sqrt{\sum_n\sum_m \langle B\varphi_n, \varphi_m\rangle\langle \varphi_m, B\varphi_n\rangle} \\
&= \sqrt{\sum_m \langle \varphi_m, AA^*\varphi_m\rangle}\sqrt{\sum_n \langle \varphi_n, B^*B\varphi_n\rangle} \\
&= \operatorname{tr}\left(|A^*|^2\right)\operatorname{tr}\left(|B|^2\right) \\
&= \|A^*\|_{\text{HS}}^2\|B\|_{\text{HS}}^2\,.
\end{aligned}$$

However, since $\ker(|A|-\lambda) = \ker(|A^*|-\lambda)$ for all $\lambda \neq 0$, we get the result.

*Remark* 9.77*.* You may wonder how to define $|A|^p$. One easy answer that we will encounter later is that since $|A|$ is self-adjoint, we may apply to it the bounded measurable functional calculus (below). If $p \in \mathbb{N}$ then we can just use the polynomial functional calculus.

**Proposition 9.78.** *We have the inclusion, for any $p \geq 1$,*

$$\mathcal{J}_p(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})\,.$$

*Proof.* If the sum is finite the spectrum will be finite or countable with the origin as limit point, and hence a compact operator, via Theorem 9.60. In doing so we are invoking the spectral mapping theorem

$$\sigma(|A|^p) = \sigma(|A|)^p \qquad (p \geq 1)\,.$$

We could also mimic the proof of Claim 9.68. ☐

**Theorem 9.79.** *We have the following properties of $\mathcal{J}_p(\mathcal{H})$:*

1. *$\mathcal{J}_p(\mathcal{H})$ is a two-sided star-ideal within $\mathcal{B}(\mathcal{H})$.*
2. *If $1 \leq p \leq q \leq \infty$ then $\mathcal{J}_p(\mathcal{H}) \subseteq \mathcal{J}_q(\mathcal{H})$ (we have $\mathcal{J}_\infty(\mathcal{H}) \equiv \mathcal{B}(\mathcal{H})$), and for any $A \in \mathcal{J}_q(\mathcal{H})$, we have*
$$\|A\|_q \leq \|A\|_p .$$
3. *If $p, q, r \in [0, \infty]$ are such that*
$$\frac{1}{q} + \frac{1}{p} = \frac{1}{r}$$
*then for $A \in \mathcal{J}_p(\mathcal{H})$, $B \in \mathcal{J}_q(\mathcal{H})$ we have $AB \in \mathcal{J}_r(\mathcal{H})$ and moreover*
$$\|AB\|_r \leq \|A\|_p \|B\|_q .$$
*In particular, when $q = \infty$ (with the convention that $\|\cdot\|_\infty$ is the operator norm) we get*
$$\|AB\|_r \leq \|A\|_r \|B\|_{op} .$$
4. *The Schatten norms are unitarily invariant: if $U, V$ are unitary and $A \in \mathcal{J}_p(\mathcal{H})$ then*
$$\|UAV\|_p = \|A\|_p .$$
5. *For $p \in (0, 1)$, $\|\cdot\|_p$ is merely a* quasinorm.

*Proof.* We show first that $\mathcal{J}_p(\mathcal{H})$ is a two-sided ideal. To that end, let $A \in \mathcal{J}_p(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. Then we want to show that $AB, BA, A^* \in \mathcal{J}_p(\mathcal{H})$. As in Lemma 9.72 we may assume that $B$ is in fact unitary, and we then establish that
$$|UA|^p = |A|^p$$
whereas
$$|AU|^p = U^* |A|^p U$$
so that $UA$ and $AU$ are also Schatten-p.

Next, if $p \leq q$ we want to show that
$$\|A\|_q \quad \leq \quad \|A\|_p .$$
To that end, we want to show that
$$\left(\sum_n \sigma_n(A)^q\right)^{\frac{1}{q}} \quad \leq \quad \left(\sum_n \sigma_n(A)^p\right)^{\frac{1}{p}}$$
where $\{\sigma_n(A)\}_{n\in\mathbb{N}}$ are the singular values of $A$ (we know there are only discrete of them since $A$ is compact). This follows the usual proof that the $\ell^q$ norm is smaller than the $\ell^p$ norm: Let $x = \{\sigma_n(A)\}$ and assume WLOG that $\|x\|_q = 1$. Then $\sigma_n(A) \leq 1$ so
$$1 = \|x\|_q^q = \sum_n \sigma_n(A)^q \leq \sum_n \sigma_n(A)^p = \|x\|_p^p .$$

Hoelder: TODO. ☐

**Lemma 9.80.** *Let $p \in \mathbb{N}_{\geq 1}$ and $A \geq 0$ be of Schatten class $p$. Then given $0 < a < b \leq \|A\|$, the spectrum of $A$ with $(a, b)$, $\sigma(A) \cap (a, b)$, has a gap of size at least*
$$\frac{b - a}{1 + \|A\|_p^p / a^p} .$$

*Proof.* Since $A$ is Schatten $p$, it is in particular compact so that it may only have accumulation of eigenvalues at zero. As such there must be a finite number of eigenvalues of $A$ within the interval $(a,b)$. Say that number is $n\in\mathbb{N}$. Then since $A\geq 0$,

$$\|A\|_p^p = \operatorname{tr}(A^p)\geq a^p n\,.$$

Hence in the worst case we will have an interval of size at least, say, $(b-a)/(1+n)$ free of eigenvalues somewhere within $(a,b)$. □

**Lemma 9.81.** *For any operator $A\in\mathcal{B}(\mathcal{H})$, an ONB $\{\,\delta_x\,\}_{x\in\mathbb{Z}}$ of $\mathcal{H}$, and*

$$A_{xy} := \langle\delta_x, A\delta_y\rangle \qquad (x,y\in\mathbb{Z})$$

*we have the estimate*

$$\|A\|_p \leq \sum_{k\in\mathbb{Z}}\left(\sum_{x\in\mathbb{Z}}|A_{x+k,x}|^p\right)^{\frac{1}{p}}.$$

*where $\|A\|_p\equiv(\operatorname{tr}(|A|^p))^{\frac{1}{p}}$ is the Schatten-$p$ norm.*

*Proof.* Let us decompose $A$ to its diagonals as

$$A=\sum_{k\in\mathbb{Z}}A^{(k)}$$

defined via $\left(A^{(k)}\right)_{xy}\equiv A_{xy}\delta_{x-y,k}$ for all $k\in\mathbb{Z}$. Since $\|\cdot\|_p$ is a norm, applying the triangle inequality we find

$$\|A\|_p\leq\sum_{k\in\mathbb{Z}}\left\|A^{(k)}\right\|_p.$$

But now,

$$\begin{aligned}\left\|A^{(k)}\right\|_p &= \left(\operatorname{tr}\left(\left|A^{(k)}\right|^p\right)\right)^{\frac{1}{p}}\\ &= \left(\operatorname{tr}\left(\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}}\\ &= \left(\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}^{\frac{p}{2}}\right)^{\frac{1}{p}}\\ &= \sqrt{\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}}\,.\end{aligned}$$

But note that

$$\begin{aligned}\left(\left|A^{(k)}\right|^2\right)_{xy} &\equiv \left(\left(A^{(k)}\right)^*A^{(k)}\right)_{xy}\\ &= \sum_{z\in\mathbb{Z}}\left(\left(A^{(k)}\right)^*\right)_{xz}\left(A^{(k)}\right)_{zy}\\ &= \sum_{z\in\mathbb{Z}}\left(A_{zx}\delta_{z-x,k}\right)^*A_{zy}\delta_{z-y,k}\\ &= \delta_{x,y}\sum_{z\in\mathbb{Z}}\left(A_{zx}\delta_{z-x,k}\right)^*A_{zy}\delta_{z-y,k}\\ &= \delta_{x,y}\left|A_{x+k,x}\right|^2\,.\end{aligned}$$

Since $\left|A^{(k)}\right|^2$ is a-posteriori a diagonal operator, it is easy to calculate its Schatten-$\frac{p}{2}$ norm, since it is easy to take its powers. Indeed,

$$\left[\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right]_{xy} = \delta_{x,y}\left|A_{x+k,x}\right|^p$$

and so

$$\begin{aligned}\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}^{\frac{p}{2}} &= \operatorname{tr}\left(\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right)\\ &= \sum_{x\in\mathbb{Z}}\left[\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right]_{xx}\\ &= \sum_{x\in\mathbb{Z}}\left|A_{x+k,x}\right|^p .\end{aligned}$$

Collecting everything together we find

$$\begin{aligned}\|A\|_p &\leq \sum_{k\in\mathbb{Z}}\sqrt{\left(\sum_{x\in\mathbb{Z}}\left|A_{x+k,x}\right|^p\right)^{\frac{2}{p}}}\\ &\leq \sum_{k\in\mathbb{Z}}\left(\sum_{x\in\mathbb{Z}}\left|A_{x+k,x}\right|^p\right)^{\frac{1}{p}} .\end{aligned}$$

□

**Theorem 9.82.** *Let $\mathcal{H}$ be a separable Hilbert space. Then the Hilbert space $\mathcal{H}\otimes\mathcal{H}^*$ (where we take the Hilbert space completion of the algebraic tensor product) is isomorphic, as a Hilbert space, to the Hilbert space of Hilbert-Schmidt operators, i.e., $\mathcal{J}_2(\mathcal{H})$ with inner product*

$$\langle A,B\rangle_2 := \operatorname{tr}(A^*B) .$$

*Claim* 9.83. Let $X_n\to 0$ in operator norm and $Y\in J_1(\mathcal{H})$. Then $\|X_nY\|_1\to 0$ and $\|YX_n^*\|_1\to 0$.

### 9.7.2 The Fedosov formula

One we have the notion of a trace, we may go back and compute the Fredholm index using

**Theorem 9.84.** *(Fedosov) If $A\in\mathcal{F}(\mathcal{H})$ and $B$ is* any *parametrix of $A$ such that $\mathbb{1}-AB,\mathbb{1}-BA$ is finite rank, then*

$$\operatorname{index}(A) = \operatorname{tr}([A,B]) .$$

The following is taken from [Mur94].

*Proof.* First we note that if $B,B'$ are two such parametrii, then $B'=B+F$ for some $F$ finite rank. Then $[A,B']=[A,B]+[A,F]$ and since $F$ is finite rank, $\operatorname{tr}([A,F])=0$. So we are free to choose *any* such finite rank parametrix.

We claim there is a finite-rank parametrix $B$ such that $A=ABA$. If we can find such a parametrix, then $\mathbb{1}-AB$ and $\mathbb{1}-BA$ are idempotents, and their traces equal the dimension of the cokernel and kernel of $A$ respectively and hence the result.

To find this special $B$, since $A$ induces an isomorphism $\varphi:\ker(A)^\perp\to\operatorname{im}(A)$, by the bounded inverse theorem, $\varphi^{-1}$ is bounded. Let $B$ be any extension of $\varphi^{-1}$ to $\mathcal{H}$. Then it fulfills $A=ABA$. □

Another useful trace formula is the following:

*Claim* 9.85. If there is some $n\in\mathbb{N}$ such that $\mathbb{1}-|A|^2$ and $\mathbb{1}-|A^*|^2$ are $n$-Schatten class, then

$$\operatorname{index}(A) = \operatorname{tr}((\mathbb{1}-|A|)^n)-\operatorname{tr}((\mathbb{1}-|A^*|)^n) .$$

The proof is left as an exercise to the reader.

# 10 The spectral theorem for bounded normal operators

In this section we finally extend the functional calculus, i.e., we extend the class of functions $f : \mathbb{C} \to \mathbb{C}$ for which we can define an operator

$$f(A)$$

given some $A \in \mathcal{B}(\mathcal{H})$. To recall what we have so far:

1. The polynomial functional calculus is trivially defined for all elements in a Banach algebra.
2. The holomorphic functional calculus is defined as well for all meromorphic functions whose poles are outside the spectrum of any element $A$ of a Banach algebra.
3. The continuous functional calculus is defined for all normal elements of a C-star algebra.
4. The square root is defined for all positive elements of a C-star algebra.

We are now going to extend this as follows: For elements $A \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space, $A$ is normal ($|A|^2 = |A^*|^2$) and $f$ is Borel measurable and bounded. To define $f(A)$ will take us some steps, the most important of which will be to define the *spectral measures* for self-adjoint operators. We mostly follow [RS80],[AW15] and [Tes09].

## 10.1 Herglotz-Pick-Nevanlinna functions

**Definition 10.1** (Nevanlinna function)**.** Let

$$\mathbb{C}^+ := \{ z \in \mathbb{C} \mid \mathbb{Im}\{z\} > 0 \}$$

be the open upper half plane. A function $f : \mathbb{C}^+ \to \mathbb{C}^+$ which is holomorphic is called a Nevanlinna, Herglotz, Pick or R function.

Our main interest in such functions stems from

**Lemma 10.2.** *Let $\psi \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then*

$$\mathbb{C}^+ \ni z \mapsto \left\langle \psi, (A - z\mathbb{1})^{-1} \psi \right\rangle \equiv \langle R_A(z) \rangle_\psi$$

*is a Herglotz function.*

*Proof.* We know that $\sigma(A) \subseteq \mathbb{R}$ via Claim 8.9. So $\mathbb{C}^+ \ni z \mapsto \langle R_A(z) \rangle_\psi$ clearly makes sense, as $\mathbb{C}^+ \subseteq \rho(A)$, the resolvent set of $A$. Moreover, the function is holomorphic since its derivative exists:

$$\begin{aligned} \frac{\left\langle \psi, (A - (z+w)\mathbb{1})^{-1} \psi \right\rangle - \left\langle \psi, (A - z\mathbb{1})^{-1} \psi \right\rangle}{w} &= \frac{1}{w} \left\langle \psi, \left[ (A - (z+w)\mathbb{1})^{-1} - (A - z\mathbb{1})^{-1} \right] \psi \right\rangle \\ &= \frac{1}{w} \left\langle \psi, \left[ (A - (z+w)\mathbb{1})^{-1} (z - (z+w)) (A - z\mathbb{1})^{-1} \right] \psi \right\rangle \\ &= - \left\langle \psi, (A - (z+w)\mathbb{1})^{-1} (A - z\mathbb{1})^{-1} \psi \right\rangle \end{aligned}$$

so that taking the limit $w \to 0$ yields

$$- \left\langle \psi, (A - z\mathbb{1})^{-2} \psi \right\rangle .$$

Moreover, we have

$$\mathbb{Im}\left\{\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle\right\} \equiv \frac{1}{2\mathrm{i}}\left[\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle-\overline{\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle}\right]$$
$$= \frac{1}{2\mathrm{i}}\left[\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle-\left\langle\psi,(A-\overline{z}\mathbb{1})^{-1}\psi\right\rangle\right]$$
$$= \frac{1}{2\mathrm{i}}\left\langle\psi,\left[(A-z\mathbb{1})^{-1}-(A-\overline{z}\mathbb{1})^{-1}\right]\psi\right\rangle$$
$$= \frac{1}{2\mathrm{i}}\left\langle\psi,\left[(A-z\mathbb{1})^{-1}(z-\overline{z})(A-\overline{z}\mathbb{1})^{-1}\right]\psi\right\rangle$$
$$= \mathbb{Im}\{z\}\left\|(A-z\mathbb{1})^{-1}\psi\right\|^2 .$$

Hence if $\mathbb{Im}\{z\}>0$ and $\psi\neq 0$, we have $\mathbb{Im}\left\{\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle\right\}>0$ (note it is impossible for $(A-z\mathbb{1})^{-1}\psi=0$ since $A-z\mathbb{1}$ is invertible!). ☐

**Example 10.3.** Other interesting examples of Herglotz functions are

1. $z\mapsto c+\mathrm{i}d$, $z\mapsto c+dz$ for $c\in\mathbb{R}$, $d>0$:
2. $z\mapsto z^r$ for $r\in(0,1)$ with the appropriate branch.
3. log with the appropriate branch.
4. tan, $-$cot
5. $z\mapsto\frac{a+bz}{c+dz}$ for $\begin{bmatrix}c & d\\ a & b\end{bmatrix}=M$ which is Hermitian-symplectic: $M^*JM=J$ with $J\equiv\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$. A special case if $z\mapsto-\frac{1}{z}$.

*Claim* 10.4. If $m,n:\mathbb{C}^+\to\mathbb{C}^+$ are Herglotz functions then so are $m+n$ and $m\circ n$.

The main point of identifying the Herglotz property is the following representation theorem for such functions, whose proof we skip.

**Theorem 10.5.** *Let $f:\mathbb{C}^+\to\mathbb{C}^+$ be Herglotz. Then there exists a* unique *Borel measure $\mu_f$ on $\mathbb{R}$ satisfying*

$$\int_{x\in\mathbb{R}}\frac{1}{1+x^2}\mathrm{d}\mu_f(x)<\infty$$

*such that $f$ may be represented as*

$$f(z)=a+bz+\int_{x\in\mathbb{R}}\left[\frac{1}{x-z}-\frac{x}{1+x^2}\right]\mathrm{d}\mu_f(x)\qquad\left(z\in\mathbb{C}^+\right)$$

*where $a:=\mathbb{Re}\{f(\mathrm{i})\}$ and*

$$b:=\lim_{\eta\to\infty}\frac{f(\mathrm{i}\eta)}{\mathrm{i}\eta}\geq 0 .$$

Note it may well *happen* that $b=0$ as well as that

$$a=\int_{x\in\mathbb{R}}\frac{x}{1+x^2}\mathrm{d}\mu_f(x)$$

in which case we would have

$$f(z)=\int_{x\in\mathbb{R}}\frac{1}{x-z}\mathrm{d}\mu_f(x)\qquad\left(z\in\mathbb{C}^+\right) .$$

The RHS of the above equation is called the Borel-Stieltjes (or just Borel) transform of the measure $\mu_f$. The sub-class of Herglotz functions which are the Borel-Stieltjes transform of some measure turns out to play an important role in spectral theory.

Actually, in our application where

$$f(z) \quad := \quad \langle R_A(z)\rangle_\psi \qquad \left(z\in\mathbb{C}^+\right)$$

this indeed happens, as we shall see. We merely note that

$$\begin{aligned}
\mathbb{R}\mathrm{e}\{f(\mathrm{i}\eta)\} &\equiv \mathbb{R}\mathrm{e}\left\{\left\langle\psi,(A-\mathrm{i}\eta\mathbb{1})^{-1}\psi\right\rangle\right\}\\
&= \frac{1}{2}\left(\left\langle\psi,(A-\mathrm{i}\eta\mathbb{1})^{-1}\psi\right\rangle+\overline{\left\langle\psi,(A-\mathrm{i}\eta\mathbb{1})^{-1}\psi\right\rangle}\right)\\
&= \frac{1}{2}\left\langle\psi,\left[(A-\mathrm{i}\eta\mathbb{1})^{-1}+(A+\mathrm{i}\eta\mathbb{1})^{-1}\right]\psi\right\rangle\\
&= \frac{1}{2}\left\langle\psi,\left[(A-\mathrm{i}\eta\mathbb{1})^{-1}-(-A-\mathrm{i}\eta\mathbb{1})^{-1}\right]\psi\right\rangle\\
&= \frac{1}{2}\left\langle\psi,\left[(A-\mathrm{i}\eta\mathbb{1})^{-1}(-2A)(-A-\mathrm{i}\eta\mathbb{1})^{-1}\right]\psi\right\rangle\\
&= \left\langle\psi,\left[(A-\mathrm{i}\eta\mathbb{1})^{-1}A(A+\mathrm{i}\eta\mathbb{1})^{-1}\right]\psi\right\rangle\\
&= \left\langle\psi,\left[A\left(A^2+\eta^2\mathbb{1}\right)^{-1}\right]\psi\right\rangle
\end{aligned}$$

Moreover, we may calculate

$$\begin{aligned}
\mathbb{R}\mathrm{e}\left\{\int_{x\in\mathbb{R}}\frac{1}{x-\mathrm{i}\eta}\mathrm{d}\mu_f(x)\right\} &= \frac{1}{2}\left[\int_{x\in\mathbb{R}}\frac{1}{x-\mathrm{i}\eta}\mathrm{d}\mu_f(x)+\overline{\int_{x\in\mathbb{R}}\frac{1}{x-\mathrm{i}\eta}\mathrm{d}\mu_f(x)}\right]\\
&= \int_{x\in\mathbb{R}}\frac{1}{2}\left[\frac{1}{x-\mathrm{i}\eta}+\overline{\frac{1}{x-\mathrm{i}\eta}}\right]\mathrm{d}\mu_f(x)\\
&= \int_{x\in\mathbb{R}}\frac{x}{x^2+\eta^2}\mathrm{d}\mu_f(x)
\end{aligned}$$

We know that $\langle R_A(\mathrm{i}\eta)\rangle_\psi\to 0$ as $\eta\to\infty$ so that $b=0$ for us. Indeed:

*Remark* 10.6. For $A=A^*$, the function $\mathbb{C}^+\ni z\mapsto\langle R_A(z)\rangle_\psi$ obeys

$$\left|\langle R_A(z)\rangle_\psi\right| = \left|\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle\right| \leq \left\|(A-z\mathbb{1})^{-1}\right\|\|\psi\|^2 \leq \frac{\|\psi\|^2}{\mathbb{I}\mathrm{m}\{z\}}\,.$$

For this reason instead of proving the general representation theorem stated above we contend ourselves with a restricted version.

**Theorem 10.7.** *Let $f:\mathbb{C}^+\to\mathbb{C}^+$ be Herglotz such that there exists some $M\in(0,\infty)$ for which*

$$|f(z)|\leq\frac{M}{\mathbb{I}\mathrm{m}\{z\}} \qquad \left(z\in\mathbb{C}^+\right)\,. \tag{10.1}$$

*Then there is a Borel measure $\mu$ on on $\mathbb{R}$ such that $\mu(\mathbb{R})\leq M$ and such that $f$ is the Borel transform of $\mu$:*

$$f(z)=\int_{\lambda\in\mathbb{R}}\frac{1}{\lambda-z}\mathrm{d}\mu(\lambda)\,.$$

*Proof.* We mainly follow [Tes09].

We want to define a sequence of positive measures $\mu_\varepsilon$ for each $\varepsilon>0$ via the formula

$$\mu_\varepsilon(A) \quad := \quad \int_{x\in A}\frac{1}{\pi}\,\mathbb{I}\mathrm{m}\{f(\lambda+\mathrm{i}\varepsilon)\}\,\mathrm{d}\lambda \qquad (A\subseteq\mathbb{R}\text{ Borel measurable})\,.$$

To do so, let us first establish a few basic facts about the RHS integral. Let $x\in\mathbb{R}$ and $y>0$ be given. Then for $\varepsilon\in(0,y)$ and $R>0$ large, define the contour

$$\Gamma:=\{\,x+\mathrm{i}\varepsilon+\lambda\mid\lambda\in[-R,R]\,\}\cup\left\{\,x+\mathrm{i}\varepsilon+R\mathrm{e}^{\mathrm{i}\theta}\;\middle|\;\theta\in[0,\pi]\,\right\}\,.$$

Then $z := x + \mathrm{i}y$ lies within the interior of the contour whereas $\overline{z} + 2\mathrm{i}\varepsilon$ does not. Hence Cauchy's formula implies

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\zeta\in\Gamma} \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - \overline{z} - 2\mathrm{i}\varepsilon}\right] f(\zeta)\, \mathrm{d}\zeta\,.$$

The explicit form of $\Gamma$ implies

$$\begin{aligned} f(z) &= \frac{1}{\pi}\int_{-R}^{R} \frac{y-\varepsilon}{\lambda^2 + (y-\varepsilon)^2} f(x + \mathrm{i}\varepsilon + \lambda)\,\mathrm{d}\lambda + \\ &\quad + \frac{\mathrm{i}}{\pi}\int_0^{\pi} \frac{y-\varepsilon}{R^2\mathrm{e}^{2\mathrm{i}\theta} + (y-\varepsilon)^2} f\left(x + \mathrm{i}\varepsilon + R\mathrm{e}^{\mathrm{i}\theta}\right) R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\,. \\ &\overset{R\to\infty}{\to} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y-\varepsilon}{(\lambda - x)^2 + (y-\varepsilon)^2} f(\lambda + \mathrm{i}\varepsilon)\,\mathrm{d}\lambda\,. \end{aligned}$$

In the last limit, we have used (10.1):

$$\begin{aligned} \left|\frac{\mathrm{i}}{\pi}\int_0^{\pi} \frac{y-\varepsilon}{R^2\mathrm{e}^{2\mathrm{i}\theta} + (y-\varepsilon)^2} f\left(x + \mathrm{i}\varepsilon + R\mathrm{e}^{\mathrm{i}\theta}\right) R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\right| &\leq \frac{R(y-\varepsilon)M}{\left(R^2 - (y-\varepsilon)^2\right)\varepsilon} \\ &\to 0\,. \end{aligned}$$

Hence taking the imaginary part of both sides of the equation we learn that

$$\mathbb{Im}\{f(z)\} = \int_{-\infty}^{\infty} \frac{y-\varepsilon}{(\lambda - x)^2 + (y-\varepsilon)^2} \frac{1}{\pi} \mathbb{Im}\{f(\lambda + \mathrm{i}\varepsilon)\}\,\mathrm{d}\lambda\,.$$

But by (10.1) the LHS has the bound

$$\mathbb{Im}\{f(z)\} \leq |f(z)| \leq \frac{M}{y}\,.$$

Hence,

$$\begin{aligned} M &\geq y\,\mathbb{Im}\{f(z)\} \\ &= \int_{-\infty}^{\infty} \frac{(y-\varepsilon)y}{(\lambda - x)^2 + (y-\varepsilon)^2} \frac{1}{\pi} \mathbb{Im}\{f(\lambda + \mathrm{i}\varepsilon)\}\,\mathrm{d}\lambda \\ &\overset{y\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} \frac{1}{\pi} \mathbb{Im}\{f(\lambda + \mathrm{i}\varepsilon)\}\,\mathrm{d}\lambda \\ &= \mu_\varepsilon(\mathbb{R})\,. \end{aligned}$$

In particular, since

$$g_\varepsilon(\lambda) := \frac{y-\varepsilon}{\lambda^2 + (y-\varepsilon)^2}$$

has

$$|g_\varepsilon(\lambda) - g_0(\lambda)| \leq \varepsilon\left|\frac{y^2 - y\varepsilon}{y^4 - 2y^3\varepsilon}\right|$$

we have

$$\begin{aligned} \mathbb{Im}\{f(z)\} &= \int_{-\infty}^{\infty} g_\varepsilon(\lambda - x)\,\mathrm{d}\mu_\varepsilon(\lambda) \\ &= \int_{-\infty}^{\infty} g_0(\lambda - x)\,\mathrm{d}\mu_\varepsilon(\lambda) + \mathcal{O}\left(\varepsilon\left|\frac{y^2 - y\varepsilon}{y^4 - 2y^3\varepsilon}\right| M\right) \\ &\overset{\varepsilon\to 0^+}{\longrightarrow} \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} g_0(\lambda - x)\,\mathrm{d}\mu_\varepsilon(\lambda)\,. \end{aligned}$$

Now we have $\mu_\varepsilon(\mathbb{R}) \leq M$ so there is a subsequence of measures which converges weakly to some measure $\mu$, and so we have

$$\mathbb{Im}\{f(z)\} = \int_{\mathbb{R}} g_0(\lambda - x)\, \mathrm{d}\mu(\lambda)\,.$$

Since $f$ and $\int \frac{1}{\lambda - \cdot}\mathrm{d}\mu(\lambda)$ are two analytic functions on $\mathbb{C}_+$ that have the same imaginary part, they may only differ by a real constant. But the bound (10.1) implies this constant must be zero. ☐

*Claim* 10.8. Actually we have

$$\mathbb{Im}\{f(z)\} = \mathbb{Im}\{z\}\int_{\mathbb{R}} \frac{1}{|\lambda - z|^2}\mathrm{d}\mu(\lambda)$$

and

$$\lim_{\lambda\to\infty} \lambda\, \mathbb{Im}\{f(\mathrm{i}\lambda)\} = \mu(\mathbb{R})\,.$$

**Theorem 10.9.** *Let $f$ be the Borel transform of some finite Borel measure $\mu$. Then $\mu$ is unique and can be reconstructed via*

$$\frac{1}{2}\left(\mu\left((\lambda_1,\lambda_2)\right) + \mu\left([\lambda_1,\lambda_2]\right)\right) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \mathbb{Im}\{f(\lambda + \mathrm{i}\varepsilon)\}\,\mathrm{d}\lambda\,. \tag{10.2}$$

*Proof.* By Fubini we have

$$\begin{aligned}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \mathbb{Im}\{f(\lambda+\mathrm{i}\varepsilon)\}\,\mathrm{d}\lambda &= \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\left(\int_{x\in\mathbb{R}} \frac{\varepsilon}{(x-\lambda)^2+\varepsilon^2}\mathrm{d}\mu(x)\right)\mathrm{d}\lambda \\ &= \frac{1}{\pi}\int_{x\in\mathbb{R}}\left(\int_{\lambda_1}^{\lambda_2} \frac{\varepsilon}{(x-\lambda)^2+\varepsilon^2}\mathrm{d}\lambda\right)\mathrm{d}\mu(x)\,.\end{aligned}$$

But

$$\begin{aligned}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \frac{\varepsilon}{(x-\lambda)^2+\varepsilon^2}\mathrm{d}\lambda &= \frac{1}{\pi}\left(\arctan\left(\frac{\lambda_2 - x}{\varepsilon}\right) - \arctan\left(\frac{\lambda_1 - x}{\varepsilon}\right)\right) \\ &\overset{\varepsilon\to 0}{\to} \frac{1}{\pi}\left(\chi_{[\lambda_1,\lambda_2]}(x) + \chi_{(\lambda_1,\lambda_2)}(x)\right) \in [0,1]\,.\end{aligned}$$

pointwise. Thus we conclude by the dominated convergence theorem. ☐

It turns out that a lot about the measure can be deduced from the limit of the Borel transform on the real line, as the following statements show us. We should keep these statements in mind when discussing spectral measures of self-adjoint operators, in connection with their resolvents.

**Lemma 10.10.** *Let $\mu$ be a finite Borel measure and $f$ its Borel transform. Then*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}}(\lambda) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\mathbb{Im}\{f(\lambda+\mathrm{i}\varepsilon)\} \qquad (\lambda\in\mathbb{R})$$

*whenever the limit on the RHS exists, where $\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}}$ is the Radon-Nikodym derivative of $\mu$ w.r.t. the Lebesgue measure $\mathcal{L}$.*

We conclude that when $\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\mathbb{Im}\{f(\lambda+\mathrm{i}\varepsilon)\}$ exists, the associated measure $\mu$ to $f$ is absolutely-continuous w.r.t. the Lebesgue measure $\mathcal{L}$.

**Lemma 10.11** (Characterization of measure type via the Borel transform)**.** *Let $\mu$ be a finite Borel measure and $f$ its Borel transform. Then*

$$\lim_{\varepsilon\to0^+}\frac{1}{\pi}\mathbb{Im}\left\{f\left(\lambda+\mathrm{i}\varepsilon\right)\right\}$$

*exists a.e. w.r.t. both $\mu$ and $\mathcal{L}$. Moreover,*

$$\left\{\,\lambda\in\mathbb{R}\,\middle|\,\mathbb{Im}\left\{f\left(\lambda+\mathrm{i}0^+\right)\right\}=\infty\,\right\}$$

*and*

$$\left\{\,\lambda\in\mathbb{R}\,\middle|\,0<\mathbb{Im}\left\{f\left(\lambda+\mathrm{i}0^+\right)\right\}<\infty\,\right\}$$

*are the support of the singular and absolutely continuous parts of $\mu$ respectively. Moreover, the set of point masses of $\mu$ is given by*

$$\left\{\lambda\,\middle|\,\lim_{\varepsilon\to0^+}\varepsilon\,\mathbb{Im}\left\{f\left(\lambda+\mathrm{i}\varepsilon\right)\right\}>0\right\}.$$

*Proof.* See [Jak06]. □

## 10.2 The spectral measures associated to bounded self-adjoint operators

**Definition 10.12** (spectral measure of an operator and a vector)**.** Given the above considerations, given a self-adjoint operator $A=A^*$ and any $\psi\in\mathscr{H}$, we have a unique *positive* measure $\mu_{A,\psi}$ on $\mathbb{R}$, called the spectral measure of the duo $A,\psi$, such that $\mathbb{C}_+\ni z\mapsto\left\langle\psi,\left(A-z\mathbb{1}\right)^{-1}\psi\right\rangle$ is the Borel transform of $\mu_{A,\psi}$, i.e.,

$$\left\langle\psi,\left(A-z\mathbb{1}\right)^{-1}\psi\right\rangle=\int_{\lambda\in\mathbb{R}}\frac{1}{\lambda-z}\mathrm{d}\mu_{A,\psi}\left(\lambda\right)\qquad\left(z\in\mathbb{C}^+\right).$$

It has the property that

$$\mu_{A,\psi}\left(\mathbb{R}\right)=\left\|\psi\right\|^2$$

so that if $\|\psi\|=1$ we get a *probability measure* on $\mathbb{R}$. Moreover,

*Claim* 10.13. The support of $\mu_{A,\psi}$ is contained in the spectrum of $A$.

*Proof.* We have the inversion formula (10.2), and

$$\mathbb{Im}\left\{\left\langle\psi,\left(A-\left(\lambda+\mathrm{i}\varepsilon\right)\mathbb{1}\right)^{-1}\psi\right\rangle\right\}=\varepsilon\left\|\left(A-\left(\lambda+\mathrm{i}\varepsilon\right)\mathbb{1}\right)^{-1}\psi\right\|^2$$

and if $\lambda\notin\sigma\left(A\right)$, $A-\lambda\mathbb{1}$ is invertible so that

$$\sup_{\varepsilon>0}\left\|\left(A-\left(\lambda+\mathrm{i}\varepsilon\right)\mathbb{1}\right)^{-1}\psi\right\|^2<\infty\,.$$

As a result, these points will not be contained in the support of the unique measure defined above. □

Recall that we have the *polarization identity*,

$$\left\langle\psi,Z\varphi\right\rangle\equiv\frac{1}{4}\sum_{k=0}^{3}\mathrm{i}^k\left\langle\psi+\mathrm{i}^k\varphi,Z\left(\psi+\mathrm{i}^k\varphi\right)\right\rangle\,.\tag{10.3}$$

Using this identity we may define the spectral measures associated to *triplets* of an operator and two vectors:

**Definition 10.14** (Spectral measure of an operator and two vectors)**.** Given a self-adjoint operator $A = A^*$ and any two $\psi, \varphi \in \mathcal{H}$, we have a unique *complex measure* $\mu_{A,\psi,\varphi}$ on $\mathbb{R}$ given by

$$\mu_{A,\psi,\varphi} := \frac{1}{4}\left[\mu_{A,\psi+\varphi} - \mu_{A,\psi-\varphi} - \mathrm{i}\mu_{A,\psi+\mathrm{i}\varphi} + \mathrm{i}\mu_{A,\psi-\mathrm{i}\varphi}\right]$$

which obeys the property that

$$\left\langle \psi, (A - z\mathbb{1})^{-1}\varphi\right\rangle = \int_{\lambda\in\mathbb{R}} \frac{1}{\lambda - z}\mathrm{d}\mu_{A,\psi,\varphi}(\lambda) \qquad \left(z \in \mathbb{C}^+\right) .$$

## 10.3 The bounded measurable functional calculus

**Definition 10.15** (The bounded measurable functional calculus)**.** Given any measurable $f \in L^\infty(\mathbb{R} \to \mathbb{C})$, $A = A^* \in \mathcal{B}(\mathcal{H})$ we want to define an operator $f(A)$ via its inner products. To that end, for any $\psi, \varphi \in \mathcal{H}$, we define

$$\langle \psi, f(A)\varphi\rangle := \int_{\lambda\in\mathbb{R}} f(\lambda)\,\mathrm{d}\mu_{A,\psi,\varphi}(\lambda) .$$

Clearly we have

$$\left|\int_{\lambda\in\mathbb{R}} f(\lambda)\,\mathrm{d}\mu_{A,\psi,\varphi}(\lambda)\right| \le \|f\|_\infty \|\psi\|\|\varphi\|$$

by the polarization identity.

Once we have the operator $f(A)$ defined in the inner-product for any pair $\psi, \varphi \in \mathcal{H}$, it is uniquely defined via Theorem 7.14, so we get an actual operator $f(A) \in \mathcal{B}(\mathcal{H})$, with

$$\begin{aligned}\|f(A)\| &= \sup(\{\, |\langle \psi, f(A)\psi\rangle| : \|\psi\| = 1 \,\}) \\ &= \sup_{\lambda\in\sigma(A)} |f(\lambda)|\end{aligned}$$

where in the last step we have used Claim 10.13.

**Theorem 10.16.** *The functional calculus obeys*

$$\begin{aligned}(f+g)(A) &= f(A) + g(A) \\ (fg)(A) &= f(A)\,g(A) \\ \left(\overline{f}\right)(A) &= (f(A))^*\end{aligned}$$

*i.e., the functional calculus is a star-homomorphism from functions to operators, and $f \ge 0$ leads to $f(A) \ge 0$. Moreover, we have*

$$(x \mapsto x)(A) = A$$

*and if $f_n \to f$ pointwise and $\sup_n \|f_n\|_{L^\infty} < \infty$ then $f_n(A) \to f(A)$ strongly. Next, if $[A, B] = 0$ then $[f(A), B] = 0$. Finally,*

$$\ker(A - \lambda\mathbb{1}) = \ker(f(A) - f(\lambda)\mathbb{1}) .$$

*Proof.* The linearity and star property may be deduced from

$$\langle \psi, f(A)\varphi\rangle \equiv \int_{\lambda\in\mathbb{R}} f(\lambda)\,\mathrm{d}\mu_{A,\psi,\varphi}(\lambda) .$$

The multiplicative property is proven using

$$\begin{aligned}\langle\psi, f(A)\, g(A)\,\varphi\rangle &\equiv \left\langle\psi, f(A)\sum_n e_n\otimes e_n^* g(A)\,\varphi\right\rangle \\ &= \sum_n\int_{\lambda\in\mathbb{R}} f(\lambda)\,\mathrm{d}\mu_{A,\psi,e_n}(\lambda)\int_{\lambda\in\mathbb{R}} g(\lambda)\,\mathrm{d}\mu_{A,e_n,\varphi}(\lambda)\end{aligned}$$

and now we'd like to show that

$$\sum_n \mathrm{d}\mu_{A,\psi,e_n}(\lambda)\mathrm{d}\mu_{A,e_n,\varphi}\left(\tilde{\lambda}\right) = \mathrm{d}\mu_{A,\psi,\varphi}(\lambda)\delta\left(\lambda-\tilde{\lambda}\right)\,.$$

This is left as an exercise to the reader.

Now, the measure is fully determined by

$$\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\mathbb{Im}\left\{\left\langle\psi,(A-\lambda-\mathrm{i}\varepsilon)^{-1}\varphi\right\rangle\right\}\mathrm{d}\lambda\,.$$

If $f\geq 0$ we have

$$\langle\psi, f(A)\,\psi\rangle\geq 0$$

so that $f(A)\geq 0$. Clearly,

$$\langle\psi, A\varphi\rangle = \langle\psi,(x\mapsto x)(A)\,\varphi\rangle\equiv\int_{\lambda\in\mathbb{R}}\lambda\mathrm{d}\mu_{A,\psi,\varphi}(\lambda)\,.$$

If $[A,B]=0$ then

$$[R_A(z)\,,B]=0$$

which implies the same thing about the spectral projections:

$$\langle\psi, R_A(z)\,B\varphi\rangle = \langle\psi, BR_A(z)\,\varphi\rangle\ .$$

If $\psi\in\ker(A-\lambda\mathbb{1})$ then $A\psi=\lambda\psi$ so that

$$\langle\varphi, R_A(z)\,\psi\rangle = \frac{1}{\lambda-z}\langle\varphi,\psi\rangle$$

and from there the same thing for the functional calculus.

The limit of functions implies the strong limit due to the fact the spectral measures are defined diagonally. ☐

## 10.4 Projection-valued measures

We begin with a new notion

**Definition 10.17** (Projection-valued measures)**.** Let $\{\,P_\Omega\,\}_{\Omega\subseteq\mathbb{R}}$ be a sequence of self-adjoint projections on $\mathscr{H}$ indexed by measurable subsets of $\mathbb{R}$, which obey

1. $P_\varnothing = 0$ and $P_{(-a,a)}=\mathbb{1}$ for some $a>0$.
2. If $\Omega=\bigcup_{n=1}^\infty\Omega_n$ with $\Omega_n\cap\Omega_m=\varnothing$ for all $n\neq m$ then

$$P_\Omega = \operatorname*{s-lim}_{N\to\infty}\sum_{n=1}^N P_{\Omega_n}\,.$$

3. For any $\Omega_1,\Omega_2$ measurable,

$$P_{\Omega_1}P_{\Omega_2}=P_{\Omega_1\cap\Omega_2}\,.$$

Then $\{\,P_\Omega\,\}_{\Omega\subseteq\mathbb{R}}$ is called a *projection-valued measure.*

These properties allow us now to define the

**Definition 10.18** (Projection-valued measures)**.** Given any $A = A^* \in \mathcal{B}(\mathcal{H})$, we define a map

$$S \mapsto \mathcal{B}(\mathcal{H})$$

from Borel measurable sets to self-adjoint projections on Hilbert space as follows. The characteristic function

$$\chi_S(\lambda) := \begin{cases} 1 & \lambda \in S \\ 0 & \lambda \notin S \end{cases}$$

is bounded and measurable for measurable subsets, and so $\chi_S(A)$ makes sense. *According to the above definition, $\chi_\cdot(A)$ is a projection-valued measure.*

It is comforting to know that the two guises of the spectral theorem agree, in the sense that

**Theorem 10.19.** *We have for any $\psi, \varphi \in \mathcal{H}$,*

$$\langle \psi, \chi_\cdot(A)\varphi \rangle = \mu_{A,\psi,\varphi}\,.$$

*Proof.* We have, by definition, for any Borel set $S \subseteq \mathbb{R}$,

$$\langle \psi, \chi_S(A)\varphi \rangle \equiv \int_{x\in S} \mathrm{d}\mu_{A,\psi,\varphi}(x) \equiv \mu_{A,\psi,\varphi}(S)\,.$$

□

We may now also define the functional calculus now using the projection-valued measures alone, without reference for $\mu_{A,\psi,\varphi}$ itself. This is done as follows: Let $\chi_\cdot(A)$ be the projection-valued measure of $A = A^*$. Then for any $f \in L^\infty(\mathbb{R})$, we have

$$f(A) \equiv \int_{x\in\mathbb{R}} f(x)\,\mathrm{d}\chi_x(A)\,. \tag{10.4}$$

**Theorem 10.20** (Stone's formula)**.** *We have for any $A = A^* \in \mathcal{B}(\mathcal{H})$,*

$$\operatorname*{s-lim}_{\varepsilon\to 0^+} \frac{1}{2\pi\mathrm{i}} \int_{\lambda_1}^{\lambda_2} \left(R_A(\lambda+\mathrm{i}\varepsilon) - R_A(\lambda-\mathrm{i}\varepsilon)\right)\mathrm{d}\lambda = \frac{1}{2}\left(\chi_{(\lambda_1,\lambda_2)}(A) + \chi_{[\lambda_1,\lambda_2]}(A)\right)\,.$$

*Proof.* First we note that for $E \in \mathbb{R}$,

$$\frac{1}{2\mathrm{i}}\left[\frac{1}{E-\lambda-\mathrm{i}\varepsilon} - \frac{1}{E-\lambda+\mathrm{i}\varepsilon}\right] = \mathbb{Im}\left\{\frac{1}{E-\lambda-\mathrm{i}\varepsilon}\right\}\,.$$

Next, note that

$$\lim_{\varepsilon\to 0^+} \int_{\lambda=\lambda_1}^{\lambda_2} \frac{1}{\pi}\,\mathbb{Im}\left\{\frac{1}{E-\lambda-\mathrm{i}\varepsilon}\right\}\mathrm{d}\lambda = \begin{cases} 1 & E \in (\lambda_1,\lambda_2) \\ 0 & E \notin [\lambda_1,\lambda_2] \\ \frac{1}{2} & E \in \{\lambda_1,\lambda_2\} \end{cases}\,.$$

This can be deduced, e.g., using the "Sokhotski–Plemelj theorem" or "Kramers-Kronig relation".

But now, invoking Theorem 10.16, in particular, that if $f_n \to f$ in $L^\infty$ then $f_n(A) \to f(A)$ strongly, we obtain the desired result. □

## 10.5 The multiplicative form of the spectral theorem

**Definition 10.21** (Cyclic vectors)**.** A vector $\psi \in \mathcal{H}$ is called *cyclic* for $A \in \mathcal{B}(\mathcal{H})$ iff

$$\operatorname{span}(\{A^n\psi\}_n) = \operatorname{span}\left(\left\{(A - z\mathbb{1})^{-1}\psi \,\middle|\, z \in \mathbb{C}\setminus\mathbb{R}\right\}\right)$$

(in the sense of finite-linear combinations) is dense in $\mathcal{H}$.

**Lemma 10.22.** *If $A=A^\ast\in\mathcal{B}(\mathcal{H})$ and $\psi\in\mathcal{H}$ is a cyclic vector for $A$ then there is a unitary*

$$U:\mathcal{H}\to L^2(\mathbb{R},\mu_{A,\psi})$$

*such that*

$$(UAU^\ast f)(\lambda)=\lambda f(\lambda)\qquad(\lambda\in\sigma(A))\,.$$

*Proof.* Define $U$ by mapping

$$\mathcal{H}\ni A^n\psi\mapsto(\lambda\mapsto\lambda^n)\qquad(n\in\mathbb{N}_{\geq 0})\,.$$

Since $\psi\in\mathcal{H}$ is cyclic, this definition (and its extension by linearity) covers a dense subspace of vectors in $\mathcal{H}$. Moreover, clearly this map is an isometry:

$$\begin{aligned}\|\lambda\mapsto\lambda^n\|_{L^2(\mathbb{R},\mu_{A,\psi})} &= \int_{\lambda\in\mathbb{R}}|\lambda|^{2n}\,\mathrm{d}\mu_{A,\psi}(\lambda)\\ &= \left\langle\psi,|A^n|^2\psi\right\rangle\\ &= \|A^n\psi\|^2\,.\end{aligned}$$

To extend it to all of $\mathcal{H}$ we use the "bounded linear extension theorem" (9.1). It extends as an isometry. It is clearly injective, because if $f=g$ $\mu_{A,\psi}$-a.e., then

$$\begin{aligned}\int|f|^2\,\mathrm{d}\mu_{A,\psi} &= 0\\ &\updownarrow\\ \|f(A)\psi\|^2 &= 0\\ &\updownarrow\\ f(A)\psi &= 0\,.\end{aligned}$$

Hence $U$ defined as such is a unitary. □

**Lemma 10.23.** *If $A=A^\ast\in\mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ is separable, then $\mathcal{H}$ is graded as*

$$\mathcal{H}=\bigoplus_{j=1}^{N}\mathcal{H}_j$$

*with $N\in\mathbb{N}_{\geq 1}\cup\{\infty\}$ such that:*

1. *$A\mathcal{H}_j\subseteq\mathcal{H}_j$.*
2. *For each $j$, $A|_{\mathcal{H}_j}$ has a cyclic vector:*

$$\mathcal{H}_j=\overline{\operatorname{span}(\{A^l\varphi\mid l\in\mathbb{N}\})}$$

*for some $\varphi\in\mathcal{H}_j$.*

*Proof.* By Zorn's lemma. □

**Theorem 10.24** (Spectral theorem in its multiplicative form)**.** *Let $A=A^\ast\in\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ separable. Then there is some $N\in\mathbb{N}_{\geq 1}\cup\{\infty\}$ and measures $\{\mu_n\}_{n=1}^N$ on $\mathbb{R}$ such that there is a unitary operator*

$$U:\mathcal{H}\to\bigoplus_{n=1}^{N}L^2(\mathbb{R},\mu_n)$$

*such that*

$$(UAU^\ast f)_n(\lambda)=\lambda f_n(\lambda)\qquad\left(f\in L^2(\mathbb{R},\mu_n)\right)\,.$$

*Proof.* Apply the spectral theorem on each invariant cyclic subspace separately. ☐

We note that these measures are *not* uniquely defined.

**Example 10.25.** Let $R$ be the bilateral right shift operator on $\ell^2(\mathbb{Z})$. Define

$$A := R + R^* .$$

It is self-adjoint. How to realize it as a multiplication operator? Set up the Fourier series as the unitary map

$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$$

which realizes $\mathcal{F} A \mathcal{F}^* : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ as a multiplication operator by the map

$$[0, 2\pi] \ni \theta \mapsto 2\cos(\theta) .$$

To realize $A$ as a multiplication operator by a map

$$\lambda \mapsto \lambda$$

we must take two direct sum factors. These are the two possible directions of momentum.

**Example 10.26.** On $L^2(\mathbb{R})$, we have $P := -\mathrm{i}\partial$. This operator is unbounded, but via the (unitary) Fourier transform we get a multiplication operator by the function

$$\mathbb{R} \ni p \mapsto p .$$

## 10.6 Decomposition of spectrum

The Lebesgue decomposition theorem says that any regular measure $\mu$ on $\mathbb{R}$ may be decomposed as

$$\mu = \mu_{\text{pp}} + \mu_{\text{ac}} + \mu_{\text{sc}}$$

where $\mu_\sharp$ is the pure-point, absolutely-continuous and singular-continuous parts respectively. Indeed, we know that any regular measure $\mu$ on $\mathbb{R}$ may be written as

$$\mu = \mu_{\text{sing}} + \mu_{\text{ac}}$$

w.r.t. the Lebesgue measure, where $\mu_{\text{ac}}$ obeys: if a measurable set $S \subseteq \mathbb{R}$ has measure zero then $\mu_{\text{ac}}(S) = 0$. $\mu_{\text{sing}}$ is singular w.r.t. the Lebesgue measure, meaning that there is a decomposition of $\mathbb{R}$ into two disjoint parts, such that $\mu_{\text{sing}}$ is zero on all measurable subsets of the first constituent and the Lebesgue measure is zero on all measurable subsets of the second constituent. Now $\mu_{\text{sing}}$ is further decomposed into the point masses (pure point) and the rest, i.e.,

$$\mu_{\text{pp}} = \sum_{j\in\mathbb{N}} a_j \delta(\cdot - x_j) .$$

These three pieces are mutually singular, which implies

$$L^2(\mathbb{R}, \mu) = \bigoplus_{\sharp\in\{\text{ pp,ac,sc }\}} L^2(\mathbb{R}, \mu_\sharp) .$$

We thus define, for any fixed $A = A^* \in \mathcal{B}(\mathcal{H})$,

$$\mathcal{H}_\sharp := \{ \psi \in \mathcal{H} \mid \mu_{A,\psi} \text{ is } \sharp \}$$

and obtain

$$\mathcal{H} = \bigoplus_{\sharp\in\{\text{ pp,ac,sc }\}} \mathcal{H}_\sharp .$$

## 10.7 Extension to normal operators

So far we have obtained the spectral theorem for self-adjoint operators. We now want to extend this to normal operators. We do it in two steps: (1) generalize the spectral theorem to any number of commuting operators, and (2) write any normal as a sum of two commuting self-adjoints (its real and imaginary parts):

$$A = \underbrace{\frac{1}{2}\left(A + A^*\right)}_{\equiv \mathbb{Re}\{A\}} + \mathrm{i}\underbrace{\frac{1}{2\mathrm{i}}\left(A - A^*\right)}_{\equiv \mathbb{Im}\{A\}}\,.$$

Normality of $A$, $[A, A^*] = 0$, guarantees that

$$\left[\mathbb{Re}\{A\}, \mathbb{Im}\{A\}\right] = 0\,.$$

This construction has been delegated to the homework (in particular, see HW11).

## 10.8 Kuiper's theorem

It is a striking fact, due to Kuiper, that any invertible operator can be path-connected (within invertibles) to $\mathbb{1}$:

**Theorem 10.27.** *(Kuiper's theorem) The invertible operators are contractible within $\mathcal{B}(\mathcal{H})$.*

*Proof.* Let $A \in \mathcal{B}(\mathcal{H})$. We wish to find a continuous path to $\mathbb{1}$. Using the polar decomposition, write

$$A \;=\; U|A|$$

where $|A| \equiv \sqrt{A^*A}$ and $U = A|A|^{-1}$ is unitary. Since

$$\log : \mathbb{S}^1 \to \mathrm{i}\,[0, 2\pi]$$

is a bounded measurable function, we get some self-adjoint operator $\frac{1}{\mathrm{i}}\log(U) =: H$. Thanks to the functional calculus, it obeys $U = \mathrm{e}^{\mathrm{i}H}$. Define the map

$$\gamma : [0,1] \;\to\; \mathrm{GL}(\mathcal{B}(\mathcal{H}))$$

via

$$\gamma(t) \;=\; \left((1-t)|A| + t\mathbb{1}\right)\mathrm{e}^{\mathrm{i}(1-t)H}\,.$$

Then $\gamma(0) = A$ and $\gamma(1) = \mathbb{1}$, $\gamma$ is norm continuous and $\gamma(t)$ is invertible because $|A| > 0$. □

# 11 Unbounded operators and their spectral theorem

The material for this chapter is mainly taken from [RS80, Jak06].

It turns out that there are many situations in mathematics and in physics for which bounded operators are insufficient. The most typical example is that of the position operator $X$ on $L^2(\mathbb{R})$:

$$(X\psi)(x) \equiv x\psi(x)\,.$$

Coming from quantum mechanics, we interpret $X$ as the position operator, i.e., the observable whose measurement yields the position of the particle. As such its spectrum are all possible position measurement outcomes, so that

$$\sigma(X) = \mathbb{R}$$

and we expect that $X = X^*$. But since the spectral radius $r(X)$ equals $\|X\|$ for a self-adjoint operator, it must be that

$$\|X\| = \infty\,.$$

Said differently, we know e.g. that $x \mapsto \frac{1}{\sqrt{1+x^2}}$ belongs in $L^2(\mathbb{R})$. However, applying the position operator on this function yields

$$x \mapsto \frac{x}{\sqrt{1+x^2}}$$

which is *not* in $L^2(\mathbb{R})$. So it can't be that $X : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as defined above is a well-defined map. We rather need to define it on a smaller subset of $L^2(\mathbb{R})$.

## 11.1 The domain of unbounded operators

Let $\mathcal{H}$ be a separable Hilbert space. The first thing we want to see is that necessarily we *must* allow unbounded operators to be a defined on a *subset* of $\mathcal{H}$ so that they may well-defined linear functions.

**Theorem 11.1** (Hellinger–Toeplitz theorem)**.** *Let $A : \mathcal{H} \to \mathcal{H}$ be a* symmetric *linear map, in the sense that*

$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle \qquad (\varphi, \psi \in \mathcal{H}) \ .$$

*Then $\|A\| < \infty$.*

*Proof.* Recall the closed graph theorem Theorem 3.37 which says that $\|A\| < \infty$ iff $\Gamma(A) \in \text{Closed}\left(\mathcal{H}^2\right)$, where

$$\Gamma(A) \equiv \left\{ (\varphi, \psi) \in \mathcal{H}^2 \ \middle|\ \psi = A\varphi \right\} \ .$$

is the graph of $A$. Let then $\{ (\varphi_n, \psi_n) \}_n \subseteq \Gamma(A)$. Then $\psi_n = A\varphi_n$ by definition. Assume that

$$\lim_n (\varphi_n, \psi_n) = (\varphi, \psi) \in \mathcal{H}^2$$

for some $(\varphi, \psi)$. We want to show that $(\varphi, \psi) \in \Gamma(A)$, i.e., that $\psi = A\varphi$.

To that end, let $\xi \in \mathcal{H}$. Then

$$\langle \xi, \psi \rangle = \left\langle \xi, \lim_n \psi_n \right\rangle = \left\langle \xi, \lim_n A\varphi_n \right\rangle = \lim_n \langle \xi, A\varphi_n \rangle = \lim_n \langle A\xi, \varphi_n \rangle = \left\langle A\xi, \lim_n \varphi_n \right\rangle = \langle A\xi, \varphi \rangle = \langle \xi, A\varphi \rangle \ .$$

Since this is true for arbitrary $\xi$, we conclude $\psi = A\varphi$ and hence the conclusion. □

As a result, we conclude that if, for a symmetric operator $A$, $\sigma(A)$ is *unbounded* and hence $\|A\| = \infty$ then we must allow $A$ to be defined on a *subset* of $\mathcal{H}$. Since we are mostly interested in symmetric operators, we are left to the following generalized definition of an operator, which applies also to *unbounded* operators. It supersedes our previous definition of a bounded linear operator.

**Definition 11.2** (operator)**.** An (possibly unbounded) operator on a separable Hilbert space $\mathcal{H}$ is a pair $(\mathcal{D}, A)$ where $\mathcal{D}$ is a vector subspace of $\mathcal{H}$ (closed or not) and $A$ is linear map $A : \mathcal{D} \to \mathcal{H}$. The space $\mathcal{D}$ is called *the domain* of the operator, and sometimes is denoted by $\mathcal{D}(A)$. We say that $A$ is *densely defined* iff $\mathcal{D}$ is dense in $\mathcal{H}$. We set

$$\begin{aligned} \ker(A) &:= \{ \psi \in \mathcal{D} \mid A\psi = 0 \} \\ \operatorname{im}(A) &:= \{ A\psi \in \mathcal{H} \mid \psi \in \mathcal{D} \} \ . \end{aligned}$$

We likewise extend the definition of the operator norm as

$$\|A\| \equiv \sup\left(\{ \|A\psi\| \mid \psi \in \mathcal{D} : \|\psi\| = 1 \}\right) \ .$$

Since operators on Hilbert space have natural operations (of addition, composition, etc) the question of the domain of the resulting operators arises. We *set*

$$\begin{aligned} \mathcal{D}(A+B) &:= \mathcal{D}(A) \cap \mathcal{D}(B) \\ \mathcal{D}(AB) &:= \{ \psi \in \mathcal{D}(B) \mid B\psi \in \mathcal{D}(A) \} \ . \end{aligned}$$

We emphasize that what specifies an unbounded operator is the *duo* of both the domain of definition and the map on which it is defined. As such, changing the domain results in a *new* unbounded operator. Be that as it may, it is very inconvenient to keep track of two rather than one symbols for each operator, and so, from now on, we will rather refer to an operator by $A$ (with the understanding that we really mean $(\mathcal{D}(A), A)$) and when necessary to refer to its domain use $\mathcal{D}(A)$.

**Definition 11.3** (Extension)**.** The operator $B$ is an *extension* of $A$ iff

$$\mathcal{D}(A) \subseteq \mathcal{D}(B)$$

and

$$A\psi = B\psi \qquad (\psi \in \mathcal{D}(A)) \; .$$

We then denote this situation by $A \subseteq B$.

The graph of this generalized notion of an operator is defined in a natural way as

$$\Gamma(A) \equiv \left\{ (\psi, A\psi) \in \mathcal{H}^2 \,\middle|\, \psi \in \mathcal{D}(A) \right\} \; .$$

(This notation means the pair of vectors $\psi$ and $A\psi$, *not* their inner product).

*Claim* 11.4. For two unbounded operators, $A \subseteq B$ iff $\Gamma(A) \subseteq \Gamma(B)$.

*Proof.* Assume that $A \subseteq B$. Let $(\psi, \varphi) \in \Gamma(A)$. Then $\psi \in \mathcal{D}(A)$. But $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, so $\psi \in \mathcal{D}(B)$. Also, $\varphi = A\psi$. But on $\mathcal{D}(A)$, $A = B$, so we also have $\varphi = B\psi$. Thus $(\psi, \varphi) \in \Gamma(B)$.

Conversely, if $\Gamma(A) \subseteq \Gamma(B)$, if $\psi \in \mathcal{D}(A)$ then $(\psi, A\psi) \in \Gamma(A)$ and so $(\psi, A\psi) \in \Gamma(B)$. But then $\psi \in \mathcal{D}(B)$. Also, that means previous equation means $A\psi = B\psi$ so that $A = B$ on $\mathcal{D}(A)$ as desired. □

**Definition 11.5** (Bounded (generalized) operator)**.** We say that $A$ is bounded iff $\mathcal{D}(A) = \mathcal{H}$ and

$$\|A\| < \infty \, .$$

We then recover the previous definition from Section 9. Note that some sources don't require $\mathcal{D}(A) = \mathcal{H}$ to be bounded and instead only ask that $\|A\| < \infty$.

*Claim* 11.6 ("BLT lemma"). If $A$ is an operator such that $\overline{\mathcal{D}(A)} = \mathcal{H}$ (i.e. it is *densely defined*) and

$$\|A\psi\| \leq C\|\psi\| \qquad (\psi \in \mathcal{D}(A))$$

then $A$ has a *unique* extension to a bounded operator on $\mathcal{H}$.

*Proof.* We follow the same proof as in Theorem 9.25: Define $\tilde{A} : \mathcal{H} \to \mathcal{H}$ as follows. For all $\psi \in \mathcal{D}(A)$, define

$$\tilde{A}\psi := A\psi \, .$$

Now let $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$ and we need to define $\tilde{A}\psi$. By density, there is some $\{ \varphi_n \}_n \subseteq \mathcal{D}(A)$ such that $\varphi_n \to \psi$. Thus define

$$\tilde{A}\psi := \lim_n A\varphi_n \, .$$

The limit exists because

$$\|A\varphi_n - A\varphi_m\| = \|A(\varphi_n - \varphi_m)\| \leq C\|\varphi_n - \varphi_m\| \, .$$

For uniqueness, assume that $B \in \mathcal{B}(\mathcal{H})$ is some other (bounded) operator such that $B\psi = A\psi$ for all $\psi \in \mathcal{D}(A)$. Then by density, for $\psi = \lim_n \varphi_n$ with $\varphi_n \in \mathcal{D}(A)$,

$$B\psi = B \lim_n \varphi_n \stackrel{\star}{=} \lim_n B\varphi_n = \lim_n A\varphi_n \equiv \tilde{A}\psi$$

where in $\star$ we have used that $B$ is bounded, and hence continuous: Theorem 2.24. □

**Example 11.7** (The position operator revisited)**.** On $\mathcal{H} = L^2(\mathbb{R})$, we return to the position operator discussed above, $X$. We define

$$\mathcal{D}(X) := \left\{ \varphi \in L^2(\mathbb{R}) \,\middle|\, \int_{x\in\mathbb{R}} x^2 |\varphi(x)|^2 \, \mathrm{d}x < \infty \right\} \, .$$

Clearly, this is a vector subspace. We can make $\|X\varphi\|$ arbitrarily large while keeping $\|\varphi\| = 1$ by taking, e.g.,

$$\varphi_a(x) := \exp\left(-\frac{1}{2}\pi(x-a)^2\right)$$

whence $\|\varphi_a\| = 1$ and

$$\begin{aligned}\|X\varphi_a\|^2 &\equiv \int_{x\in\mathbb{R}} x^2 \exp\left(-\pi(x-a)^2\right)\mathrm{d}x \\ &= \int_{x\in\mathbb{R}} (x+a)^2 \exp\left(-\pi x^2\right)\mathrm{d}x \\ &= a^2 + \frac{1}{2\pi}.\end{aligned}$$

As a result, we conclude $\|X\|$ cannot be finite. It is also clear that there are no extensions of $X$ since $\mathcal{D}(X)$ is the largest space on which $X$ will land vectors from $L^2$ back in $L^2$.

**Definition 11.8** (Closed and closable operator)**.** An operator $A$ is called *closed* iff its graph

$$\Gamma(A) \equiv \left\{ (\psi, A\psi) \in \mathcal{H}^2 \,\middle|\, \psi \in \mathcal{D}(A) \right\}$$

is a closed subset of $\mathcal{H}^2$. *We emphasize that having a closed graph does NOT imply the domain needs to be closed.* $A$ is called *closable* iff it has a closed extension. If an operator is closable, its *smallest closed extension* is called its *closure* and is denoted by $\overline{A}$ (with domain $\mathcal{D}(\overline{A})$).

The significance of closed operators stems from the close graph theorem Theorem 3.37, which gives us some sense of boundedness of $A : \mathcal{D}(A) \to \mathcal{H}$ on its domain. Be careful, however, that even if $A$ is "closed", $\mathcal{D}(A)$ may not be closed and hence is not even a Banach space. So one cannot apply the closed graph theorem directly.

*Claim* 11.9. If $A : \mathcal{D}(A) \to \mathcal{H}$ is bounded, then $\mathcal{D}(A)$ is closed iff $A$ is closed.

*Proof.* Assume $\mathcal{D}(A)$ is closed. Then since $A : \mathcal{D}(A) \to \mathcal{H}$ is a bounded map between two Banach spaces, by the closed graph theorem Theorem 3.37, it has $\Gamma(A)$ closed, i.e., $A$ is closed. Conversely, if $\Gamma(A)$ is closed, taking $\{\psi_n\}_n \in \mathcal{D}(A)$ such that $\psi_n \to \psi$ for some $\psi \in \mathcal{H}$, we want to show $\psi \in \mathcal{D}(A)$. We have

$$(\psi_n, A\psi_n) \in \Gamma(A) \qquad (n \in \mathbb{N})$$

and

$$\lim_n (\psi_n, A\psi_n) = \left(\psi, \lim_n A\psi_n\right) \overset{\text{cont. of } A}{=} (\psi, A\psi)$$

this latter point lies in $\Gamma(A)$ since it is closed, and hence $\psi \in \mathcal{D}(A)$. ☐

**Corollary 11.10.** *Any bounded operator defined on a non-closed domain is not closed.*

*Claim* 11.11. The unbounded operator $A$ is closable iff $\overline{\Gamma(A)}$ is the graph of a linear operator, in which case

$$\Gamma(\overline{A}) = \overline{\Gamma(A)}.$$

*Proof.* First assume that

$$\overline{\Gamma(A)} = \Gamma(B)$$

for some operator $B$. Then by definition $B$ is closed and as

$$\Gamma(A) \subseteq \overline{\Gamma(A)},$$

we have that $B$ is an extension of $A$, hence $A$ is closable. Since $B$ is that operator whose graph equals the closure of $\Gamma(A)$, it is the *smallest* closed extension of $A$ and hence $\overline{A} = B$ as desired.

In the reverse direction, assume that $A$ is closable. Then let $(\mathcal{D}(B), B)$ be a closed extension of $(\mathcal{D}(A), A)$. Then by definition, $\Gamma(A) \subseteq \Gamma(B)$ and since $\Gamma(B)$ is closed,

$$\overline{\Gamma(A)} \subseteq \Gamma(B)$$

as $\overline{\Gamma(A)}$ is the *smallest* closed subset containing $\Gamma(A)$. Hence if $(0, \psi) \in \overline{\Gamma(A)}$ then $\psi = 0$. Indeed, then

$$\begin{aligned} (0, \psi) \in \Gamma(B) &\iff B0 = \psi \\ &\iff \psi = 0\,. \end{aligned}$$

Define now $R : \mathcal{D}(R) \to \mathcal{H}$ via

$$\begin{aligned} \mathcal{D}(R) &:= \left\{ \psi \,\middle|\, (\psi, \varphi) \in \overline{\Gamma(A)} \exists \varphi \in \mathcal{H} \right\} \\ R\psi &:= \varphi \qquad \left(\varphi \text{ is the unique vector so that } (\psi, \varphi) \in \overline{\Gamma(A)}\right)\,. \end{aligned}$$

First we show that $R$ is indeed well defined. This follows since if there are two $\varphi_1, \varphi_2$ such that

$$(\psi, \varphi_1) = (\psi, \varphi_2)$$

then we'd have $(0, \varphi_1 - \varphi_2) \in \Gamma(B)$ which implies $\varphi_1 = \varphi_2$. By construction,

$$\Gamma(R) = \overline{\Gamma(A)}$$

so $R$ is a closed extension of $A$. But $R \subseteq B$, where $B$ is an arbitrary closed extension, so $R = \overline{A}$. □

Not every operator is even closable, as the following example shows:

**Example 11.12** (Non-closable operator)**.** Let $\{ \varphi_n \}_n$ be an ONB for $\mathcal{H}$ and $\psi \in \mathcal{H}$ be some vector which is not a *finite* linear combination of these basis vectors. Let $\mathcal{D}$ be the set of finite linear combinations of elements of $\{ \varphi_n \}_n$ as well as $\psi$ (so that it is a vector subspace) and on it define

$$A\left(a\psi + \sum_{j=1}^{N} a_j \varphi_j\right) := a\psi$$

for all $a, a_1, \dots, a_N \in \mathbb{C}$. Then since $\{ \varphi_j \}_{j=1}^{\infty}$ is an ONB, we may, via approximation, exhibit the set

$$\{ (a\psi + \varphi, a\psi) \mid a \in \mathbb{C}, \varphi \in \mathcal{H} \}$$

within $\overline{\Gamma(A)}$, so the latter contains both $(\psi + 0, \psi) = (\psi, \psi)$ as well as $(0\psi + \psi, 0\psi) = (\psi, 0)$. But no linear map $B$ may contain both of these points in its graph, since that would imply

$$B\psi = 0$$

and

$$B\psi = \psi$$

i.e., $B$ is not well-defined.

**Definition 11.13** (Core of an operator)**.** If $A$ is a closed operator then a subset $\mathcal{C} \subseteq \mathcal{D}(A)$ is called a *core* for $A$ iff

$$\overline{A|_{\mathcal{C}}} = A\,.$$

**Definition 11.14** (Adjoint)**.** Given a *densely-defined* operator $A$, we define its adjoint using the following procedure. First, however we end up defining the adjoint $A^*$ of $A$, it better obey the equation

$$\langle \varphi, A\psi \rangle = \langle A^*\varphi, \psi \rangle \qquad (\psi \in \mathcal{D}(A), \varphi \in ?) .$$

But it might not be the case that there is a solution $\xi$ (which is our tentative definition for $A^*\varphi$) to the equation

$$\langle \varphi, A\psi \rangle = \langle \xi, \psi \rangle \qquad (\psi \in \mathcal{D}(A))$$

for all $\varphi$. The solution is thus to define

$$\mathcal{D}(A^*) := \{ \varphi \in \mathcal{H} \mid \exists \xi \in \mathcal{H} : \forall \psi \in \mathcal{D}(A), \langle \varphi, A\psi \rangle = \langle \xi, \psi \rangle \} .$$

If such a solution $\xi \in \mathcal{H}$ exists then it is unique. Indeed, if $\tilde{\xi} \in \mathcal{H}$ also obeys

$$\langle \varphi, A\psi \rangle = \left\langle \tilde{\xi}, \psi \right\rangle$$

then

$$\left\langle \tilde{\xi}, \psi \right\rangle = \langle \varphi, A\psi \rangle = \langle \xi, \psi \rangle \qquad (\psi \in \mathcal{D}(A)) .$$

As such, $\tilde{\xi} - \xi \in \mathcal{D}(A)^\perp$. But $\mathcal{D}(A)$ is dense, so using Claim 7.8, we have

$$\mathcal{D}(A)^\perp = \left(\overline{\mathcal{D}(A)}\right)^\perp = \mathcal{H}^\perp = \{ 0 \} .$$

Hence, $\xi = \tilde{\xi}$. Note it may happen that

$$\mathcal{D}(A^*) := \{ 0 \} .$$

If $\overline{\mathcal{D}(A^*)} = \mathcal{H}$ then we *define* $A^{**} := (A^*)^*$. Otherwise, $(A^*)^*$ does not exist as an operator (we need a densely defined operator to define the adjoint!).

*Claim* 11.15. For any densely defined operator $A$,

$$\mathcal{D}(A^*) = \left\{ \varphi \in \mathcal{H} \;\middle|\; \sup_{\psi \in \mathcal{D}(A)} \frac{|\langle \varphi, A\psi \rangle|}{\|\psi\|} < \infty \right\} .$$

*Proof.* First let $\varphi \in \mathcal{D}(A^*)$. Then we know $A^*\varphi \in \mathcal{H}$ exists which obeys

$$\langle \varphi, A\psi \rangle = \langle A^*\varphi, \psi \rangle \qquad (\psi \in \mathcal{D}(A)) .$$

Then

$$|\langle \varphi, A\psi \rangle| = |\langle A^*\varphi, \psi \rangle| \leq \|A^*\varphi\| \|\psi\|$$

so that

$$\sup_{\psi \in \mathcal{D}(A)} \frac{|\langle \varphi, A\psi \rangle|}{\|\psi\|} \leq \|A^*\varphi\| < \infty .$$

Conversely, if $\sup_{\psi \in \mathcal{D}(A)} \frac{|\langle \varphi, A\psi \rangle|}{\|\psi\|} < \infty$, then

$$\psi \mapsto \langle \varphi, A\psi \rangle$$

is a bounded linear functional and hence by Riesz' lemma, Theorem 7.10, there corresponds to it a unique vector (which ends up being $A^*\varphi$), i.e., $\varphi \in \mathcal{D}(A^*)$. □

**Example 11.16** (Example where $\mathcal{D}(A^*)$ is not dense)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be bounded and measurable, but *not* $L^2(\mathbb{R})$. Define

$$\mathcal{D}(A) := \left\{ \psi \in L^2(\mathbb{R}) \;\middle|\; \int |f(x)\psi(x)| \, dx < \infty \right\} .$$

Now clearly $\mathcal{D}(A)$ contains all $L^2$ functions with compact support so that $\mathcal{D}(A)$ is dense in $L^2$. Now fix some $\psi_0 \in L^2$

and define

$$A\psi := \langle f,\psi\rangle_{L^2}\,\psi_0 \qquad (\psi\in\mathcal{D}(A))\ .$$

Then for $\varphi\in\mathcal{D}(A^*)$, we have

$$\langle\psi,A^*\varphi\rangle \equiv \langle A\psi,\varphi\rangle = \langle\langle f,\psi\rangle\,\psi_0,\varphi\rangle = \langle\psi,f\rangle\,\langle\psi_0,\varphi\rangle\,\langle\psi,\langle\psi_0,\varphi\rangle\,f\rangle \qquad (\psi\in\mathcal{D}(A))\ .$$

Hence

$$A^*\varphi = \langle\psi_0,\varphi\rangle\,f\ .$$

But $f\notin L^2$, so $\langle\psi_0,\varphi\rangle = 0$, i.e., $\varphi\perp\psi_0$ and hence $\mathcal{D}(A^*) = \{\,\varphi \mid \varphi\perp\psi_0\,\}$ cannot be dense. So on that domain, $A^*$ is the zero operator.

**Theorem 11.17.** *[R&S Thm. VIII.1] Let $A$ be a densely defined operator. Then*

1. $A^*$ is closed.
2. $A$ is closable iff $\mathcal{D}(A^*)$ is dense, and in this case $\overline{A} = A^{**}$.
3. If $A$ is closable then $\left(\overline{A}\right)^* = A^*$.

*Proof.* Define a unitary $V$ on $\mathcal{H}^2$ via

$$V(\varphi,\psi) := (\varphi,-\psi)\ .$$

It is indeed unitary since it can be written in operator block form as

$$V = \begin{bmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{bmatrix}\ .$$

The unitarity of $V$ guarntees that for any subspace $E$, $V\left(E^\perp\right) = (V(E))^\perp$. We claim that

$$\Gamma(A^*) = V(\Gamma(A))^\perp\ . \tag{11.1}$$

This would imply the first statement thanks to Theorem 7.7. But (11.1) holds, via

$$\begin{aligned}(\varphi,\psi)\in\Gamma(A^*) &\iff \varphi\in\mathcal{D}(A^*)\wedge\psi = A^*\varphi\\ &\iff \langle\varphi,A\eta\rangle = \langle\psi,\eta\rangle \qquad (\eta\in\mathcal{D}(A))\\ &\iff \langle(\psi,\varphi),(\eta,-A\eta)\rangle \qquad (\eta\in\mathcal{D}(A))\\ &\iff ((\psi,\varphi),V\langle\eta,A\eta\rangle) \qquad (\eta\in\mathcal{D}(A))\\ &\iff (\psi,\varphi)\perp V(\Gamma(A))\ .\end{aligned}$$

For the second statement, we have via Claim 7.8,

$$\begin{aligned}\overline{\Gamma(A)} &= \left(\Gamma(A)^\perp\right)^\perp\\ &\overset{V^2=\mathbb{1}}{=} \left(V^2\Gamma(A)^\perp\right)^\perp\\ &\overset{\Gamma(A^*)=V(\Gamma(A))^\perp}{=} (V\Gamma(A^*))^\perp\\ &\overset{\text{same equation again}}{=} \Gamma\left((A^*)^*\right)\ .\end{aligned}$$

But if $A^*$ is densely defined, $(A^*)^*$ is an operator so that $\Gamma\left((A^*)^*\right) = \Gamma(A^{**})$ is the graph of an operator, and hence $A$ is closable. Conversely, if $A^*$ is not densely defined, let $\psi\in\mathcal{D}(A^*)^\perp$. Then

$$(\psi,0) \quad\in\quad \Gamma(A^*)^\perp$$

hence $V(\Gamma(A^*))^\perp$ is not the graph of a single-valued operator. But $(V\Gamma(A^*))^\perp = \overline{\Gamma(A)}$, so $A$ is not closable.

Finally, if $A$ is closable,

$$\begin{aligned} A^* &\overset{A^* \text{ is closed}}{=} \overline{A^*} \\ &\overset{\text{second statement}}{=} (A^*)^{**} \\ &= A^{***} \\ &= \left(\overline{A}\right)^* . \end{aligned}$$

□

**Definition 11.18** (The spectrum of an unbounded densely defined operator)**.** Let a closed operator $A$ be given. Then

$$\rho(A) := \{ z \in \mathbb{C} \mid (A - z\mathbb{1}) : \mathcal{D}(A) \to \mathcal{H} \text{ is a bijection} \}$$

is the *resolvent set*. Since $\Gamma(A)$ is closed, by the closed graph theorem Theorem 3.37, for all $z \in \rho(A)$, $(A - z\mathbb{1})^{-1} : \mathcal{H} \to \mathcal{D}(A)$ is continuous, and hence

$$R_A(z) := (A - z\mathbb{1})^{-1} \in \mathcal{B}(\mathcal{H}) \qquad (z \in \rho(A)) .$$

The spectrum of a closed operator is defined as

$$\sigma(A) := \mathbb{C} \setminus \rho(A) .$$

We use the same definition of point spectrum

$$\sigma_p(A) := \{ z \in \mathbb{C} \mid \ker(A - z\mathbb{1}) \neq \{ 0 \} \} .$$

and residual spectrum from Definition 9.14. The spectrum of a closable operator is the spectrum of the closure of the operator.

**Theorem 11.19.** *Let $A$ be a closed densely defined operator. Then $\rho(A) \in \operatorname{Open}(\mathbb{C})$ and $R_A : \rho(A) \to \mathcal{H}$ is an analytic operator-valued function. Moreover,*

$$\{ R_A(\lambda) \mid \lambda \in \rho(A) \}$$

*is a commuting family of bounded operators for which*

$$R_A(\lambda) - R_A(\mu) = (\mu - \lambda) R_A(\lambda) R_A(\mu) .$$

The proof is the same as in the bounded case and hence is not repeated.

**Example 11.20** (Spectrum depends on domain of definition)**.** Let

$$\mathcal{A} := \left\{ \psi : [0,1] \to \mathbb{C} \,\middle|\, \psi \text{ is ac and } \psi' \in L^2([0,1]) \right\} .$$

Recall that $\psi$ is called absolutely-continuous iff $\psi'$ exists Lebesgue-almost-everywhere on $[0,1]$, $\psi' \in L^1$, and

$$\psi(x) = \psi(0) + \int_0^x \psi' \qquad (x \in [0,1]) .$$

We then define $A_1, A_2$ both with

$$\begin{aligned} \mathcal{D}(A_1) &:= \mathcal{A} \\ \mathcal{D}(A_2) &:= \{ \varphi \in \mathcal{A} \mid \varphi(0) = 0 \} \end{aligned}$$

and with the joint operator

$$\psi \mapsto -\mathrm{i}\psi' .$$

We claim that both $A_1, A_2$ have densely defined and closed, but:

$$\sigma(A_1) = \mathbb{C} \text{ yet } \sigma(A_2) = \varnothing .$$

To see the spectral claim, let us calculate

$$\left(A_1 - \lambda \mathbb{1}\right) \mathrm{e}^{\mathrm{i}\lambda \cdot} = 0$$

and observe $\mathrm{e}^{\mathrm{i}\lambda\cdot} \in \mathcal{D}(A_1)$ and this is true for all $\lambda \in \mathbb{C}$. For the second spectrum, we exhibit an inverse for $A_2 - \lambda\mathbb{1}$ for all $\lambda \in \mathbb{C}$:

$$\left(B_\lambda g\right)(x) \quad := \quad -\mathrm{i} \int_0^x \mathrm{e}^{\mathrm{i}\lambda(x-y)} g(y)\,\mathrm{d}y\,.$$

Then

$$\begin{aligned} \left(A_2 - \lambda\mathbb{1}\right) B_\lambda &= \mathbb{1}_{\mathcal{H}} \\ B_\lambda \left(A_2 - \lambda\mathbb{1}\right) &= \mathbb{1}_{\mathcal{D}(A_2)}\,. \end{aligned}$$

## 11.2 Symmetric and self-adjoint operators

**Definition 11.21** (symmetric operator)**.** A densely defined operator $A$ is called *symmetric* iff $A \subseteq A^*$. That means $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $A\varphi = A^*\varphi$ for all $\varphi \in \mathcal{D}(A)$.

*Claim* 11.22. $A$ is symmetric iff

$$\langle A\varphi, \psi\rangle = \langle \varphi, A\psi\rangle \qquad (\varphi, \psi \in \mathcal{D}(A))\,.$$

*Proof.* Assume that $A$ is symmetric. Then $A \subseteq A^*$. Then for $\varphi, \psi \in \mathcal{D}(A)$, we have

$$\langle \varphi, A\psi\rangle = \langle A^*\varphi, \psi\rangle \overset{A^*\varphi = A\varphi \text{ for } \varphi\in\mathcal{D}(A)}{=} \langle A\varphi, \psi\rangle\,.$$

Conversely, assume

$$\langle A\varphi, \psi\rangle = \langle \varphi, A\psi\rangle \qquad (\varphi, \psi \in \mathcal{D}(A))\,.$$

Then, given $\varphi \in \mathcal{D}(A)$, we have $\xi = A\varphi$ as the solution of

$$\langle \varphi, A\psi\rangle = \langle \xi, \psi\rangle$$

by symmetry, so that $\varphi \in \mathcal{D}(A^*)$. Moreover, $A\varphi = A^*\varphi$ then so that indeed $A \subseteq A^*$. □

**Definition 11.23** (self-adjoint operator)**.** A densely defined operator $A$ is called *self-adjoint* iff $A = A^*$, that is, iff $A$ is symmetric and $\mathcal{D}(A) = \mathcal{D}(A^*)$.

Note that a symmetric operator is always closable by Theorem 11.17, since $\mathcal{D}(A^*) \supseteq \mathcal{D}(A)$, the latter being dense by definition of symmetric operator. Moreover, for a symmetric operator, $A^*$ is a closed extension of $A$, so by Theorem 11.17 again, $\overline{A} = A^{**}$ is contained in $A^*$. Hence, for symmetric operators

$$A \subseteq A^{**} \subseteq A^*\,.$$

For closed symmetric operators,

$$A = A^{**} \subseteq A^*\,.$$

Finally, for self-adjoint operators,

$$A = A^{**} = A^*\,.$$

Hence for $A$ closed symmetric, $A$ is self-adjoint iff $A^*$ is symmetric.

**Definition 11.24** (essentially self-adjoint operator)**.** A symmetric operator $A$ is called essentially self-adjoint iff its closure $\overline{A}$ is self-adjoint.

*Claim* 11.25. If $A$ is essentially self-adjoint (i.e. if $\left(\overline{A}\right)^* = \overline{A}$), then it has a *unique* self-adjoint extension.

*Proof.* Let $B$ be *any* self-adjoint extension of $A$. We claim that actually $B = A^{**} = \overline{A}$ (last equality via Theorem 11.17, recall that symmetric is always closable). We show this in two ways: First $B \supseteq A^{**}$. Indeed, $\overline{A} = A^{**}$ is the *smallest* closed extension of $A$. But $B$ is also closed and is an extension of $A$. For the other direction, first note that if $C \subseteq D$ and both are densely defined then $D^* \subseteq C^*$. Indeed, we have $C \subseteq D$ iff $\Gamma(C) \subseteq \Gamma(D)$ via Claim 11.4. But in a proof above that

$$\Gamma(C^*) = V(\Gamma(C))^{\perp}$$

for some unitary operator. Using that fact, we have then for the other direction,

$$B = B^* \subseteq (A^{**})^* = \left(\overline{A}\right)^* = \overline{A} = A^{**}$$

where we have used the fact that $A$ is essentially self-adjoint in the penultimate equality. □

Thus we see that a non-closed but symmetric operator has a uniquely defined self-adjoint operator associated to it, and many times, it is easier to specify a self-adjoint operator by specifying a non-closed symmetric operator whose domain doesn't have to be determined precisely, but just the core on which it is essentially self-adjoint!

Note also that the converse is also true: if $A$ has a unique self-adjoint extension then it is essentially self-adjoint.

**Theorem 11.26.** *Let $A$ be a* symmetric *operator. Then the following are equivalent:*

1. *$A$ is self-adjoint.*
2. *$A$ is closed and* $\ker(A^* \pm \mathrm{i}\mathbb{1}) = \{0\}$. *(In fact, here $\pm\mathrm{i}$ may be replaced by $z, \overline{z}$ for any $z \in \mathbb{C} \setminus \mathbb{R}$).*
3. $\operatorname{im}(A \pm \mathrm{i}\mathbb{1}) = \mathcal{H}$.

*Proof.* Assume that $A$ is self-adjoint. Then $A = A^*$ and since Theorem 11.17 says $A^*$ is closed, so is $A$. Assume that

$$A\psi = \mathrm{i}\psi \qquad (\psi \in \mathcal{D}(A))\ .$$

Then

$$-\mathrm{i}\langle\psi,\psi\rangle = \langle\mathrm{i}\psi,\psi\rangle = \langle A\psi,\psi\rangle = \langle\psi,A\psi\rangle = \mathrm{i}\langle\psi,\psi\rangle$$

so $\psi = 0$. I.e., $\ker(A^* \pm \mathrm{i}\mathbb{1}) = \{0\}$.

Next, we show that the second statement implies the third. Assume that

$$A^*\varphi \quad = \quad -\mathrm{i}\varphi$$

has no solutions. We want to show that $\operatorname{im}(A - \mathrm{i}\mathbb{1}) = \mathcal{H}$ which we do in two steps: show $\operatorname{im}(A - \mathrm{i}\mathbb{1})$ is dense and then that it is closed. For the first step, assume that $\psi \in \operatorname{im}(A - \mathrm{i}\mathbb{1})^{\perp}$. Then

$$\langle(A - \mathrm{i}\mathbb{1})\varphi,\psi\rangle = 0 \qquad (\varphi \in \mathcal{D}(A))$$

so $\psi \in \mathcal{D}(A^*)$ and hence

$$(A - \mathrm{i}\mathbb{1})^*\psi = (A^* + \mathrm{i}\mathbb{1})\psi = 0$$

which is impossible as we said $A^*\psi = -\mathrm{i}\psi$ has no solutions. To show $\operatorname{im}(A - \mathrm{i}\mathbb{1})$ is closed, note that for any $\varphi \in \mathcal{D}(A)$,

$$\|(A - \mathrm{i}\mathbb{1})\varphi\|^2 = \|A\varphi\|^2 + \|\varphi\|^2\,.$$

Hence if $\varphi_n \in \mathcal{D}(A)$ and $(A - \mathrm{i}\mathbb{1})\varphi_n \to \xi$ then $\varphi_n \to \varphi_0$ and $A\varphi_n$ converges too. But $A$ is closed, so $\varphi_0 \in \mathcal{D}(A)$ and $(A - \mathrm{i}\mathbb{1})\varphi_0 = \xi$. Hence $\operatorname{im}(A - \mathrm{i}\mathbb{1})$ is closed.

Finally, let us show the third statement implies self-adjointness. Let $\varphi \in \mathcal{D}(A^*)$. Since $\operatorname{im}(A - \mathrm{i}\mathbb{1}) = \mathcal{H}$, there is some $\eta \in \mathcal{D}(A)$ with

$$(A - \mathrm{i}\mathbb{1})\eta = (A^* - \mathrm{i})\varphi\,.$$

But $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$, so

$$\varphi - \eta \in \mathcal{D}(A^*)$$

and

$$(A^* - \mathrm{i}\mathbb{1})(\varphi - \eta) = 0\,.$$

But $\operatorname{im}(A + \mathrm{i}\mathbb{1}) = \mathcal{H}$ too, so $\ker(A^* - \mathrm{i}\mathbb{1}) = \{0\}$, so that $\varphi = \eta \in \mathcal{D}(A)$. Hence $\mathcal{D}(A^*) = \mathcal{D}(A)$ and hence $A$ is self-adjoint. □

**Corollary 11.27.** *Let $A$ be a symmetric operator. Then the following are equivalent:*

1. *$A$ is essentially self-adjoint.*
2. $\ker(A^* \pm \mathrm{i}\mathbb{1}) = \{0\}$ .
3. $\overline{\operatorname{im}(A \pm \mathrm{i}\mathbb{1})} = \mathcal{H}$.

*Proof.* Assume that $A$ is essentially self-adjoint. Then $\overline{A} = A^{**}$ is self-adjoint. Applying of the above lemma on $\overline{A}$ then yields that $\overline{A}$ is closed (trivial) and that

$$\ker\left(\left(\overline{A}\right)^* \pm \mathrm{i}\mathbb{1}\right) = \{0\}\,.$$

But by Theorem 11.17 we know that $\overline{A}^* = A^*$. Next, another application of the lemma on $\overline{A}$ yields that $\operatorname{im}\left(\overline{A} \pm \mathrm{i}\mathbb{1}\right) = \mathcal{H}$. We then claim that

$$\operatorname{im}\left(\overline{A} \pm \mathrm{i}\mathbb{1}\right) = \overline{\operatorname{im}(A \pm \mathrm{i}\mathbb{1})}\,. \tag{11.2}$$

Indeed, let $\xi \in \overline{\operatorname{im}(A \pm \mathrm{i}\mathbb{1})}$. Then $\{\varphi_n\}_n \subseteq \mathcal{H}$ with

$$A\varphi_n \pm \mathrm{i}\varphi_n \to \xi\,.$$

But

$$\begin{aligned} \|A\varphi_n + z\varphi_n\|^2 &= \langle A\varphi_n + z\varphi_n, A\varphi_n + z\varphi_n\rangle \\ &= \langle A\varphi_n, A\varphi_n\rangle + |z|^2\langle\varphi_n,\varphi_n\rangle + \overline{z}\langle\varphi_n, A\varphi_n\rangle + z\langle A\varphi_n,\varphi_n\rangle \\ &\overset{A\text{ symm.}}{=} \|A\varphi_n\|^2 + |z|^2\|\varphi_n\|^2 + 2\,\mathbb{Re}\{z\}\langle\varphi_n, A\varphi_n\rangle \end{aligned}$$

so that if $z = \pm\mathrm{i}$ we get

$$\|A\varphi_n \pm \mathrm{i}\varphi_n\| = \sqrt{\|A\varphi_n\|^2 + \|\varphi_n\|^2}$$

imply that if $\{A\varphi_n \pm \mathrm{i}\varphi_n\}_n$ is Cauchy, so are $\{A\varphi_n\}_n$ and $\{\varphi_n\}_n$. Thus $\varphi_n \to \varphi$ and $A\varphi_n \to \psi$ for some $\varphi, \psi \in \mathcal{H}$. I.e., $(\varphi, \psi) \in \overline{\Gamma(A)}$. But $A$ is closable, so

$$\Gamma\left(\overline{A}\right) = \overline{\Gamma(A)}$$

and hence $(\varphi,\psi) \in \Gamma\left(\overline{A}\right)$. I.e., $\psi = \overline{A}\varphi$ and $\varphi \in \mathcal{D}\left(\overline{A}\right)$. Hence

$$A\varphi_n \pm \mathrm{i}\varphi_n \to \overline{A}\varphi \pm \mathrm{i}\varphi\,.$$

But the limit is unique, so

$$\overline{A}\varphi \pm \mathrm{i}\varphi = \xi\,,$$

i.e., $\xi \in \operatorname{im}\left(\overline{A} \pm \mathrm{i}\mathbb{1}\right)$.

Conversely, if $\xi \in \operatorname{im}\left(\overline{A} \pm \mathrm{i}\mathbb{1}\right)$, we write $\xi = \overline{A}\varphi \pm \mathrm{i}\varphi$ for some $\varphi \in \mathcal{D}\left(\overline{A}\right)$. Then

$$\overline{\Gamma(A)} = \Gamma\left(\overline{A}\right) \ni \left(\varphi, \overline{A}\varphi\right) = (\varphi, \xi \mp \mathrm{i}\varphi)$$

so there is a sequence $\{\psi_n\}_n \subseteq \mathcal{D}(A)$ such that $\psi_n \to \varphi$ and $A\psi_n \to \xi \mp \mathrm{i}\varphi$. But

$$\|\xi - (A\psi_n \pm \mathrm{i}\psi_n)\| \leq \|(\xi \mp \mathrm{i}\varphi) - A\psi_n\| + \|\pm\mathrm{i}\varphi \mp \mathrm{i}\psi_n\| \to 0$$

so $\xi \in \overline{\mathrm{im}\,(A \pm \mathrm{i}\mathbb{1})}$.

Thanks to (11.2) we now have the third item.

Finally, assuming the third item, using (11.2) and applying the lemma again on $\overline{A}$ we obtain that $\overline{A}$ is self-adjoint, i.e., $A$ is essentially self-adjoint. □

An operator which is merely symmetric but *not* essentially self-adjoint may have either no self-adjoint extensions or many self-adjoint extensions.

**Example 11.28** (many self-adjoint extensions)**.** Let $P := -\mathrm{i}\partial$ with

$$\mathcal{D}(P) := \{ \varphi \in \mathcal{A} \mid \varphi(0) = \varphi(1) = 0 \}$$

with $\mathcal{A}$ as above. Integration by parts shows that $P$ is symmetric:

$$\langle \varphi, P\psi \rangle = \int \overline{\varphi} P\psi = (-\mathrm{i}) \int \overline{\varphi}\psi' = \mathrm{i} \int \overline{\varphi'}\psi = \int \overline{(-\mathrm{i})\,\varphi'}\psi = \int \overline{P\varphi}\psi = \langle P\varphi, \psi \rangle \,.$$

We claim that $P^*$ is given by $P^* = -\mathrm{i}\partial$ as well, but with

$$\mathcal{D}(P^*) = \mathcal{A}\,.$$

[TODO: prove this] But $P$ is not essentially self-adjoint, since $\mathrm{e}^{\pm\cdot} \in \mathcal{D}(A)$ and

$$P\mathrm{e}^{\pm\cdot} = \pm \mathrm{i}\mathrm{e}^{\pm\cdot}\,.$$

Actually $P$ is a closed symmetric but not self-adjoint operator. It has uncountably many self-adjoint extensions indexed by the unit circle: for any $\alpha \in \mathbb{S}^1$,

$$\mathcal{D}(P_\alpha) := \{ \varphi \in \mathcal{A} \mid \varphi(0) = \alpha\varphi(1) \}$$

and $P_\alpha := -\mathrm{i}\partial$.

**Theorem 11.29.** *Let $A$ be self-adjoint. Then*

1. *If $z \in \mathbb{C}$,*
$$\|(A - z\mathbb{1})\,\psi\|^2 = \|(A - \mathbb{Re}\{z\}\,\mathbb{1})\,\psi\|^2 + \mathbb{Im}\{z\}^2\,\|\psi\|^2 \qquad (\psi \in \mathcal{D}(A))\,.$$
2. *$\sigma(A) \subseteq \mathbb{R}$ and for all $z \in \mathbb{C} \setminus \mathbb{R}$,*
$$\left\|(A - z\mathbb{1})^{-1}\right\| \leq \frac{1}{|\mathbb{Im}\{z\}|} \qquad (z \in \mathbb{C} \setminus \mathbb{R})\,.$$
3. *For any $x \in \mathbb{R}$ and $\psi \in \mathcal{H}$,*
$$\lim_{y\to\infty} \mathrm{i}y\,(A - (x + \mathrm{i}y)\,\mathbb{1})^{-1}\,\psi = -\psi\,.$$
4. *If $\lambda_1, \lambda_2 \in \sigma_p(A)$ with $\lambda_1 \neq \lambda_2$ and $\psi_1, \psi_2$ are the corresponding eigenvectors, then $\psi_1 \perp \psi_2$.*

*Proof.* We have

$$\begin{aligned}
\|(A - (x + \mathrm{i}y)\,\mathbb{1})\,\psi\|^2 &= \langle (A - (x + \mathrm{i}y)\,\mathbb{1})\,\psi, (A - (x + \mathrm{i}y)\,\mathbb{1})\,\psi \rangle \\
&= \left\langle \psi, (A - (x + \mathrm{i}y)\,\mathbb{1})^*\,(A - (x + \mathrm{i}y)\,\mathbb{1})\,\psi \right\rangle \\
&= \langle \psi, (A - (x - \mathrm{i}y)\,\mathbb{1})\,(A - (x + \mathrm{i}y)\,\mathbb{1})\,\psi \rangle \\
&= \left\langle \psi, \left((A - x\mathbb{1})^2 + y^2\mathbb{1}\right)\psi \right\rangle \\
&= \|(A - x\mathbb{1})\,\psi\|^2 + y^2\|\psi\|^2\,.
\end{aligned}$$

For the next statement, let $z \in \mathbb{C} \setminus \mathbb{R}$. If $(A - z\mathbb{1})\,\psi = 0$ then

$$\|(A - x\mathbb{1})\,\psi\|^2 = y^2\|\psi\|^2 = 0$$

which implies $\psi = 0$. So

$$A - z\mathbb{1} : \mathcal{D}(A) \to \mathcal{H}$$

is injective. By the third condition in Theorem 11.26 (using it at $z \in \mathbb{C} \setminus \mathbb{R}$ rather than $z = \pm\mathrm{i}$), since $A$ is assumed to be self-adjoint, $\mathrm{im}\,(A - z\mathbb{1}) = \mathcal{H}$. Hence $A - z\mathbb{1}$ is bijective. By Definition 11.18, this implies $z \in \rho\,(A)$. Next, from the first statement of the claim, we have

$$\|\psi\| \quad \leq \quad \frac{1}{|\mathbb{Im}\,\{z\}|}\|(A - z\mathbb{1})\,\psi\| \qquad (\psi \in \mathcal{D}\,(A))\ .$$

Now let $\varphi \in \mathcal{H}$. Then $(A - z\mathbb{1})^{-1}\varphi \in \mathcal{D}\,(A)$ as $(A - z\mathbb{1})^{-1} : \mathcal{H} \to \mathcal{D}\,(A)$ is a bijection. Hence we may write

$$\left\|(A - z\mathbb{1})^{-1}\varphi\right\| \leq \frac{1}{|\mathbb{Im}\,\{z\}|}\|\varphi\| \qquad (\varphi \in \mathcal{H})$$

which is what we wanted to prove. We learn that $(A - z\mathbb{1})^{-1}$ is bounded in the sense that its operator norm has the (so-called) "trivial bounded"

$$\left\|(A - z\mathbb{1})^{-1}\right\| \leq \frac{1}{|\mathbb{Im}\,\{z\}|}\,.$$

Of course we may re-define its codomain as $\mathcal{H}$ in which case it would be bounded in the sense of Definition 11.5.

Finally, for the third statement, let $B := A - x\mathbb{1}$ for $x \in \mathbb{R}$. Then $B$ is also self-adjoint. Its domain equals

$$\mathcal{D}\,(A - x\mathbb{1}) \equiv \mathcal{D}\,(A) \cap \mathcal{D}\,(-x\mathbb{1}) = \mathcal{D}\,(A) \cap \mathcal{H} = \mathcal{D}\,(A)\ .$$

It is symmetric:

$$\langle \varphi, B\psi \rangle = \langle \varphi, (A - x\mathbb{1})\,\psi \rangle = \langle \varphi, A\psi \rangle - x\,\langle \varphi, \psi \rangle = \langle A\varphi, \psi \rangle - \langle x\varphi, \psi \rangle = \langle (A - x\mathbb{1})\,\varphi, \psi \rangle\ .$$

As such, via Theorem 11.26 it is self-adjoint since we can invoke the third condition

$$\mathrm{im}\,(A \pm \mathrm{i}\mathbb{1}) = \mathcal{H}$$

with *any* complex number off the real axis, rather than merely $\pm\mathrm{i}$. Let us thus prove the statement for $B$. First, let $\psi \in \mathcal{D}\,(B)$. Then we have

$$\begin{aligned}
(B - \mathrm{i}y\mathbb{1})\,(B - \mathrm{i}y\mathbb{1})^{-1} &\equiv \mathbb{1} \\
B\,(B - \mathrm{i}y\mathbb{1})^{-1} - \mathrm{i}y\,(B - \mathrm{i}y\mathbb{1})^{-1} &= \mathbb{1} \\
\mathrm{i}y\,(B - \mathrm{i}y\mathbb{1})^{-1} + \mathbb{1} &= (B - \mathrm{i}y\mathbb{1})^{-1}\,B
\end{aligned}$$

and so

$$\begin{aligned}
\left\|\mathrm{i}y\,(B - \mathrm{i}y\mathbb{1})^{-1}\psi + \psi\right\| &= \left\|(B - \mathrm{i}y\mathbb{1})^{-1}\,B\psi\right\| \\
&\leq \left\|(B - \mathrm{i}y\mathbb{1})^{-1}\right\|\|B\psi\| \\
&\leq \frac{1}{|y|}\|B\psi\|\,.
\end{aligned}$$

Taking now the limit $y \to \infty$ yields the desired limit for all $\psi \in \mathcal{D}\,(B)$. Now, if $\psi \notin \mathcal{D}\,(B)$, since $B$ is self-adjoint, we have via Theorem 11.17 that $\mathcal{D}\,(B)$ is dense. So let $\{\,\psi_n\,\}_n \subseteq \mathcal{D}\,(B)$ be such that $\|\psi_n - \psi\| \leq \frac{1}{n}$. Then

$$\begin{aligned}
\left\|\mathrm{i}y\,(B - \mathrm{i}y\mathbb{1})^{-1}\psi + \psi\right\| &\leq \left\|\mathrm{i}y\,(B - \mathrm{i}y\mathbb{1})^{-1}(\psi - \psi_n)\right\| + \|\psi - \psi_n\| + \\
&\quad + \left\|\mathrm{i}y\,(B - \mathrm{i}y\mathbb{1})^{-1}\psi_n + \psi_n\right\| \\
&\leq 2\|\psi - \psi_n\| + \frac{\|B\psi_n\|}{|y|} \\
&\leq \frac{2}{n} + \frac{\|B\psi_n\|}{|y|}\,.
\end{aligned}$$

Taking now $\limsup_{y\to\infty}$ and then $n \to \infty$ we find the result. The last claim's proof is identical to the one in the case of bounded operators, shown in Theorem 9.21. □

## 11.3 Direct sums and invariant subspaces

### 11.3.1 Direct sum of operators

Let $A_i : \mathcal{D}(A_i) \to \mathcal{H}_i$ for $i = 1, 2$. Then we can form the map $A := A_1 \oplus A_2 : \mathcal{D}(A_1) \oplus \mathcal{D}(A_2) \to \mathcal{H}_1 \oplus \mathcal{H}_2$ using the notion of direct sum of Hilbert spaces as in Section 7.4. Since $\mathcal{D}(A_i)$ is not required to be closed, the direct sum within $\mathcal{D}(A_1) \oplus \mathcal{D}(A_2)$ is meant in the sense of vector spaces, not Hilbert spaces. It is clear that if $A_i$ is self-adjoint for $i = 1, 2$ then so is $A$, and

$$(A - z\mathbb{1})^{-1} \quad = \quad (A_1 - z\mathbb{1})^{-1} \oplus (A_2 - z\mathbb{1})^{-1} \ .$$

**Definition 11.30** (invariant subspace)**.** Let $A$ be a self-adjoint operator on Hilbert space. We say that a closed vector subspace $\mathcal{G} \subseteq \mathcal{H}$ is *invariant* under $A$ iff

$$(A - z\mathbb{1})^{-1} \mathcal{G} \subseteq \mathcal{G} \qquad (z \in \mathbb{C} \setminus \mathbb{R}) \ .$$

Observe that if $\mathcal{G}$ is invariant under $A$, so is $\mathcal{G}^\perp$: let $\psi \in (A - z\mathbb{1})^{-1} \mathcal{G}^\perp$. Then

$$\psi = (A - z\mathbb{1})^{-1} \varphi \qquad \exists \varphi \in \mathcal{G}^\perp \, .$$

Now let $\eta \in \mathcal{G}$. Then

$$\left\langle \eta, (A - z\mathbb{1})^{-1} \varphi \right\rangle = \left\langle \underbrace{(A - \overline{z}\mathbb{1})^{-1} \eta}_{\in \mathcal{G}}, \underbrace{\varphi}_{\in \mathcal{G}^\perp} \right\rangle = 0$$

so that $\mathcal{G}^\perp$ is invariant as well. Let us define for any *closed* vector subspace $\mathcal{V}$,

$$\mathcal{D}(A_\mathcal{V}) \quad := \quad \mathcal{D}(A) \cap \mathcal{V}$$

and $A_\mathcal{V} : \mathcal{D}(A_\mathcal{V}) \to \mathcal{V}$ an operator on the Hilbert space $\mathcal{V}$ via

$$A_\mathcal{V} \psi := A\psi \qquad (\psi \in \mathcal{D}(A_\mathcal{V})) \ .$$

*Claim* 11.31. If $\mathcal{G}$ is invariant under the self-adjoint operator $A$ then $A_\mathcal{G}$ as defined above is also self-adjoint.

*Proof.* Clearly $\mathcal{D}(A_\mathcal{V})$ is dense in $\mathcal{G}$ if $\mathcal{D}(A)$ is dense in $\mathcal{H}$, since $\mathcal{V}$ is closed. We first show that $A_\mathcal{G}$ is symmetric: we want

$$\langle A_\mathcal{G} \varphi, \psi \rangle = \langle \varphi, A_\mathcal{G} \psi \rangle \qquad (\varphi, \psi \in \mathcal{D}(A_\mathcal{G})) \ .$$

Since $A_\mathcal{G} = A$ on $\mathcal{G}$, and $\mathcal{D}(A_\mathcal{G}) \subseteq \mathcal{G}$, we have

$$\langle A_\mathcal{G} \varphi, \psi \rangle = \langle A\varphi, \psi \rangle \overset{A \text{ symmetric}}{=} \langle \varphi, A\psi \rangle = \langle \varphi, A_\mathcal{G} \psi \rangle \ .$$

Next, note that

$$\begin{aligned} \Gamma(A_\mathcal{G}) &\equiv \{ (\varphi, A_\mathcal{G}\varphi) \mid \varphi \in \mathcal{D}(A_\mathcal{G}) \} \\ &= \{ (\varphi, A\varphi) \mid \varphi \in \mathcal{D}(A) \cap \mathcal{G} \} \\ &= \{ (\varphi, A\varphi) \mid \varphi \in \mathcal{D}(A) \} \cap (\mathcal{G} \times \mathcal{H}) \\ &= \Gamma(A) \cap (\mathcal{G} \times \mathcal{H}) \ . \end{aligned}$$

Hence using the same unitary from Theorem 11.17, we have

$$\begin{aligned}
\Gamma\left(A_{\mathcal{I}}^{*}\right) &= \left(V\Gamma\left(A_{\mathcal{I}}\right)\right)^{\perp}\\
&= \left(V\left(\Gamma\left(A\right)\cap\left(\mathcal{I}\times\mathcal{H}\right)\right)\right)^{\perp}\\
&= \left(\left(V\Gamma\left(A\right)\right)\cap\left(\mathcal{I}\times\left(-\mathcal{H}\right)\right)\right)^{\perp}\\
&\overset{\mathcal{H}=-\mathcal{H}}{=} \left(\left(V\Gamma\left(A\right)\right)\cap\left(\mathcal{I}\times\mathcal{H}\right)\right)^{\perp}\\
&= \left(\Gamma\left(A^{*}\right)^{\perp}\cap\left(\mathcal{I}\times\mathcal{H}\right)\right)^{\perp}\\
&= \left(\Gamma\left(A\right)^{\perp}\cap\left(\mathcal{I}\times\mathcal{H}\right)\right)^{\perp}\\
&= \left\{\,\left(\varphi,A\varphi\right)\mid\varphi\in\mathcal{D}\left(A\right)\cap\mathcal{I}\,\right\}\\
&= \Gamma\left(A_{\mathcal{I}}\right)\,.
\end{aligned}$$

Hence, $A_{\mathcal{I}}^{*}=A_{\mathcal{I}}$. □

**Proposition 11.32.** *Let* $\left\{\,A_n:\mathcal{D}\left(A_n\right)\to\mathcal{H}_n\,\right\}_n$ *be a sequence of self-adjoint operators. Define the operator* $A:\mathcal{D}\left(A\right)\to\mathcal{H}$ *on*

$$\mathcal{H}:=\bigoplus_n\mathcal{H}_n$$

*with the direct sum as in Section 7.4, via the domain*

$$\mathcal{D}\left(A\right):=\left\{\psi\in\mathcal{H}\;\middle|\;\psi_n\in\mathcal{D}\left(A_n\right)\forall n\wedge\sum_n\left\|A_n\psi_n\right\|_{\mathcal{H}_n}^2<\infty\right\}$$

*and action*

$$A\psi:=\left\{\,A_n\psi_n\,\right\}\qquad\left(\psi\in\mathcal{D}\left(A\right)\right)\,.$$

*Then:*

1. *$A$ is a self-adjoint operator on $\mathcal{H}$.*
2. *For all $z\in\mathbb{C}\setminus\mathbb{R}$,*
$$\left(A-z\mathbb{1}\right)^{-1}=\bigoplus_n\left(A_n-z\mathbb{1}_n\right)^{-1}$$
3. *The spectrum obeys*
$$\sigma\left(A\right)=\overline{\bigcup_n\sigma\left(A_n\right)}\,.$$

*Proof.* We have

$$A^{*}=\bigoplus_n A_n^{*}=\bigoplus_n A_n=A$$

since the adjoint of the direct sum is the direct sum of the adjoints, by definition. The same statement holds for the inverse, which shows the second claim. Finally, for the spectrum, let $z\in\sigma\left(A\right)$. Then

$$\left(A-z\mathbb{1}\right):\mathcal{D}\left(A\right)\to\mathcal{H}$$

is *not* a bijection. Thanks to the direct sum decomposition, this implies there exists at least one $n$ for which

$$\left(A_n-z\mathbb{1}_n\right):\mathcal{D}\left(A_n\right)\quad\to\quad\mathcal{H}_n$$

is not a bijection for otherwise the whole $A$ would be. But that implies that $z\in\sigma\left(A_n\right)\subseteq\overline{\bigcup_n\sigma\left(A_n\right)}$. Conversely. if

$$z\in\overline{\bigcup_n\sigma\left(A_n\right)}$$

then there is a sequence $\{ w_j \}_j \subseteq \bigcup_n \sigma(A_n)$ such that $w_j \to z$. Then for each $j$ there is some $n_j$ such that

$$\left(A_{n_j} - w_j \mathbb{1}_{n_j}\right) : \mathcal{D}\left(A_{n_j}\right) \to \mathcal{H}_{n_j}$$

is not a bijection. That means that $A - w_j \mathbb{1}$ is not a bijection for each $j$, i.e., $w_j \in \sigma(A)$ for each $j$. But since the spectrum is closed, $z \in \sigma(A)$ as well. □

## 11.4 Cyclic spaces and the decomposition theorem

**Definition 11.33.** Let $A$ be a self-adjoint operator on $\mathcal{H}$. A collection $\{ \psi_n \}_n \subseteq \mathcal{H}$ is called *cyclic* for $A$ iff

$$\overline{\operatorname{span}\left(\left\{ (A - z\mathbb{1})^{-1} \psi_n \,\middle|\, z \in \mathbb{C} \setminus \mathbb{R} \right\}\right)} = \mathcal{H}\,.$$

Note that a cyclic set always exists since we can always take as the initial set an orthonormal basis for $\mathcal{H}$. If it happens that the whole set is just one single vector, then that vector is called cyclic for $A$.

**Theorem 11.34.** *Let $A$ be a self-adjoint operator on $\mathcal{H}$. Then there exists a sequence of closed subspaces*

$$\{ \mathcal{H}_n \}_n \subseteq \mathcal{H}$$

*which are mutually orthogonal ($\mathcal{H}_n \perp \mathcal{H}_m$ for all $n \neq m$) and self-adjoint operators $\{ A_n : \mathcal{D}(A_n) \to \mathcal{H}_n \}_n$ such that*

1. *For all $n$, there exists a vector $\psi_n \in \mathcal{H}_n$ cyclic for $A_n$.*
2. *$\mathcal{H} = \bigoplus_n \mathcal{H}_n$ and $A = \bigoplus_n A_n$.*

*Proof.* Let $\{ \varphi_n \}_n$ be any cyclic set for $A$. Define

$$\psi_1 \quad := \quad \varphi_1$$

and let $\mathcal{H}_1$ be the cyclic space generated by $A$ and $\psi_1$:

$$\mathcal{H}_1 \quad := \quad \overline{\operatorname{span}\left(\left\{ (A - z\mathbb{1})^{-1} \psi_1 \,\middle|\, z \in \mathbb{C} \setminus \mathbb{R} \right\}\right)}\,.$$

We know that $\psi_1 \in \mathcal{H}_1$ via the third item in Theorem 11.29. Using the same claim, we see that $\mathcal{H}_1$ is invariant under $A$. We set

$$A_1 \quad := \quad A_{\mathcal{H}_1}$$

with the notation as in the section above about invariant subspaces.

Let now $\tilde{\varphi}_2$ be the first element of $\{ \varphi_2, \varphi_3, \dots \}$ which is not in $\mathcal{H}_1$. Decompose

$$\tilde{\varphi}_{n_2} =: \tilde{\varphi}_{n_2}^{(1)} \oplus \tilde{\varphi}_{n_2}^{(2)} \in \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}\,.$$

Define

$$\psi_2 \quad := \quad \tilde{\varphi}_{n_2}^{(2)}$$

and let $\mathcal{H}_2$ be the cyclic subspace generated by $A$ and $\psi_2$. We claim that $\mathcal{H}_1 \perp \mathcal{H}_2$. Indeed,

$$\begin{aligned}\left\langle (A - z\mathbb{1})^{-1} \psi_1, (A - w\mathbb{1})^{-1} \psi_2 \right\rangle &= \left\langle \psi_1, (A - \overline{z}\mathbb{1})^{-1} (A - w\mathbb{1})^{-1} \psi_2 \right\rangle \\ &= \frac{1}{w - \overline{z}} \left\langle \psi_1, \left[ (A - \overline{z}\mathbb{1})^{-1} - (A - w\mathbb{1})^{-1} \right] \psi_2 \right\rangle\end{aligned}$$

and now use the fact that $\psi_2 \perp \mathcal{H}_1$. Define $A_2$ are the restriction of $A$ to $\mathcal{H}_2$, which is also invariant. By construction and Proposition 11.32, the rest of the claims follow. □

## 11.5 The spectral theorem for unbounded operators

**Theorem 11.35.** *Let $(X,\mathcal{F},\mu)$ be a finite measure space, with $\mu$ a finite positive measure. Let*

$$f : X \to \mathbb{R}$$

*be measurable and define $M_f : \mathcal{D}(M_f) \to L^2(X,\mu)$ with*

$$\begin{aligned}\mathcal{D}(M_f) &:= \left\{ \psi \in L^2(X,\mu) \,\middle|\, f\psi \in L^2(X,\mu) \right\} \\ M_f\psi &:= f\psi\,.\end{aligned}$$

*Then*

1. *$M_f$ is self-adjoint.*
2. *$M_f$ is bounded iff $f \in L^\infty(X,\mu)$, in which case*
$$\|M_f\| = \|f\|_\infty\,.$$
3. *$\sigma(M_f)$ equals the essential range of $f$, that is,*
$$\sigma(M_f) = \left\{ \lambda \in \mathbb{R} \,\middle|\, \mu\left(f^{-1}(B_\varepsilon(\lambda))\right) > 0 \forall \varepsilon > 0 \right\}\,.$$

*Proof.* First we want to show that $M_f$ is densely defined. This is clear since the compactly supported functions are dense in $L^2$. Next, $M_f$ is symmetric, since

$$\langle \varphi, M_f\psi\rangle_{L^2} \equiv \int_X \overline{\varphi} f\psi \mathrm{d}\mu \overset{\mathrm{im}(f)\subseteq\mathbb{R}}{=} \int_X \overline{f\varphi}\psi \mathrm{d}\mu = \langle M_f\varphi, \psi\rangle_{L^2}\,.$$

According to Definition 11.23 we are left to show that $\mathcal{D}\left(M_f^*\right) \subseteq \mathcal{D}(M_f)$. Divide $X$ according to the decomposition

$$Y_N(x) := \begin{cases} 1 & |f(x)| \leq N \\ 0 & \text{else} \end{cases} \qquad (x \in X)\,.$$

Through a limiting argument with $Y_N$ we may show that

$$\left\|M_f^*\psi\right\| = \|f\psi\|$$

so that $f\psi \in L^2$, i.e., $\psi \in \mathcal{D}(M_f)$. The proof of the spectral statement follows as in the bounded case (this was HW10Q6). □

**Definition 11.36** (unitary equivalence)**.** Let $A_i : \mathcal{D}(A_i) \to \mathcal{H}_i$ be given for $i = 1,2$. We say that $A_1$ is unitarily equivalent to $A_2$ iff there exists a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U\mathcal{D}(A_1) = \mathcal{D}(A_2)$$

and

$$UA_1U^{-1} = A_2\,.$$

**Theorem 11.37** (Spectral theorem for self-adjoint operators)**.** *Let $A$ be a self-adjoint operator on $\mathcal{H}$. Then there exists a measure space $(X,\mathcal{F},\mu)$ with $\mu$ a finite positive measure and a measurable functions $f : M \to \mathbb{R}$ such that $A$ is unitarily equivalent to the operator $M_f$ on $L^2(X,\mu)$.*

We will prove this theorem in two steps. The first step is to decompose $\mathcal{H}$ into the cyclic subspaces of $A$ as in Theorem 11.34. Then we shall prove the unitary equivalence to a multiplication operator in each cyclic subspace separately.

### 11.5.1 Proof of the spectral theorem in the cyclic case

**Theorem 11.38.** *Let $V:\mathbb{C}_+\to\mathbb{R}$ be a positive harmonic function. Then there is a constant*

$$c \geq 0$$

*and a positive measure $\mu$ on $\mathbb{R}$ such that*

$$V(z) = c\,\mathbb{Im}\{z\} + \mathbb{Im}\{z\}\int_{t\in\mathbb{R}}\frac{1}{(\mathbb{Re}\{z\}-t)^2+\mathbb{Im}\{z\}^2}\mathrm{d}\mu(t) \qquad \left(z\in\mathbb{C}^+\right).$$

Note that $\frac{y}{x^2+y^2} = \mathbb{Im}\left\{\frac{1}{x-\mathrm{i}y}\right\}$ so this could also be written as

$$V(z) = \mathbb{Im}\left\{cz + \int_{t\in\mathbb{R}}\frac{1}{t-z}\mathrm{d}\mu(t)\right\}$$

connecting with the previous Herglotz perspective and invoke Theorem 10.7.

*Proof.* This is Theorem 2.11 in [Jak06], which we do not reproduce here. □

**Theorem 11.39.** *Let $A$ be a self-adjoint operator on $\mathcal{H}$ and $\psi\in\mathcal{H}$ be given. Then there exists a unique finite positive Borel measure $\mu_\psi$ on $\mathbb{R}$ such that $\mu_\psi(\mathbb{R}) = \|\psi\|^2$ and*

$$\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle = \int_{\mathbb{R}}\frac{1}{t-z}\mathrm{d}\mu_\psi(t) \qquad (z\in\mathbb{C}\setminus\mathbb{R}). \tag{11.3}$$

*The measure $\mu_\psi$ is called the spectral measure of $A$ and $\psi$.*

*Proof.* Le $z\in\mathbb{C}_+$. Set

$$U(z) \quad := \quad \left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle$$

and

$$V(z) \quad := \quad \mathbb{Im}\{U(z)\}.$$

Then from the resolvent identity,

$$\begin{aligned}
V(z) &= \frac{1}{2\mathrm{i}}\left(U(z)-\overline{U(z)}\right)\\
&= \frac{1}{2\mathrm{i}}\left(\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle - \overline{\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle}\right)\\
&= \frac{1}{2\mathrm{i}}\left(\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle - \left\langle(A-z\mathbb{1})^{-1}\psi,\psi\right\rangle\right)\\
&= \frac{1}{2\mathrm{i}}\left(\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle - \left\langle\psi,\left[(A-z\mathbb{1})^{-1}\right]^*\psi\right\rangle\right)\\
&= \frac{1}{2\mathrm{i}}\left(\left\langle\psi,(A-z\mathbb{1})^{-1}\psi\right\rangle - \left\langle\psi,(A-\overline{z}\mathbb{1})^{-1}\psi\right\rangle\right)\\
&= \left\langle\psi,(A-z\mathbb{1})^{-1}\mathbb{Im}\{z\}(A-\overline{z}\mathbb{1})^{-1}\psi\right\rangle\\
&= \mathbb{Im}\{z\}\left\|(A-z\mathbb{1})^{-1}\psi\right\|^2.
\end{aligned}$$

So $V$ is strictly positive on $\mathbb{C}_+$. In fact, it is harmonic, since it is the imaginary part of a holomorphic function $U$. Invoking Theorem 11.39 implies there is a unique measure $\mu$ and $c\geq 0$ such that

$$V(z) = c\,\mathbb{Im}\{z\} + \mathbb{Im}\{z\}\int_{t\in\mathbb{R}}\frac{1}{(\mathbb{Re}\{z\}-t)^2+\mathbb{Im}\{z\}^2}\mathrm{d}\mu(t) \qquad \left(z\in\mathbb{C}^+\right).$$

Since we have

$$V(z) \leq \frac{\|\psi\|^2}{|\mathbb{Im}\{z\}|}$$

we get $c = 0$ (the other term is already $\frac{1}{|\mathbb{Im}\{z\}|}$ as $|\mathbb{Im}\{z\}| \to \infty$). Next, we have from the third iterm in Theorem 11.29 that

$$\lim_{y\to\infty} yV(x+\mathrm{i}y) = \|\psi\|^2 .$$

Hence

$$\lim_{y\to\infty} \int_{t\in\mathbb{R}} \frac{y^2}{(x-t)^2+y^2} \mathrm{d}\mu(t) = \|\psi\|^2 .$$

Applying now the DCT yields $\mu(\mathbb{R}) = \|\psi\|^2$.

Finally, the functions $\mathbb{C}_+ \ni z \mapsto \int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}\mu_\psi(t)$ and $U$ are analytic and have equal imaginary parts. So the Cauchy-Riemann equations imply they are equal up to a constant. But they both vanish at $|\mathbb{Im}\{z\}| \to \infty$, so they must be equal, and hence (11.3) holds. □

**Corollary 11.40.** *Let $\varphi, \psi \in \mathcal{H}$ and $A$ be self-adjoint. Then there exists a unique complex measure $\mu_{\varphi,\psi}$ on $\mathbb{R}$ such that*

$$\left\langle \varphi, (A - z\mathbb{1})^{-1} \psi \right\rangle = \int_{t\in\mathbb{R}} \frac{1}{t-z} \mathrm{d}\mu_{\varphi,\psi}(t) \qquad (z \in \mathbb{C} \setminus \mathbb{R}) .$$

*Proof.* We use the polarization identity to get

$$\mu_{\varphi,\psi} := \frac{1}{4} \left(\mu_{\varphi+\psi} - \mu_{\varphi-\psi} + \mathrm{i}\left(\mu_{\varphi-\mathrm{i}\psi} - \mu_{\varphi+\mathrm{i}\psi}\right)\right) .$$

□

For uniqueness, we note that if $\nu$ is any other measure obeying

$$\left\langle \varphi, (A - z\mathbb{1})^{-1} \psi \right\rangle = \int_{t\in\mathbb{R}} \frac{1}{t-z} \mathrm{d}\nu(t) \qquad (z \in \mathbb{C} \setminus \mathbb{R})$$

then we have

$$\int_{t\in\mathbb{R}} \frac{1}{t-z} \mathrm{d}\nu(t) = \int_{t\in\mathbb{R}} \frac{1}{t-z} \mathrm{d}\mu_{\varphi,\psi}(t) \qquad (z \in \mathbb{C} \setminus \mathbb{R}) .$$

But since $t \mapsto \frac{1}{t-z}$ is dense in $C_0(\mathbb{R})$ as $z$ ranges in $\mathbb{C} \setminus \mathbb{R}$, the two measures must coincide.

**Theorem 11.41.** *Assume that $\psi$ is a cyclic vector for the self-adjoint operator $A$. Then $A$ is unitarily equivalent to the operator of multiplication by $x$ on*

$$L^2(\mathbb{R}, \mu_\psi) .$$

*In particular,*

$$\sigma(A) = \operatorname{supp}(\mu_\psi) .$$

*Proof.* We note that (as long as $\psi \neq 0$, otherwise nothing makes sense),

$$(A - z\mathbb{1})^{-1} \psi = (A - w\mathbb{1})^{-1} \psi$$

iff $z = w$. For $z \in \mathbb{C} \setminus \mathbb{R}$, set

$$r_z(x) := (x - z)^{-1} .$$

Then $r_z \in L^2(\mathbb{R}, \mu_\psi)$ and the linear span of $\{ r_z \}_{z\in\mathbb{C}\setminus\mathbb{R}}$ is dense in $L^2(\mathbb{R}, \mu_\psi)$. Define

$$U (A - z\mathbb{1})^{-1} \psi := r_z .$$

If $\overline{z} \neq w$, then

$$\langle r_z, r_w \rangle_{L^2(\mathbb{R},\mu_\psi)} = \int_{\mathbb{R}} \overline{r_z} r_w \mathrm{d}\mu_\psi = \int_{\mathbb{R}} r_{\overline{z}} r_w \mathrm{d}\mu_\psi = \frac{1}{\overline{z}-w} \int_{\mathbb{R}} (r_{\overline{z}} - r_w) \,\mathrm{d}\mu_\psi = \frac{1}{\overline{z}-w} \left[ \int_{\mathbb{R}} r_{\overline{z}} \mathrm{d}\mu_\psi - \int_{\mathbb{R}} r_w \mathrm{d}\mu_\psi \right] .$$

But

$$\int_{\mathbb{R}} r_w \mathrm{d}\mu_\psi \equiv \left\langle \psi, (A - w\mathbb{1})^{-1} \psi \right\rangle \qquad (w \in \mathbb{C} \setminus \mathbb{R})$$

by definition of $\mu_\psi$. Thus,

$$\begin{aligned} \langle r_z, r_w \rangle_{L^2(\mathbb{R},\mu_\psi)} &= \frac{1}{\overline{z}-w} \left[ \left\langle \psi, (A - \overline{z}\mathbb{1})^{-1} \psi \right\rangle - \left\langle \psi, (A - w\mathbb{1})^{-1} \psi \right\rangle \right] \\ &= \left\langle (A - z\mathbb{1})^{-1} \psi, (A - w\mathbb{1})^{-1} \psi \right\rangle . \end{aligned}$$

If $\overline{z} = w$, take a sequence of points to get the relation for all $z, w \in \mathbb{C} \setminus \mathbb{R}$. Thanks to cyclicity and (9.1), $U$ extends to a unitary

$$U : \mathcal{H} \quad \to \quad L^2(\mathbb{R}, \mu_\psi) .$$

Note that

$$U (A - z\mathbb{1})^{-1} (A - w\mathbb{1})^{-1} \psi \quad \overset{\text{resol. id.}}{=} \quad r_z(x) r_w(x) = r_z(x) U (A - w\mathbb{1})^{-1} \psi$$

or in other words,

$$U (A - z\mathbb{1})^{-1} \quad = \quad r_z(x) U$$

which means that $(A - z\mathbb{1})^{-1}$ is unitarily equivalent to multiplication by $r_z$. Next, for any $\varphi \in \mathcal{H}$,

$$\begin{aligned} UA (A - z\mathbb{1})^{-1} \varphi &= U\varphi + zU (A - z\mathbb{1})^{-1} \varphi \\ &= \left(\mathbb{1} + z (x - z)^{-1}\right) U\varphi \\ &= x (x - z)^{-1} U\varphi \\ &= xU (A - z\mathbb{1})^{-1} \varphi \end{aligned}$$

or in other words

$$UA \quad = \quad xU$$

i.e., $A$ is unitarily equivalent to multiplication by $x$. □

Note that a unitary satisfying

$$UAU^{-1} \quad = \quad \text{multiplication by } x \mapsto x \text{ on } L^2(\mathbb{R}, \mu_\psi)$$

is *not* unique, but it can be made unique by further requiring that

$$U\psi \quad = \quad \frac{\|\psi\|}{\|f\|} f .$$

### 11.5.2 The proof of the spectral theorem in the general case

Using the decomposition theorem Theorem 11.34, we have for any self-adjoint $A$, the sequence

$$\{ A_n : \mathcal{D}(A_n) \to \mathcal{H}_n \}$$

with $\psi_n \in \mathcal{H}_n$ cyclic for $A_n$, and so applying the cyclic spectral theorem pointwise in $n$ we find a sequence of unitary maps

$$U_n : \mathcal{H}_n \quad \to \quad L^2(\mathbb{R}, \mu_{\psi_n}) .$$

We thus define

$$U \quad := \quad \bigoplus_n U_n$$

which is also unitary and find

**Theorem 11.42.** *The map*

$$U:\mathcal{H}\to\bigoplus_n L^2\left(\mathbb{R},\mu_{\psi_n}\right)$$

*is unitary and $A$ is unitarily equivalent to $\bigoplus_n M_n$ where $M_n$ is multiplication by $x\mapsto x$ on $L^2\left(\mathbb{R},\mu_{\psi_n}\right)$. In particular,*

$$\sigma\left(A\right)=\overline{\bigcup_n \operatorname{supp}\left(\mu_{\psi_n}\right)}\,.$$

*Moreover, if $\varphi\in\mathcal{H}$ and $U\varphi:=\left\{\,\varphi_n\,\right\}_n$ then $\mu_\varphi=\sum_n\mu_{\varphi_n}$.*

## 11.6 Schrödinger operators

The material for this section is taken from [Tes09] among other sources.

### 11.6.1 The Laplacian–free dynamics

Consider the operator $-\Delta$ on $L^2\left(\mathbb{R}^d\right)$. This represents the Hamiltonian of a free particle in three-dimensional space. Indeed,

$$H=\frac{P^2}{2m}+V\left(X\right)$$

in quantum mechanics where $m$ is the mass of the particle and $V:\mathbb{R}^d\to\mathbb{R}$ is the potential. Here $X$ is the position operator on $L^2\left(\mathbb{R}^d\right)$ and $P\equiv-\mathrm{i}\nabla$ (we use $\hbar\equiv1$). Hence, with $V=0$ (free particle) and $m=\frac{1}{2}$ we get

$$H=-\Delta\equiv\sum_{j=1}^{d}-\partial_j^2\,.$$

Clearly this operator requires some explanation since functions in $L^2\left(\mathbb{R}^d\right)$ are *not* differentiable! By the way, recall we are using $L^2$ because quantum mechanics stipulates that if $\psi:\mathbb{R}^d\to\mathbb{C}$ is the wave function of a particle, then

$$\mathbb{R}^d\ni x\mapsto\left|\psi\left(x\right)\right|^2$$

is a probability density function (which yields the probability to find the particle in a given position).

The problem of functions in $L^2$ not being twice-differentiable can be understood as their derivatives (or second derivatives) being infinite, or having infinite $L^2$ norm. Hence, to make sense of $-\Delta$ we need to first give it a domain. A natural choice would be those $L^2$ functions which are twice differentiable and whose derivatives land in $L^2$:

$$\mathcal{D}\left(-\Delta\right)\overset{?}{:=}\left\{\,\psi\in L^2\left(\mathbb{R}^d\right)\,\middle|\,\psi\text{ is twice differentiable with derivatives in }L^2\,\right\}\,.$$

But this definition turns out to be too restrictive. Instead we need the notion of

**Definition 11.43** (Weak derivatives)**.** A function $f\in L^2\left(\mathbb{R}^d\to\mathbb{C}\right)$ is said to be *weakly-differentiable* with weak-$j$-derivative $\psi\in L^2\left(\mathbb{R}^d\right)$ iff

$$\int_{\mathbb{R}^d}f\partial_j\varphi=-\int_{\mathbb{R}^d}\psi\varphi\qquad\left(\varphi\in C_c^\infty\left(\mathbb{R}^d\right)\right)\,.$$

We then say that $\psi=\partial_j\varphi$ *weakly*.

*Claim* 11.44. The weak derivative is unique if it exists.

Hence we rather define

$$\mathcal{D}\left(-\Delta\right):=\left\{\,\psi\in L^2\left(\mathbb{R}^d\right)\,\middle|\,\psi\text{ has two weak derivatives in }L^2\,\right\}\,.$$

This space actually has a name: the second Sobolev space.

**The Fourier transform** There is yet another way to characterize the second Sobolev space, via the Fourier transform

$$\mathcal{F} : L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$$

given by

$$\left(\mathcal{F}\psi\right)(p) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x}\psi(x)\,\mathrm{d}x\,.$$

In fact, this integral is initially only defined for functions in $L^1$, but not necessarily in $L^2$! A convenient thing to do in this case is to define $\mathcal{F}$ only on the Schwarz space

$$\mathcal{S}\left(\mathbb{R}^d\right) := \left\{ \psi \in C^\infty\left(\mathbb{R}^d\right) \,\middle|\, \sup_{x\in\mathbb{R}^d} \left|x^\alpha \left(\partial_\beta \psi\right)(x)\right| < \infty \forall \alpha,\beta \in \mathbb{N}_{\geq 0}^d \right\}.$$

This space is a vector space which is dense in $L^2\left(\mathbb{R}^d\right)$ (recall the functions of compact support are dense in $L^2$ and $C_c^\infty \subseteq \mathcal{S}$). Then one may prove that

$$\mathcal{F} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right)$$

is a well-defined bijection with inverse

$$\left(\mathcal{F}^{-1}\hat{\psi}\right)(x) = (2\pi)^{-\frac{d}{2}} \int_{p\in\mathbb{R}^d} \mathrm{e}^{\mathrm{i}p\cdot x}\hat{\psi}(p)\,\mathrm{d}p\,.$$

Moreover, $\mathcal{F}^4 = \mathbb{1}$. Since we have Parseval's theorem

$$\|\psi\|_{L^2} = \|\mathcal{F}\psi\|_{L^2} \qquad \left(\psi \in \mathcal{S}\left(\mathbb{R}^d\right)\right)$$

then (9.1) allows us to extend $\mathcal{F}$ to the entirety of $L^2\left(\mathbb{R}^d\right)$ by setting

$$\left(\mathcal{F}(\psi)\right)(p) \quad := \quad (2\pi)^{-\frac{d}{2}} \lim_{R\to\infty} \int_{x\in B_R(0)} \mathrm{e}^{-\mathrm{i}p\cdot x}\psi(x)\,\mathrm{d}x$$

and the limit is to be understood in the $L^2$ norm. Hence $\mathcal{F}$ extends to a unitary operator

$$\mathcal{F} : L^2 \to L^2$$

with spectrum

$$\sigma(\mathcal{F}) = \{\pm 1, \pm\mathrm{i}\}\,.$$

**The Sobolev space** With the Fourier transform at hand, it is easier to characterize the Sobolev spaces encountered above as

**Definition 11.45** (Sobolev space)**.** For $r \in \mathbb{N}$, we define the $r$th Sobolev space as

$$H^r\left(\mathbb{R}^d\right) := \left\{ \psi \in L^2 \,\middle|\, p \mapsto \|p\|^r \mathcal{F}(\psi)(p) \in L^2\left(\mathbb{R}^d\right) \right\}$$

which is essentially a statement about existence of weak derivatives. This space turns out to be a Hilbert space, with inner product

$$\langle\varphi,\psi\rangle_{H^r} := \sum_{\alpha,\beta\in\mathbb{N}_{\geq 0}^d : |\alpha|=|\beta|\leq r} \left\langle \partial^\alpha\varphi, \partial^\beta\psi\right\rangle_{L^2}\,.$$

**Back to the domain of $-\Delta$** We thus identify

$$\mathcal{D}(-\Delta) \equiv H^2\left(\mathbb{R}^d\right)\,.$$

We have seen that this domain is a dense vector space within $L^2$. In fact using $\mathcal{F}$ we see that $-\Delta$ is unitarily equivalent to $M_{p\mapsto\|p\|^2}$ on $L^2$. Its domain is

$$\mathcal{D}\left(M_{p\mapsto\|p\|^2}\right) = \left\{ \hat{\psi} \in L^2 \,\middle|\, p \mapsto \|p\|^2\hat{\psi}(p) \in L^2 \right\}.$$

We have seen in HW11 that this domain is the maximal domain of definition for this operator. Thanks to this, we know that $-\Delta$ is self-adjoint.

*Claim* 11.46. We have

$$\sigma(-\Delta) = \sigma_{\mathrm{ac}}(-\Delta) = [0,\infty)\,.$$

*Proof.* We calculate the spectral measure associated to some $\psi$. We start with the resolvent

$$\begin{aligned}\left\langle\psi,\left(-\Delta-z\mathbb{1}\right)^{-1}\psi\right\rangle &= \left\langle\mathcal{F}\psi,\mathcal{F}\left(-\Delta-z\mathbb{1}\right)^{-1}\mathcal{F}^{*}\mathcal{F}\psi\right\rangle\\ &= \int_{\mathbb{R}^{d}}\left|\hat{\psi}\left(p\right)\right|^{2}\frac{1}{\left\|p\right\|^{2}-z}\mathrm{d}p\\ &= C_{d}\int_{r=0}^{\infty}\mathrm{d}r r^{d-1}\int_{\omega\in\mathbb{S}^{d-1}}\mathrm{d}\Omega\left(\omega\right)\left|\hat{\psi}\left(r\omega\right)\right|^{2}\frac{1}{r^{2}-z}\\ &= \int_{r=0}^{\infty}\frac{1}{r^{2}-z}r^{d-1}\left(C_{d}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{\psi}\left(r\omega\right)\right|^{2}\mathrm{d}\Omega\left(\omega\right)\right)\mathrm{d}r\\ &= \int_{r=-\infty}^{\infty}\frac{1}{r^{2}-z}\underbrace{\chi_{[0,\infty)}\left(r\right)r^{d-1}\left(C_{d}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{\psi}\left(r\omega\right)\right|^{2}\mathrm{d}\Omega\left(\omega\right)\right)\mathrm{d}r}_{=:\mathrm{d}\tilde{\mu}(r)}\,.\end{aligned}$$

Hence we identify

$$\mathrm{d}\tilde{\mu}\left(r\right)\equiv\chi_{[0,\infty)}\left(r\right)r^{d-1}\left(C_{d}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{\psi}\left(r\omega\right)\right|^{2}\mathrm{d}\Omega\left(\omega\right)\right)\mathrm{d}r$$

as the spectral measure associated to $-\Delta$ and $\psi$. Make now a change of variable $\lambda:=r^{2}$ to get

$$\left\langle\psi,\left(-\Delta-z\mathbb{1}\right)^{-1}\psi\right\rangle=\int_{\lambda\in\mathbb{R}}\frac{1}{\lambda-z}\mathrm{d}\mu_{\psi}\left(\lambda\right)$$

with

$$\mathrm{d}\mu_{\psi}\left(\lambda\right):=\frac{1}{2}\chi_{[0,\infty)}\left(\lambda\right)\lambda^{\frac{d}{2}-1}\left(C_{d}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{\psi}\left(\sqrt{\lambda}\omega\right)\right|^{2}\mathrm{d}\Omega\left(\omega\right)\right)\mathrm{d}\lambda\,.$$

In particular $\mu_{\psi}$ is absolutely-continuous w.r.t. the Lebesgue measure. Since $\psi$ was arbitrary, this implies there is only ac part to the spectrum. ☐

*Claim* 11.47. $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ is a core for $-\Delta$.

*Proof.* In fact $\mathcal{S}\left(\mathbb{R}^{d}\right)$ is a core for $-\Delta$ and hence suffice to show $-\Delta|_{C_{c}^{\infty}}$ contains $-\Delta|_{\mathcal{S}}$. To that end, let $\varphi\in C_{c}^{\infty}$ which is 1 for $\|x\|\leq 1$ and vanishes for $\|x\|\geq 2$. Set

$$\varphi_{n}\left(x\right)\quad:=\quad\varphi\left(\frac{1}{n}x\right)\,.$$

Then $\psi_{n}\left(x\right):=\varphi_{n}\left(x\right)\psi\left(x\right)$ is in $C_{c}^{\infty}$ for any $\psi\in\mathcal{S}$ and $\psi_{n}\to\psi$, as well as

$$-\Delta\psi_{n}\to-\Delta\psi\,.$$

☐

Hence we know that $-\Delta$ *has* a unique self-adjoint extension.

*Claim* 11.48. For any $t>0$ the operator $\exp\left(-\mathrm{i}t\left(-\Delta\right)\right)$ exists and is given with the integral kernel on $L^{1}\cap L^{2}$ via

$$\exp\left(-\mathrm{i}t\left(-\Delta\right)\right)\left(x,y\right)=\frac{1}{\left(4\pi\mathrm{i}t\right)^{\frac{d}{2}}}\exp\left(\mathrm{i}\frac{\left\|x-y\right\|^{2}}{4t}\right)\qquad\left(x,y\in\mathbb{R}^{d}\right)\,.$$

and for any $\psi\in L^{2}$, $\Omega\subseteq\mathbb{R}^{d}$ compact,

$$\lim_{t\to\infty}\left\|\chi_{\Omega}\left(X\right)\mathrm{e}^{-\mathrm{i}t(-\Delta)}\psi\right\|^{2}=0\,.$$

In fact one may also calculate the heat kernel

$$\exp\left(-t\left(-\Delta\right)\right)\left(x,y\right)=\frac{1}{\left(4\pi t\right)^{\frac{d}{2}}}\exp\left(-\frac{\left\|x-y\right\|^{2}}{4t}\right)\qquad\left(t>0;\,x,y\in\mathbb{R}^{d}\right)\,.$$

Using the identity

$$\frac{1}{\lambda - z} = \int_{t=0}^{\infty} \mathrm{e}^{-t(\lambda - z)} \mathrm{d}t \qquad (\lambda \in \mathbb{R}, z \in \mathbb{C} : \mathbb{Re}\{z\} < 0)$$

we can calculate the resolvent out of the heat kernel.

*Claim* 11.49. For any $z \in \mathbb{C}$ with $\mathbb{Re}\{z\} < 0$, the resolvent of the Laplacian at spectral parameter $z$, i.e.,

$$(-\Delta - z\mathbb{1})^{-1}$$

is given by an integral kernel which has the integral form

$$(-\Delta - z\mathbb{1})^{-1}(x, y) = \int_{t=0}^{\infty} \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{1}{4t}\|x - y\|^2 + zt\right) \mathrm{d}t\,.$$

This integral may be evaluated to yield

$$(-\Delta - z\mathbb{1})^{-1}(x, y) = \frac{1}{2\pi}\left(\frac{\sqrt{-z}}{2\pi\|x - y\|}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(\sqrt{-z}\|x - y\|\right)$$

where $K_\nu$ is the modified Bessel function, given by the solution to the differential equation

$$\left(\partial^2 + \frac{1}{X}\partial - 1 - \frac{\nu^2}{X^2}\right) K_\nu = 0\,.$$

It has the asymptotic form

$$K_\nu(x) = \begin{cases} \frac{\Gamma(\nu)}{2}\left(\frac{x}{2}\right)^{-\nu} + \mathcal{O}\left(x^{-\nu+2}\right) & \nu > 0 \\ -\log\left(\frac{x}{2}\right) + \mathcal{O}(1) & \nu = 0 \end{cases} \qquad (|x| \to 0)$$

and

$$K_\nu(x) \quad = \quad \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) \qquad (|x| \to 0)\,.$$

In particular, $(-\Delta - z\mathbb{1})^{-1}(x, y)$ has an analytic continuation to $z \in \mathbb{C} \setminus [0, \infty)$. In fact, for odd $d$ we find the explicit form

$$(-\Delta - z\mathbb{1})^{-1}(x, y) = \begin{cases} \frac{1}{2\sqrt{-z}}\mathrm{e}^{-\sqrt{-z}\|x-y\|} & d = 1 \\ \frac{1}{4\pi\|x-y\|}\mathrm{e}^{-\sqrt{-z}\|x-y\|} & d = 3 \\ \cdots & d \geq 5 \end{cases}.$$

We see in particular that for $z = 0$ there is still decay for $d \geq 3$ but not for $d = 1$.

### 11.6.2 The Laplacian on more complicated domains

Studying the Laplacian on some other domain $\Omega \subseteq \mathbb{R}^d$ rather than the whole space leads to some interesting questions, e.g., those of boundary conditions, only some of which yield a self-adjoint operator. We have seen some of this phenomenon in Example 11.28.

[TODO: elaborate on this]

### 11.6.3 The Harmonic oscillator

Another exactly solvable model is the harmonic oscillator. While it may be solved in any $d \in \mathbb{N}_{\geq 0}$, we concentrate on $d = 1$ for simplicity, the generalization being clear. We thus consider on $L^2(\mathbb{R} \to \mathbb{C})$ the operator

$$\begin{aligned} H_{\text{SHO}} &= P^2 + \frac{1}{2}\omega^2 X^2 \\ &= -\Delta + \frac{1}{2}\omega^2 X^2\,. \end{aligned}$$

For its domain we again need to be careful. We pick

$$\mathcal{D}(H_{\text{SHO}}) := \operatorname{span}\left(\left\{\, x \mapsto x^\alpha \mathrm{e}^{-\frac{1}{2}x^2} \,\middle|\, \alpha \in \mathbb{N}_{\geq 0} \right\}\right)$$

for reasons that will become clear momentarily. To diagonalize this Hamiltonian, it is convenient to introduce the shift operators

$$R := \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega}{2}}X - \sqrt{\frac{2}{\omega}}\partial\right)$$

with the same domain. One then may verify that

$$[R^*, R] = \mathbb{1}$$

and

$$H_{\text{SHO}} = \frac{1}{\sqrt{2}}\omega\left(2RR^* + \mathbb{1}\right) .$$

In fact, $N := RR^*$ is called the number operator which obeys $[N, R] = R$ and $[N, R^*] = -R^*$. Moreover, if $\psi$ is an eigenvector of $N$ with eigenvalue $n$, $N\psi = n\psi$, then

$$NR\psi = (n+1)\, R\psi \, .$$

Moreover, $\|R\psi\|^2 = \langle\psi, R^*R\psi\rangle = (n+1)\,\|\psi\|^2$ and $\|R^*\psi\|^2 = n\|\psi\|^2$. Hence

$$\sigma_p\,(N) \subseteq \mathbb{N}_{\geq 0}\, .$$

If $N\psi_0 = 0$ for some $\psi_0 \neq 0$ then $R^*\psi_0 = 0$ which is a simple ODE whose solution is

$$\psi_0\,(x) = \left(\frac{\omega}{\sqrt{2}\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}\frac{1}{\sqrt{2}}\omega x^2} \qquad (x \in \mathbb{R})$$

which lies in $\mathcal{D}$. Hence

$$\psi_n\,(x) \quad := \quad \frac{1}{\sqrt{n!}}R^n\psi_0$$

is a normalized eigenfucntion of $N$ with eigenvalue $n$. In fact, these generate polynomials, so $\operatorname{span}(\psi_n) = \mathcal{D}\,(H_{\text{SHO}})$. These polynomials are called the Hermite polynomials. We find

> *Claim* 11.50. $H_{\text{SHO}}$ is essentially self-adjoint on $\mathcal{D}\,(H_{\text{SHO}})$ and has an orthonormal basis of eigenfunctions. The spectrum is given by
>
> $$\sigma\,(H_{\text{SHO}}) = \sigma_p\,(H_{\text{SHO}}) = \left\{ \frac{1}{\sqrt{2}}\omega\,(2n+\mathbb{1}) \,\middle|\, n \in \mathbb{N}_{\geq 0} \right\} .$$

### 11.6.4 One particle Schrödinger operators

These are operators on $L^2\left(\mathbb{R}^d\right)$ of the form

$$H = -\Delta + V\,(X)$$

for some $V : \mathbb{R}^d \to \mathbb{R}$. The first question we'd like to ask is what conditions on $V$ guarantee the self-adjointness of $H$, and what conditions on $V$ imply certain spectral properties of $H$.

[TODO: elaborate on this]

# A Useful identities and inequalities

# B Glossary of mathematical symbols and acronyms

Sometimes it is helpful to include mathematical symbols which can function as valid grammatical parts of sentences. Here is a glossary of some which might appear in the text:

- The bracket $\langle\cdot,\cdot\rangle_V$ means an inner product on the inner product space $V$. For example,
$$\langle u,v\rangle_{\mathbb{R}^2} \equiv u_1 v_1 + u_2 v_2 \qquad \left(u,v\in\mathbb{R}^2\right)$$
and
$$\langle u,v\rangle_{\mathbb{C}^2} \equiv \overline{u_1} v_1 + \overline{u_2} v_2 \qquad \left(u,v\in\mathbb{C}^2\right) \, .$$
- Sometimes we denote an integral by writing the integrand without its argument. So if $f:\mathbb{R}\to\mathbb{R}$ is a real function, we sometimes in shorthand write
$$\int_a^b f$$
when we really mean
$$\int_{t=a}^b f(t)\,\mathrm{d}t\,.$$
This type of shorthand notation will actually also apply for contour integrals, in the following sense: if $\gamma:[a,b]\to\mathbb{C}$ is a contour with image set $\Gamma:=\operatorname{im}(\gamma)$ and $f:\mathbb{C}\to\mathbb{C}$ is given, then the contour integral of $f$ along $\gamma$ will be denoted equivalently as
$$\int_\Gamma f \equiv \int_\Gamma f(z)\,\mathrm{d}z \equiv \int_{t=a}^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t$$
depending on what needs to be emphasized in the context. Sometimes when the contour is clear one simply writes
$$\int_{z_0}^{z_1} f(z)\,\mathrm{d}z$$
for an integral along *any* contour from $z_0$ to $z_1$.
- iff means "if and only if", which is also denoted by the symbol $\Longleftrightarrow$.
- WLOG means "without loss of generality".
- CCW means "counter-clockwise" and CW means "clockwise".
- $\exists$ means "there exists" and $\nexists$ means "there does not exist". $\exists!$ means "there exists *a unique*".
- $\forall$ means "for all" or "for any".
- : (i.e., a colon) may mean "such that".
- ! means negation, or "not".
- $\wedge$ means "and" and $\vee$ means "or".
- $\Longrightarrow$ means "and so" or "therefore" or "it follows".
- $\in$ denotes set inclusion, i.e., $a\in A$ means $a$ is an element of $A$ or $a$ lies in $A$.
- $\ni$ denotes set inclusion when the set appears first, i.e., $A\ni a$ means $A$ includes $a$ or $A$ contains $a$.
- Speaking of set inclusion, $A\subseteq B$ means $A$ is contained within $B$ and $A\supseteq B$ means $B$ is contained within $A$.
- $\varnothing$ is the empty set $\{\ \}$.
- While $=$ means equality, sometimes it is useful to denote types of equality:
  - $a:=b$ means "this equation is now the instant when $a$ is defined to equal $b$".
  - $a\equiv b$ means "at some point above $a$ has been defined to equal $b$".
  - $a=b$ will then simply mean that the result of some calculation *or* definition stipulates that $a=b$.
  - Concrete example: if we write $\mathrm{i}^2=-1$ we don't specify anything about *why* this equality is true but writing $\mathrm{i}^2\equiv-1$ means this is a matter of definition, not calculation, whereas $\mathrm{i}^2:=-1$ is the first time you'll see this definition. So this distinction is meant to help the reader who wonders *why* an equality holds.

### B.1 Important sets

1. The unit circle

$$\mathbb{S}^1 \equiv \{ z \in \mathbb{C} \mid |z| = 1 \} .$$

2. The (open) upper half plane

$$\mathbb{H} \equiv \{ z \in \mathbb{C} \mid \mathbb{Im}\{z\} > 0 \} .$$

## C Reminder from complex analysis

**Lemma C.1.** *Given* $\Omega \in \mathrm{Open}(\mathbb{C})$ *and* $K \in \mathrm{Compact}(\mathbb{C})$ *such that* $K \subseteq \Omega$*. Then there exists some collection* $\gamma_j : [0,1] \to \Omega$*,* $j = 1, \dots, n$ *of simple loops none of whose range intersects* $K$*, and such that*

$$\frac{\mathrm{i}}{2\pi} \sum_{j=1}^{n} \oint_{\gamma_j} \frac{1}{\alpha - z} \mathrm{d}z = \begin{cases} 1 & \alpha \in K \\ 0 & \alpha \notin \Omega \end{cases} .$$

## D Vocabulary from topology

**Definition D.1.** Given a set $S$, a *topology* on $S$ is a set of subsets $\mathcal{T}$ of $S$ (i.e., it is a subset of the power set $\mathcal{P}(S)$) with the properties that:

1. $S, \varnothing \in \mathcal{T}$.
2. $A \cap B \in \mathcal{T}$ for any $A, B \in \mathcal{T}$.
3. $\left(\bigcup_{\alpha \in \mathcal{I}} A_\alpha\right) \in \mathcal{T}$ for any $\{ A_\alpha \}_{\alpha \in \mathcal{I}} \subseteq \mathcal{T}$. Here $\mathcal{I}$ is any index set, which need not be countable.

If we have a space $S$ which we know is a topological space and we want to refer to its topology, we denote this by $\mathrm{Open}(S)$.

**Definition D.2.** A *neighborhood of a point* $x \in S$ is any open set $U \in \mathrm{Open}(S)$ that contains $x$: $x \in U \in \mathrm{Open}(S)$. We denote the set of neighborhoods of a point $x$ as $\mathrm{Nbhd}(x) \subseteq \mathrm{Open}(S)$.

**Definition D.3.** A topological space $S$ is called *Hausdorff* iff for any $x, y \in S$ such that $x \neq y$, there are $U \in \mathrm{Nbhd}(x), V \in \mathrm{Nbhd}(y)$ such that $U \cap V = \varnothing$.

**Definition D.4.** A subset $\mathcal{B} \subseteq \mathrm{Open}(S)$ is called a *base* or *basis* for $\mathrm{Open}(S)$ iff any $U \in \mathrm{Open}(S)$ may be written as

$$U = \bigcup_{\alpha \in \mathcal{I}} B_\alpha$$

for some $\{ B_\alpha \}_{\alpha \in \mathcal{I}} \subseteq \mathcal{B}$. Here $\mathcal{I}$ is some (not necessarily countable) index set.

**Definition D.5.** A set $T \subseteq S$ is *compact* iff every open cover of $T$ has a finite sub-cover.

**Definition D.6** (Metric)**.** Given a set $S$, a metric on $S$ is a map $d : S^2 \to [0, \infty)$ such that

1. $d(x,y) = 0$ implies that $x = y$ for all $x, y \in S$.
2. $d(x,y) = d(y,x)$ for all $x, y \in S$.
3. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in S$.

**Definition D.7** (Norm)**.** A vector space $V$ is called *a normed vector space* iff there is a map

$$\|\cdot\| : V \to [0,\infty)$$

which obeys the following axioms:

1. Absolute homogeneity:
$$\|\alpha\psi\| = |\alpha| \, \|\psi\| \qquad (\alpha \in \mathbb{C}, \psi \in V) \; .$$
2. Triangle inequality:
$$\|\psi + \varphi\| \leq \|\psi\| + \|\varphi\| \qquad (\psi, \varphi \in V) \; .$$
3. Injectivity at zero: If $\|\psi\| = 0$ for some $\psi \in V$ then $\psi = 0$.

To any norm $\|\cdot\|$ a metric is associated via

$$\begin{array}{rcl} d : V^2 & \to & [0,\infty) \\ (\psi,\varphi) & \mapsto & \|\psi - \varphi\| \, . \end{array}$$

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