Functional Analysis
Princeton University MAT520
HW9, Due Nov 24th 2024

November 15, 2024

1 Compact operators

1. In an infinite dimensional Hilbert space, show that if A is invertible then it cannot be compact.

2. Show that if A_n is a sequence of operators such that for any bounded $B \subseteq \mathcal{H}$, $\overline{A_n(B)}$ is a *compact set* in $\mathcal{H}$, and $A_n \to A$ in norm, then also $\overline{A(B)}$ is a compact subset.

3. Show that the following are equivalent for a given $A \in \mathcal{B}(\mathcal{H})$:

   (a) A is compact.

   (b) For any bounded sequence $\{\psi_n\}_n \subseteq \mathcal{H}$, $\{A\psi_n\}_n$ has a subsequence which converges.

   (c) For any bounded $B \subseteq \mathcal{H}$, $\overline{A(B)}$ is a *compact set* in $\mathcal{H}$.

4. Let $A \in \mathcal{B}(\mathcal{H})$ be compact and $\{\varphi_n\}_n \subseteq \mathcal{H}$ converge *weakly* (in the sense of the Banach space weak topology on $\mathcal{H}$) to some $\varphi \in \mathcal{H}$. Show that $A\varphi_n \to A\varphi$ in norm.

5. Figure out if the following operators are compact or not (and prove what you think):

   (a) $\mathbb{1}$.

   (b) $u \otimes v^*$ for some $u, v \in \mathcal{H}$.

   (c) On *the Banach space* $X := C([0,1] \to \mathbb{C})$ with $\|\cdot\|_\infty$, let $A : X \to X$ be given

   $$(A\varphi)(x) := \int_{y=0}^{1} K(x,y)\varphi(y)\,\mathrm{d}y$$

   where $K : [0,1]^2 \to \mathbb{C}$ is some *continuous* function.

   (d) $A := \frac{1}{1+X^2}$ on $\ell^2(\mathbb{Z})$ where X is the position operator given by

   $$(X\psi)(n) \equiv n\psi(n) \qquad \left(n \in \mathbb{Z}; \psi \in \ell^2(\mathbb{Z})\right)$$

   and we employ the holomorphic functional calculus to define A.

6. On $\mathcal{H} \oplus \mathcal{H}$, let

   $$H \quad := \quad \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$$

   for some $S \in \mathcal{B}(\mathcal{H})$. Find the polar decomposition of H.

7. Show that an idempotent is compact if and only if it is of finite rank.

8. Show that no nonzero multiplication operator on $L^2([0,1])$ is compact.

9. Show that if $A \in \mathcal{B}(\mathcal{H})$ is compact and $\{e_n\}_n$ is an ONB then $\|Ae_n\| \to 0$. Find a counter-example of the converse.

10. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded region with a smooth boundary surface $\partial\Omega$. Let $f : \partial\Omega \to \mathbb{C}$ be continuous. Fix some parameter $m > 0$. Find a function $\varphi : \overline{\Omega} \to \mathbb{C}$ which is twice differentiable in Ω and continuous on $\overline{\Omega}$ such that

$$\begin{aligned} \left(-\Delta + m^2 \mathbb{1}\right) \varphi &= 0 \\ \varphi|_{\partial\Omega} &= f\,. \end{aligned}$$

Find (and prove the properties of) a function $K : \overline{\Omega} \times \overline{\Omega} \to \mathbb{C}$ (called the Poisson kernel of $-\Delta + m^2 \mathbb{1}$ in the interior of Ω) which allows the solution of the above Dirichlet problem be written as

$$\varphi(x) = \int_{y \in \partial\Omega} K(x, y) f(y) \, \mathrm{d}y\,.$$

2 Fredholm operators

11. Provide an example of a norm continuous path of operators $[0, 1] \ni t \mapsto A_t$ for which $\dim \ker A_t$ has jump discontinuities.
12. Show that if X is the position operator on $\ell^2(\mathbb{N})$ then $\frac{1}{X}$ does not have a closed image.
13. Show that if $A : V_1 \to V_2$ is a linear map between two finite dimensional Hilbert spaces then A is Fredholm and

$$\text{index}(A) = \dim V_1 - \dim V_2\,.$$

Conclude that square matrices always have zero index.

14. Prove that a compact operator cannot be Fredholm.
15. Show that if K is compact then $\mathbb{1} - K$ is Fredholm.