Functional Analysis
Princeton University MAT520
HW8, Due Nov 17th 2024

November 9, 2024

1. Let P, Q be two orthogonal projections onto subspaces M, N in a Hilbert space $\mathcal{H}$ such that $[P, Q] = 0$.
   (a) Show $P^\perp \equiv \mathbb{1} - P$, $Q^\perp$, PQ, $P + Q - PQ$ and $P + Q - 2PQ$ are orthogonal projections.
   (b) What is the relation between the projections in the previous item and M, N?
2. Let P, Q be two orthogonal projections onto subspaces M, N in a Hilbert space $\mathcal{H}$. Show that
$$\underset{n\to\infty}{\text{s-lim}} \, (PQ)^n$$
   exists and is the orthogonal projection onto $M \cap N$.
3. Let $A \in \mathcal{B}(\mathcal{H})$. Show that the set of $\lambda \in \sigma(A)$ such that λ is not an eigenvalue of A and $\operatorname{im}(A - \lambda\mathbb{1})$ is closed but not the whole of $\mathcal{H}$ is an open subset of $\mathbb{C}$.
4. Define the numerical range $N(A)$ of $A \in \mathcal{B}(\mathcal{H})$ via
$$N(A) := \{ \langle \psi, A\psi \rangle \mid \psi \in \mathcal{H} \wedge \|\psi\| = 1 \} .$$
   (a) Show that
$$\sigma(A) \subseteq \overline{N(A)} .$$
   (b) Find an example where $N(A)$ is not closed and
$$\sigma(A) \nsubseteq N(A) .$$
   (c) Find an example where
$$\sigma(A) \neq N(A) = \overline{N(A)} .$$
5. Show that if $A \in \mathcal{B}(\mathcal{H})$ has $A = A^*$ then
$$\left\| (A - z\mathbb{1})^{-1} \right\| \leq \frac{1}{|\mathbb{Im}\{z\}|} \qquad (z \in \mathbb{C} : |\mathbb{Im}\{z\}| > 0) .$$
6. Show that if $A \in \mathcal{B}(\mathcal{H})$ is an isometry then $\operatorname{im}(A)$ is closed in $\mathcal{H}$.
7. Let $V \in \mathcal{B}\left(L^2([0,1] \to \mathbb{C})\right)$ be give by
$$V(\psi) := \int_0^{\cdot} \psi \qquad \left(\psi \in L^2\right) .$$
   (a) Show that V is well-defined (it is a bounded linear map) with
$$V^*(\psi) = \int_{\cdot}^{1} \psi \qquad \left(\psi \in L^2\right) .$$
   (b) Show that the spectral radius of V, $r(V)$, equals zero and that $\sigma(V) = \{ 0 \}$.
   (c) Show that $\|V\| = \frac{2}{\pi}$.

8. Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$ be the Fourier series given by

$$\ell^2(\mathbb{Z}) \ni \psi \mapsto \left([0, 2\pi] \ni k \mapsto \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}kn} \psi_n =: \hat{\psi}(k) \right) .$$

Let $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be the discrete Laplacian:

$$A = R + R^*$$

where R is the bilateral right shift operator

$$R\delta_n := \delta_{n+1} \qquad (n \in \mathbb{Z})$$

and $\{ \delta_n \}_{n \in \mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$. Calculate

$$\mathcal{F} A \mathcal{F}^* \in \mathcal{B}(L^2(\mathbb{S}^1)) .$$

9. Call an operator $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ *local* iff

$$\inf_{x, y \in \mathbb{Z}^d} -\frac{1}{\|x - y\|} \log(|\langle \delta_x, A\delta_y \rangle|) > 0 .$$

Show that if $A = A^*$ and A is local then $(A - z\mathbb{1})^{-1}$ is local too for any $z \in \mathbb{C} : |\mathbb{Im}\{z\}| > 0$.