Functional Analysis
Princeton University MAT520
HW7, Due Nov 11th 2024

November 5, 2024

1 Hilbert spaces

In this section $\mathscr{H}$ is a Hilbert space.

1. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$:
$$Re_j \quad := \quad e_{j+1} \qquad (j \in \mathbb{N})$$
where $\{ e_j \}_{j\in\mathbb{N}}$ is the standard basis of $\ell^2(\mathbb{N})$ and extend linearly.
   (a) Calculate R^*.
   (b) Calculate $|R|^2$ and $|R^*|^2$.
   (c) Show that R is a partial isometry.
   (d) Calculate $\sigma(R), \sigma(R^*), \sigma\left(|R|^2\right)$ and $\sigma\left(|R^*|^2\right)$.
2. Let $\hat{R}$ be the bilateral right shift operator on $\ell^2(\mathbb{Z})$:
$$\hat{R}e_j := e_{j+1} \qquad (j \in \mathbb{Z})$$
where $\{ e_j \}_{j\in\mathbb{Z}}$ is the standard basis of $\ell^2(\mathbb{Z})$ and extend linearly.
   (a) Calculate $\hat{R}^*$.
   (b) Calculate $\left|\hat{R}\right|^2$ and $\left|\hat{R}^*\right|$.
   (c) Show that $\hat{R}$ is a unitary.
   (d) Calculate $\sigma\left(\hat{R}\right), \sigma\left(\hat{R}^*\right), \sigma\left(\left|\hat{R}\right|^2\right)$ and $\sigma\left(\left|\hat{R}^*\right|^2\right)$.
3. Let $\frac{1}{X} \in \mathcal{B}\left(\ell^2(\mathbb{N})\right)$ be given by
$$\frac{1}{X}e_j := \frac{1}{j}e_j \qquad (j \in \mathbb{N})$$
and extend linearly.
   (a) Calculate $\left(\frac{1}{X}\right)^*$.
   (b) Calculate $\sigma\left(\frac{1}{X}\right)$.
   (c) Show that $\frac{1}{X}$ does *not* have closed range.
4. Show that if M is a closed linear subspace and $P_M : \mathscr{H} \to \mathscr{H}$ is given by
$$P_M\psi \quad := \quad a$$
where $\psi = a + b$ in the unique decomposition $\mathscr{H} = M \oplus M^\perp$, then P_M is a *self-adjoint projection*, i.e., show that $P_M = P_M^* = P_M^2$. Conversely, given any self-adjoint projection $P \in \mathcal{B}(\mathscr{H})$, find a closed linear subspace M such that $P = P_M$.

5. Let $\{ A_n \}_n \subseteq \mathcal{B}(\mathcal{H})$ such that for any $\varphi, \psi \in \mathcal{H}$,

$$\exists \lim_n \langle \varphi, A_n \psi \rangle \ .$$

Show there exists $A \in \mathcal{B}(\mathcal{H})$ such that $A_n \to A$ weakly.

6. For any $t > 0$, let $T_t \in \mathcal{B}\left(L^2(\mathbb{R})\right)$ be given by

$$T_t \varphi \ := \ \varphi(\cdot + t) \qquad \left(\varphi \in L^2\right) \ .$$

   (a) Calculate $\|T_t\|$.

   (b) Find a limit to which T_t converges as $t \to \infty$ (in which operator topology?).

7. Show that multiplication is not jointly continuous as a map

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

if $\mathcal{B}(\mathcal{H})$ is given the strong operator topology.

8. Let $A_n \to A, B_n \to B$ in the strong operator topology. Show that $A_n B_n \to AB$ in the strong operator topology.

9. Let $A_n \to A, B_n \to B$ in the weak operator topology. Find a counter example for $A_n B_n \to AB$ in the weak operator topology.

10. Show that for $A \in \mathcal{B}(\mathcal{H})$,

$$\|A\|_{\text{op}} = \sup\left(\{ |\langle \varphi, A\psi \rangle| \mid \|\varphi\| = \|\psi\| = 1 \}\right)$$

and if $A = A^*$ then

$$\|A\|_{\text{op}} = \sup\left(\{ |\langle \varphi, A\varphi \rangle| \mid \|\varphi\| = 1 \}\right) \ .$$

11. Show that if $A_n \geq 0$, $A_n \to A$ in norm (resp. strongly) then $\sqrt{A_n} \to \sqrt{A}$ in norm (resp. strongly).

12. Show that if $A_n \to A$ in norm then $|A_n| \to |A|$ in norm.

13. Show that if $A_n \to A$ and $A_n^* \to A^*$ strongly then $|A_n| \to |A|$ strongly.

14. Find a counter example to

$$\| |A| - |B| \| \leq \|A - B\| \ .$$