Functional Analysis
Princeton University MAT520
HW6, Due Nov 3rd 2024

November 3, 2024

Solve the maximal number of exercises which doesn't cause you misery.

1 Hilbert spaces

In the following exercises, $\mathcal{H}$ is a Hilbert space and $\varphi, \psi, \dots$ are vectors in it.

1. Show that $\ell^2(\mathbb{N} \to \mathbb{C})$ is a Hilbert space: define an inner product on it and show that the induced metric is complete.
2. Show that $L^2(\mathbb{R})$ (with the Lebesgue measure) is a Hilbert space. Define an inner product and show that the induced metric is complete.
3. Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of bounded linear operators on $\mathcal{H}$. Show (in a concrete example, e.g., $\mathcal{H} = \mathbb{C}^2$) that $\mathcal{B}(\mathcal{H})$ is *not* a Hilbert space by showing the operator norm violates the parallelogram law.
4. Show that when $\dim(\mathcal{H}) = \infty$ then
$$\mathcal{H} \otimes \mathcal{H}^* \subsetneq \mathcal{B}(\mathcal{H}) .$$
Note: this may be hard.
5. Show that if $M \subseteq \mathcal{H}$ is a closed vector subspace of it then $\left(M^\perp\right)^\perp = M$.
6. Show that if $\{ \varphi_n \}_{n\in\mathbb{N}}$ is a sequence of *pairwise orthogonal* vectors in $\mathcal{H}$, then the following are equivalent:
   (a) $\sum_{n\in\mathbb{N}} \varphi_n$ exists in $\|\cdot\|_{\mathcal{H}}$.
   (b) $\sum_{n\in\mathbb{N}} \|\varphi_n\|^2_{\mathcal{H}} < \infty$.
   (c) For any $\psi \in \mathcal{H}$, $\sum_{n\in\mathbb{N}} \langle \psi, \varphi_n \rangle_{\mathcal{H}}$ exists.
7. Show that if $\{ \varphi_n \}_{n\in\mathbb{N}}$ is a sequence of vectors in $\mathcal{H}$, then item (a) above implies item (c) above. Find an example where item (c) does *not* imply item (a).
8. Let $N \in \mathbb{N}$, $\alpha \in \mathbb{C}$ with $\alpha^N = 1$ and $\alpha^2 \neq 1$. Show that in $\mathcal{H}$, for any $\varphi, \psi \in \mathcal{H}$:
$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \frac{1}{N} \sum_{n=1}^{N} \alpha^n \|\psi + \alpha^n \varphi\|^2 .$$
Show also that
$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\theta} \left\|\psi + \mathrm{e}^{\mathrm{i}\theta} \varphi\right\|^2 \mathrm{d}\theta .$$
9. Let
$$\{ \varphi_n \}_{n\in\mathbb{N}}, \{ \psi_n \}_{n\in\mathbb{N}} \subseteq \{ \xi \in \mathcal{H} \mid \|\xi\| \leq 1 \} .$$
Assume further that $\lim_{n\in\mathbb{N}} \langle \varphi_n, \psi_n \rangle \to 1$. Show that
$$\lim_{n\in\mathbb{N}} \|\varphi_n - \psi_n\| = 0 .$$

10. Let $\{ \varphi_n \}_n \subseteq \mathcal{H}$ converge to some $\varphi \in \mathcal{H}$ weakly (i.e., for any $\xi \in \mathcal{H}$, $\langle \xi, \varphi_n \rangle \to \langle \xi, \varphi \rangle$ in $\mathbb{C}$). Assume further that $\|\varphi_n\| \to \|\varphi\|$ in $\mathbb{R}$. Show that
$$\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0\,.$$

11. Let V be an inner product space and $\{ \varphi_n \}_{n=1}^N \subseteq V$ be an orthonormal set. Show that, for fixed ψ, the functional
$$F(\alpha_1, \dots, \alpha_N) := \left\| \psi - \sum_{n=1}^N \alpha_n \varphi_n \right\|$$
of the N numbers $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ is minimized with the choice $\alpha_n := \langle \varphi_n, \psi \rangle$.

12. Prove that if A and B are two disjoint measure spaces then
$$L^2(A \sqcup B) \quad \cong \quad L^2(A) \oplus L^2(B)\ .$$

13. Prove that if A and B are two measure spaces then
$$L^2(A \times B) \quad \cong \quad L^2(A) \otimes L^2(B)\ .$$

14. Show that
$$\mathcal{H} := \ell^2(\mathbb{R}) \equiv \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; f^{-1}(\mathbb{C} \setminus \{ 0 \}) \text{ is a countable set and } \sum_{x\in\mathbb{R}} |f(x)|^2 < \infty \right\}$$
is a non-separable Hilbert space.

2 C-star algebras

In the following exercises, $\mathcal{A}$ is a C-star algebra with involution $* : \mathcal{A} \to \mathcal{A}$ and norm $\|\cdot\|$. $a, b, \dots \in \mathcal{A}$. Please see the corresponding section in the lecture notes for the definitions of algebraic conditions on elements in a C-star algebra.

When solving these exercises please don't forget that $\mathcal{B}(\mathcal{H})$ is a c-star algebra so everything you prove here will be useful for operators on Hilbert space.

15. Show that if a is a partial isometry (i.e. $|a|^2$ is an idempotent) then $a = aa^*a = aa^*aa^*a$.

16. Show that a is a partial isometry iff a^* is a partial isometry.

17. Show that if p, q are self-adjoint projections then $\|p - q\| \leq 1$.

18. Show that if u, v are unitary then $\|u - v\| \leq 2$.

19. Show that if a is self-adjoint with $\|a\| \leq 1$ then
$$a + \mathrm{i}\sqrt{1 - a^2}, \qquad a - \mathrm{i}\sqrt{1 - a^2}$$
are unitary. Conclude that any $b \in \mathcal{A}$ is the linear combination of four unitaries.

20. Two self-adjoint projections p, q are said to be orthogonal (written $p \perp q$) iff $pq = 0$. Show that the following are equivalent:
    (a) $p \perp q$.
    (b) $p + q$ is a self-adjoint projection.
    (c) $p + q \leq \mathbb{1}$.

21. Let $v_1, \dots, v_n$ be partial isometries and suppose that
$$\sum_{j=1}^n |v_j|^2 = \sum_{j=1}^n |v_j^*|^2 = \mathbb{1}\,.$$
Show that $\sum_{j=1}^n v_j$ is unitary.

22. Show that for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that if a obeys

$$\max\left(\left\{ \|a - a^*\|, \left\|a^2 - a\right\| \right\}\right) \leq \delta_\varepsilon$$

then there exists a self-adjoint projection p with $\|a - p\| \leq \varepsilon$.

23. Show that for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that if a obeys

$$\max\left(\left\{ \left\||a|^2 - \mathbb{1}\right\|, \left\||a^*|^2 - \mathbb{1}\right\| \right\}\right) \leq \delta_\varepsilon$$

then there exists a unitary u with $\|a - u\| \leq \varepsilon$.

24. Show that $\sigma(p) \subseteq \{ 0, 1 \}$ for an idempotent p.

25. Show that $\|p\| = 1$ for a non-zero self-adjoint projection p.

26. Show that the spectral radius $r(a)$ of a self-adjoint a equals its norm $\|a\|$.

27. Show that $\sigma(u) \subseteq \mathbb{S}^1$ if u is unitary (i.e. $|u|^2 = |u^*|^2 = \mathbb{1}$).

28. Show that $\sigma(a) \subseteq [0, \infty)$ if a is positive (i.e. $a = |b|^2 \, \exists b$).

29. Show that $\sigma(a) \subseteq \mathbb{R}$ if $a = a^*$.

30. Show that a is invertible if $|a|^2 \geq \varepsilon\mathbb{1}$ for some $\varepsilon > 0$ and $|a^*|^2 \geq \delta\mathbb{1}$ for some $\delta > 0$; (recall $a \geq b$ iff $a - b \geq 0$ iff $a - b = |c|^2$ for some c).