Functional Analysis
Princeton University MAT520
HW4, Due Oct 6th 2024

September 28, 2024

Here's a list of problems I thought were nice, solve as many as you like and as many as doesn't cause you sleep deprivation.

1 Weak stuff

1. Prove that any norm-closed convex bounded subset of a reflexive Banach space is weakly compact.
2. Define the Banach space $\ell^\infty(\mathbb{N}) := \{ a : \mathbb{N} \to \mathbb{C} \mid \|a\|_\infty < \infty \}$ with the norm $\|a\|_\infty \equiv \sup_n |a_n|$. Also define
$$\ell^1(\mathbb{N}) := \{ a : \mathbb{N} \to \mathbb{C} \mid \|a\|_1 < \infty \}$$
with the norm $\|a\|_1 := \sum_{n\in\mathbb{N}} |a_n|$.
Our goal here is to use the Banach-Alaoglu theorem to exhibit an element of $(\ell^\infty)^*$ which is not in ℓ^1.
   (a) Define $\{ \mu_n \}_n \subseteq (\ell^\infty)^*$ via
   $$\mu_n(a) := \frac{1}{n}\sum_{j=1}^{n} a_j \qquad (a \in \ell^\infty, n \in \mathbb{N}) .$$
   Show that $\mu_n \in (\ell^\infty)^*$ indeed and $\|\mu_n\| \leq 1$.
   (b) Show there exists some $\mu \in (\ell^\infty)^*$ that is the limit of $\{ \mu_n \}_n$ (in the weak-star topology).
   (c) Show that $(\ell^1)^* = \ell^\infty$.
   (d) Hence we may think of $J(\ell^1) \subseteq (\ell^\infty)^*$ where J is the natural isometric injection. Show that the limit μ constructed above does *not* lie in $J(\ell^1)$. That is, show that for any $x \in \ell^1$, $J(x) \neq \mu$.
3. Let $\{ f_n \}_n$ be given by
$$f_n(t) := \mathrm{e}^{int} \qquad (t \in [-\pi, \pi]) .$$
Show that if $p \in [1, \infty)$ then $f_n \to 0$ weakly in $L^p([-\pi, \pi])$, but not in the norm topology of $L^p([-\pi, \pi])$.
4. Consider $L^\infty([0,1])$ with its norm topology (the essential supremum norm), and, since $(L^1([0,1]))^* = L^\infty([0,1])$, the weak-star topology on $(L^1([0,1]))^*$, which is a topology on $L^\infty([0,1])$. Show that $C([0,1])$ (the space of all continuous functions) is dense in L^α but not in L^β, for either $(\alpha, \beta) = (1, \infty)$ or $(\alpha, \beta) = (\infty, 1)$.
5. Let X be an infinite-dimensional Banach space and define
$$S := \{ x \in X \mid \|x\| = 1 \} .$$
Show that the weak-closure of S is
$$B := \{ x \in X \mid \|x\| \leq 1 \} .$$
6. Let X be a Banach space, and $\{ L_n \}_n \subseteq X^*$ be a sequence which converges to some $L \in X^*$ in the weak-star sense. Assume that $\{ x_n \}_n \subseteq X$ converges to some $x \in X$ in norm. It is true that $L_n(x_n) \to L(x)$ in $\mathbb{C}$?
7. Find an example of a Banach space X for which there does *not* exist a Banach space Y such that $Y^* = X$.

2 Banach algebras

Here $\mathcal{A}$ is a Banach algebra and $x, y, \ldots$ are elements in it; $\mathcal{G}_{\mathcal{A}}$ is the set of invertible elements and $r : \mathcal{A} \to [0, \infty)$ is the spectral radius.

8. Use $(xy)^n = x\,(yx)^{n-1}\,y$ to show that $r\,(xy) = r\,(yx)$.

9. Show that if $x, xy \in \mathcal{G}_{\mathcal{A}}$ then $y \in \mathcal{G}_{\mathcal{A}}$.

10. Show that if $xy, yx \in \mathcal{G}_{\mathcal{A}}$ then $x, y \in \mathcal{G}_{\mathcal{A}}$.

11. On the Banach space $\ell^2\,(\mathbb{N} \to \mathbb{C})$, define the right shift operator $R \in \mathcal{B}\left(\ell^2\,(\mathbb{N} \to \mathbb{C})\right)$:

$$(Ra)_n := \begin{cases} a_{n-1} & n \geq 2 \\ 0 & n = 1 \end{cases}$$

and the right shift operator

$$(La)_n := a_{n+1} \qquad (n \in \mathbb{N})\ .$$

Calculate RL and LR. Conclude that one may have $xy = \mathbb{1}$ but $yx \neq \mathbb{1}$ in a Banach algebra.

12. Show that $xy - \mathbb{1}$ is invertible iff $yx - \mathbb{1}$ is invertible.

13. Show that if $z \in \mathbb{C} \setminus \{\, 0 \,\}$ then $z \in \sigma\,(xy)$ iff $z \in \sigma\,(yx)$. I.e.,

$$\sigma\,(xy) \cup \{\, 0 \,\} = \sigma\,(yx) \cup \{\, 0 \,\}\ .$$

Find an example where $\sigma\,(xy) \neq \sigma\,(yx)$.

14. Define $\mathcal{A} := C^2\,([0,1] \to \mathbb{C})$, the space of functions with continuous second derivative. Define, for $a, b > 0$,

$$\|f\| := \|f\|_\infty + a\|f'\|_\infty + b\|f''\|_\infty\,.$$

Show that $\mathcal{A}$ is a Banach space. Show that $\mathcal{A}$ is a Banach algebra (with pointwise multiplication) iff $a^2 \geq 2b$. You may consider the functions $x \mapsto x$ and $x \mapsto x^2$.

15. Show that if $z \in \partial\sigma\,(x)$ then $x - z\mathbb{1} \in \partial\mathcal{G}_{\mathcal{A}}$.

16. Let $x \in \partial\mathcal{G}_{\mathcal{A}}$. Show there exists some $\{\, y_n \,\}_n \subseteq \mathcal{A}$ with $\|y_n\| = 1$ and

$$\lim_{n\to\infty} xy_n = \lim_{n\to\infty} y_n x = 0\,.$$

Try to characterize the type of Banach algebras in which there are such elements x (which are called *topological divisor of zero*).

17. On $\ell^2\,(\mathbb{N} \to \mathbb{C})$ define $T \in \mathcal{B}\left(\ell^2\,(\mathbb{N} \to \mathbb{C})\right)$ via

$$T\,(a_1, a_2, a_3, a_4, \ldots) := (-a_2, a_1, -a_4, a_3, \ldots)\ .$$

Calculate $\sigma\,(T)$.

18. Show that if $x \in \mathcal{A}$ is nilpotent (i.e. $\exists n \in \mathbb{N}$ with $x^n = 0$) then $\sigma\,(x) = \{\, 0 \,\}$.

19. Show that r is upper semicontinuous.