Functional Analysis
Princeton University MAT520
HW3, Due Sep 29th 2024

September 26, 2024

1. Prove the $\mathbb{C}$-Hahn-Banach theorem using the $\mathbb{R}$-Banach theorem. In particular you have to setup the forgetful functor which maps a $\mathbb{C}$-vector space to its underlying $\mathbb{R}$-vector space to show:
Let X be a $\mathbb{C}$-vector space, $p : X \to \mathbb{R}$ be given such that

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) \qquad (x, y \in X; \alpha, \beta \in \mathbb{C} : |\alpha| + |\beta| = 1) .$$

Let $\lambda : Y \to \mathbb{C}$ linear where $Y \subseteq X$ is a subspace, and such that

$$|\lambda(x)| \leq p(x) \qquad (x \in Y) .$$

Then there exists $\Lambda : X \to \mathbb{C}$ linear such that $\Lambda|_Y = \lambda$ and such that

$$|\Lambda(x)| \leq p(x) \qquad (x \in X) .$$

2. Let $(X, \mathscr{F}, \mu)$ be a sigma-finite measure space. For $p > 0$, define

$$L^p(X) \quad \equiv \quad \left\{ f : X \to \mathbb{C} \,\middle|\, \int_X |f|^p \, \mathrm{d}\mu < \infty \right\} .$$

   (a) Show that if $p \geq 1$ then

   $$f \mapsto \|f\|_p \quad := \quad \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

   obeys all the axioms of a norm with the exception that $\|f\|_p = 0$ implying $f = 0$.

   (b) For $p \geq 1$, provide a concrete example (with $X = \mathbb{R}$ and μ the Lebesgue measure) in which $\|f\|_p = 0$ but where $f \neq 0$.

   (c) Show that Cauchy sequences converge under this norm.

   (d) Show that if $p \in (0, 1)$ then the triangle inequality fails for this norm by providing a concrete example (with $X = \mathbb{R}$ and μ the Lebesgue measure).

   (e) For (X, μ) a measure space, define

   $$L^\infty(X) \quad := \quad \{ f : X \to \mathbb{C} \text{ msrbl.} \mid \|f\|_\infty < \infty \}$$

   where

   $$\|f\|_\infty \equiv \inf \{ a \geq 0 \mid |f| \leq a \ \mu\text{-almost-everywhere} \} .$$

   Show that it is a Banach space.

   (f) Now we specify to $X = \mathbb{R}$ with μ the Lebesgue measure.

      i. For $p \in (1, \infty)$, Calculate $(L^p)^*$.

      ii. Calculate $(L^1)^*$ and show it is isomorphic to L^∞.

      iii. Calculate $(L^\infty)^*$ and show it is not isomorphic to L^1.

3. A Banach space is called *reflexive* iff $X \overset{J}{\cong} X^{**}$, i.e., if the isometric canonical injection

$$X \ni x \mapsto (X^* \ni \lambda \mapsto \lambda(x)) \in X^{**}$$

is surjective. Show that a Banach space X is reflexive iff X^* is reflexive.

4. A pair of Banach spaces are called strictly dual iff $\exists$ map $f : X \to Y^*$ which is isometric, so that the induced map $f^* : Y \to X^*$ is also isometric. Prove that if X and Y are strictly dual and X is reflexive, then $Y = X^*$ and $X = Y^*$ using the Hahn-Banach theorem.

5. In this exercise you may wish to consult the 1951 PNAS paper by James titled "*A Non-Reflexive Banach Space Isometric With Its Second Conjugate Space*".
Let $\mathscr{P}$ denote the family of all finite increasing sequences of integers of odd length. Then, for any $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and

$$p = (p_1, \cdots, p_{2n+1}) \in \mathscr{P}$$

we define

$$\|x\|_p := \left(x_{p_{2n+1}}^2 + \sum_{m=1}^{n} \left(x_{p_{2m-1}} - x_{p_{2m}} \right)^2 \right)^{\frac{1}{2}} .$$

Define the James space $\mathscr{J}$ as

$$\mathscr{J} := \left\{ x : \mathbb{N} \to \mathbb{R} \,\middle|\, \lim_{n\to\infty} x_n = 0 \wedge \|x\| < \infty \right\}$$

where we use the norm

$$\|x\| := \sup \left(\left\{ \|x\|_p \,\middle|\, p \in \mathscr{P} \right\} \right) .$$

(a) Show $\mathscr{J}$ is a Banach space.
(b) Show $\mathscr{J}$ has the property that the canonical map $J : \mathscr{J} \to \mathscr{J}^{**}$ is not surjective.
(c) Find some other isometric isomorphism $\eta : \mathscr{J} \to \mathscr{J}^{**}$.

6. Find an infinite sequence of *non-repeating* Banach spaces $\{ X_n \}_{n\in\mathbb{N}}$ obeying the condition:

$$X_{n+1} := X_n^{**} \qquad (n \geq 1) .$$

7. Let $S \subseteq L^1([0,1] \to \mathbb{C})$ be a closed linear subspace. Suppose that S is such that $f \in S$ implies $f \in L^p([0,1] \to \mathbb{C})$ for some $p > 1$. Show that $S \subseteq L^p([0,1] \to \mathbb{C})$ for some $p > 1$.

8. [In this question we use the $\mathbb{R}$-Hahn-Banach] Let L be the (unilateral) left shift operator on $\ell^\infty(\mathbb{N} \to \mathbb{R})$:

$$(L\psi)(n) \equiv \psi(n+1) \qquad (n \in \mathbb{N}) .$$

Prove that there exists *a Banach limit, i.e. some* $\Lambda : \ell^\infty(\mathbb{N} \to \mathbb{R}) \to \mathbb{R}$ *linear such that: (a)* $\Lambda L = \Lambda$, *(b)*

$$\liminf_n \psi(n) \leq \Lambda\psi \leq \limsup_n \psi(n) \qquad (\psi \in \ell^\infty) .$$

Suggestion: Define the functional Λ_n via $\Lambda_n \psi := \frac{1}{n}\sum_{j=1}^{n} \psi(j)$, the space $M := \{ \psi \in \ell^\infty \mid (\lim_{n\to\infty} \Lambda_n \psi) \exists \}$ and the convex function $p(\psi) := \limsup_n \Lambda_n \psi$.

9. Prove that the closed unit ball of an infinite-dimensional Banach space is not compact.

10. Prove that an infinite-dimensional Banach space cannot be spanned, as a vector space, by a countable subset.