Functional Analysis
Princeton University MAT520
HW1, Due Sep 22nd 2024

September 15, 2024

Note: Some of these exercises are rudimentary and you may have already encountered them in your past. If that happens to be the case, feel free to simply write "boring" and still get full credit nonetheless. [extra] questions may likewise be ignored (without impact to your grade).

1 Normed spaces

1. Let a normed vector space $(X, \|\cdot\|)$ be given. Show that there exists some sesquilinear inner product
$$\langle\cdot,\cdot\rangle : X^2 \to \mathbb{C}$$
which is compatible with the norm, in the sense that
$$\|x\| = \sqrt{\langle x, x\rangle} \qquad (x \in X)$$
if and only if the norm satisfies (any one of the equivalent) parallelogram identity. Here is one version which I like:
$$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \qquad (x, y \in X)\ .$$
(This is phrased as an inequality rather than equality since the other direction of the inequality is *always* true, so there is nothing to verify. Show this first).

2. Let an inner-product vector space $(X, \langle\cdot,\cdot\rangle)$ be given. Prove the Cauchy-Schwarz inequality
$$|\langle x, y\rangle| \leq \|x\|\|y\|\ .$$

3. Two norms $\|\cdot\|_1, \|\cdot\|_2$ on a normed space X are called *equivalent* iff $\exists a, b \in (0, \infty)$ such that
$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \qquad (x \in X)\ .$$
Show that all norms on $\mathbb{C}^n$ are equivalent.

4. Let X be a Banach space which is Banach w.r.t. two different norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that $\|\cdot\|_1 \leq C\|\cdot\|_2$ for some $C \in (0, \infty)$. Show that there exists a $D \in (0, \infty)$ such that $\|\cdot\|_2 \leq D\|\cdot\|_1$.

5. Show that if $A : X \to Y$ is a bounded linear map between Banach spaces then
$$\|Ax\|_Y \leq \|A\|_{\mathrm{op}}\|x\|_X \qquad (x \in X)\ .$$

6. Show that if $A, B : X \to X$ are bounded linear maps on a Banach space then
$$\|AB\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}\ .$$

7. Let $d : X^2 \to [0, \infty)$ be a *homogeneous* metric on a TVS X. Show that $S \subseteq X$ is bounded (in the TVS sense: for any $N \in \mathrm{Nbhd}(0_X)$ one has $S \subseteq tN$ for all $t > 0$ large) *if and only if* S is bounded in the metric sense
$$\sup_{x \in S} d(x, 0_X) < \infty\ .$$

8. Show that a linear map $A:X\to Y$ between two Banach spaces maps bounded sets of X to bounded sets of Y iff $\|A\|_{\text{op}}<\infty$.

9. Show that a linear map $A:X\to Y$ between two Banach spaces is bounded iff it is continuous iff it is continuous at zero (in showing continuity implies boundedness please do not use the TVS theorem from chapter 1 but rather do this directly in the context of Banach spaces).

10. Show that $L^\infty(\mathbb{R}\to\mathbb{C})$ is a Banach space.

11. Let a normed vector space $(X,\|\cdot\|)$ be given. Show that $(X,\|\cdot\|)$ is complete iff, for *any* sequence $\{\,x_n\,\}_{n\in\mathbb{N}}\subseteq X$,

$$\left(\sum_{n\in\mathbb{N}}\|x_n\|<\infty\right)\Longrightarrow\left(\lim_{N\to\infty}\sum_{n=1}^{N}x_n\ \exists\text{ and equals some }x\in X\right).$$

2 Completeness

12. Show that the $\frac{1}{3}$-Cantor set is nowhere dense.

13. Prove that if X is a *locally compact Hausdorff space* and $\{\,V_j\,\}_{j\in\mathbb{N}}$ are open dense sets then $\bigcap_{j\in\mathbb{N}}V_j$ is dense in X itself.

14. [extra] Show that $[0,1]^{\mathbb{R}}$ is locally compact but not metrizable. Show that $\mathbb{R}\setminus\mathbb{Q}$ with the Euclidean topology is metrizable but not locally compact.

A set $S\subseteq X$ is a Hamel basis for a vector space X iff S is a maximal linearly independent subset of X, iff any $x\in X$ has a unique representation as a *finite* linear combination of elements of S.

15. Show that if X is an infinite-dimensional TVS which is the union of countably many finite-dimensional subspaces, then X is of Baire's first category. Conclude that no infinite-dimensional Banach space has a countable Hamel basis.

16. Find a subset $S\subseteq[0,1]$ which is of Baire's first category but whose Lebesgue measure equals 1.

17. For any $f\in L^2(\mathbb{S}^1)$, let $\hat{f}:\mathbb{Z}\to\mathbb{C}$ be given by

$$\hat{f}(n):=\frac{1}{2\pi}\int_{\theta\in[-\pi,\pi]}f\left(e^{i\theta}\right)e^{-in\theta}d\theta\qquad(n\in\mathbb{Z}).$$

Define $\Lambda_n:L^2(\mathbb{S}^1)\to\mathbb{C}$ via

$$\Lambda_n f:=\sum_{k=-n}^{n}\hat{f}(k)\qquad\left(f\in L^2(\mathbb{S}^1)\right)$$

Show that

$$\left\{\,f\in L^2(\mathbb{S}^1)\,\middle|\,\lim_{n\to\infty}\Lambda_n f\ \exists\,\right\}$$

is a dense subspace of $L^2(\mathbb{S}^1)$ of Baire's first category.

18. Let X be a Banach space and Y is a subspace of X whose complement is of Baire's first category. Show that $Y=X$.

19. Let X,K be metric spaces with K compact. Assume that $f:X\to K$ is a map with $\Gamma(f)\in\text{Closed}(X\times K)$. Show that f is continuous.

20. [extra] The Banach-Schauder theorem: Let $A:X\to Y$ be a bounded linear function between Banach spaces. Then show that only one of the possible two alternatives may hold:

    (a) $AU\in\text{Open}(AX)$ for any $U\in\text{Open}(X)$, or,

    (b) AX is of Baire's first category.