DEC 24 2024

MATS20 — Functional Analysis —

Final Sol-n

<u>Legend</u>: LN ≡ Lecture notes

Q1 Kuiper's Thm. is proven in LN in Thm. 10.27.

Q2 Let $Z \in \mathcal{B}(\mathcal{H})$:

① $\mathbb{1} - |Z|^2,\ \mathbb{1} - |Z^*|^2 \in \mathcal{K}(\mathcal{H})$

② $Z \in \mathcal{F}(\mathcal{H})$: $\operatorname{index}(Z) = 0$.

<u>Claim</u>: $\exists\ \mathcal{U} \in \mathcal{U}(\mathcal{H})$: $Z - \mathcal{U} \in \mathcal{K}(\mathcal{H})$.

<u>Proof</u>: $\mathbb{1} - |Z|^2 = (\mathbb{1} - |Z|)\underbrace{(\mathbb{1} + |Z|)}_{\text{inv.}}$

$\implies \mathbb{1} - |Z| \in \mathcal{K}$ too.

$Z = \operatorname{pol}(Z)|Z|$

$\implies Z - \operatorname{pol}(Z) = \operatorname{pol}(Z)|Z| - \operatorname{pol}(Z)$

$= \operatorname{pol}(Z)(|Z| - \mathbb{1}) \in \mathcal{K}.$

Now, as $\mathrm{index}(Z)=0$,

$$\dim\ker Z = \dim \operatorname{coker} Z = \dim(\operatorname{im} Z)^\perp.$$

$\Longrightarrow \exists$ unitary finite matrix $M : \ker(Z) \to \operatorname{im}(Z)^\perp$.

But $\ker(Z) = \ker(\mathrm{pol}(Z))$ by def.

$\Longrightarrow$

$\mathrm{pol}(Z) \oplus M : \ker(Z)^\perp \oplus \ker(Z) \to \operatorname{im}(Z) \oplus \operatorname{im}(Z)^\perp$

is unitary, and

$Z - \mathrm{pol}(Z) \oplus M$ is cpt. as

$Z - \mathrm{pol}(Z)$ is and M is

finite rank. $\blacksquare$

$\boxed{Q3}$ Claim: If A is a C-star alg, $u \in \mathcal{U}(A) : \sigma(u) \neq S^1$

then $\exists$ cont. path $u \rightsquigarrow \mathbb{1}$

within $\mathcal{U}(A)$.

Proof: Let $\log_\varepsilon : \mathbb{C} \to \mathbb{C}$ be the single-valued log w/ branch cut at $\varepsilon \in \mathbb{S}^1$, where $\varepsilon \in \sigma(u)^c$. Then on $\sigma(u)$, $\log_\varepsilon$ is analytic and hence cont.

$$[0,1] \ni t \mapsto \gamma(t) := \exp\left(+i(1-t)\frac{1}{i}\log_\varepsilon(u)\right)$$

defines a cont. path within $\mathcal{U}(\mathcal{A})$ w/ $\gamma(0) = u$, $\gamma(1) = \mathbb{1}$. ∎

$\boxed{\text{Q4}}$

Question has a mistake! Need to assume index $\Lambda U \Lambda + \Lambda^\perp = 0$. Otherwise false.

Let $\Lambda = \Lambda^* = \Lambda^2 \in \mathcal{B}$

$$\dim\ker\Lambda = \dim\operatorname{im}\Lambda = \infty.$$

Let $U \in \mathcal{U}$: $[U, \Lambda] \in \mathcal{K}$.

Note that then $\Lambda U \equiv \Lambda U \Lambda + \Lambda^\perp \in \mathcal{F}$.

Indeed, ΛU^* is a parametrix:

$$
\begin{aligned}
\mathbb{1} - (\Lambda U^*)\Lambda U &= \Lambda + \Lambda^\perp - (\Lambda U^* \Lambda + \Lambda^\perp)(\Lambda U \Lambda + \Lambda^\perp) \\
&= \Lambda - \Lambda U^* \Lambda U \Lambda \\
&= \Lambda(\mathbb{1} - U^* \Lambda U)\Lambda \\
&= \Lambda(U^* U - U^* \Lambda U)\Lambda \\
&= \Lambda U^* (\mathbb{1} - \Lambda) U \Lambda \\
&= \Lambda U^* \Lambda^\perp U \Lambda \\
&= \Lambda U^* \Lambda^\perp [U, \Lambda] \qquad \Lambda^\perp \Lambda = 0 \\
&\in \mathcal{K}.
\end{aligned}
$$

$\Longrightarrow$ ΛU has an index.

Assume it is zero.

<u>Claim</u>: $\exists$ cont. $[0,1] \ni t \mapsto \gamma(t) \in \mathcal{U}$:

$$\gamma(0) = U \quad \wedge \quad \gamma(1) = \mathbb{1}.$$

<u>Proof</u>: Write $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ in

$\operatorname{im}(\Lambda^\perp) \oplus \operatorname{im}(\Lambda)$ decomp.

$$\Longrightarrow \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{bmatrix}.$$

$$\Rightarrow \quad [U, \Lambda] \in \mathcal{K} \Leftrightarrow U_{12}, U_{21} \in \mathcal{K}.$$

$$\Lambda U = \begin{bmatrix} U_{11} & 0 \\ 0 & \mathbb{1} \end{bmatrix}$$

$$\Rightarrow \quad \text{index}(\Lambda U) = \text{index}(U_{11} \oplus \mathbb{1})$$

$$= \text{index}(U_{11}) + \text{index}(\mathbb{1})$$

$$= \text{index}(U_{11}).$$

But $\text{index}(U) = \text{index}(U_{11} \oplus U_{22})$

$$= \text{index}(U_{11}) + \text{index}(U_{22}).$$

$U_{12}, U_{21} \in \mathcal{K}$
and index is
cpt. stable

Since U is unitary, $\text{index}(U) = 0$.

$$\Rightarrow \quad 0 = \underbrace{\text{index}(U_{11})}_{=0 \text{ by hypo.}} + \text{index}(U_{22})$$

$$\Rightarrow \quad \text{index}(U_{22}) = 0 \quad \text{too}.$$

Moreover, $|U|^2 = |U^*|^2 = \mathbb{1}.$ $\Rightarrow$

$$\mathbb{1} - |U_{ii}|^2, \; \mathbb{1} - |U_{ii}^*|^2 \in \mathcal{K}$$

for $i = 1, 2$ as $U_{12}, U_{21} \in \mathcal{K}$.

By $\boxed{Q2}$, V_{ii} may be extended to B_{ii}, an honest unitary : $V_{ii} - B_{ii} \in \mathcal{K}$.

$\Longrightarrow$ $B := B_{11} \oplus B_{22}$ is a unitary cpt.-ly away from U.

Define $A := UB^*$. $\Rightarrow$ A unitary

and $\mathbb{1} - A = BB^* - UB^*$

$= (B-U)B^* \in \mathcal{K}$.

$\Longrightarrow$ $\sigma_{ess}(A) = \sigma_{ess}(\mathbb{1}) = \{1\}$.

$\Longrightarrow$ A cannot have $\sigma(A) = \mathbb{S}^1$

The cpt. op. $\mathbb{1} - A$ can only have accum. near zero $\Leftrightarrow$ A has accum. near 1.

So deform B diag. w.r.t. Λ and A via $\boxed{Q3}$.

Counter-example to show original phrasing of Q is wrong:

R bilat. right shift on $\ell^2(\mathbb{Z})$

$\Lambda \equiv \chi_{\mathbb{N}}(X)$ proj. to right on $\ell^2(\mathbb{Z})$.

Then
$$[R,\Lambda]_{xy} = (\Lambda(x)-\Lambda(y))R_{xy}$$
$$= (\Lambda(x)-\Lambda(y))\delta_{x,y+1}$$
$$= -\delta_{x,0}\,\delta_{y,-1}$$

finite rank $\Rightarrow$ cpt.

But $\quad \text{index}\,\Lambda R = \text{index}\,\hat{R} = -1.$

unilat. shift on $\ell^2(\mathbb{N})$

Now, R_{22} has empty ker and one-dim coker, so it cannot be extended to a unitary cpt.-ly away! Indeed if that were possible its index would be zero!

Moreover, it is impossible to deform R to $\mathbb{1}$ within unitaries which essentially commute w/ Λ via Carey-Hurst-Obrien '82

FFA.

SORRY :(

Q5 This is proven in the MAT595 LN Lemma 3.11:

Lemma 3.11. *Let P be local as in (1.2) and such that $\|P\| \leq 1$ and $f \in \ell^\infty(\mathbb{Z}^2)$ be such that there exists some $D < \infty$ with which*

$$|f(x) - f(y)| \leq D\frac{\|x-y\|}{1+\|x\|}. \tag{3.6}$$

Then $[P, f(X)]$ is Schatten-3. In particular it is compact.

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Proof. We have $[P, f(X)]_{xy} = P_{xy}(f(x) - f(y))$ and using Lemma 3.12 just below, we have

$$\|[P, f(X)]\|_3 \leq \sum_{b\in\mathbb{Z}^2}\left(\sum_{x\in\mathbb{Z}^2}\|P_{x+b,x}\|^3 |f(x)-f(x+b)|^3\right)^{1/3}.$$

Now we have

$$\|P_{x+b,x}\|^3 \leq C^3 e^{-3\mu\|b\|}$$

so that together with (3.6) we have the estimate

$$\begin{aligned}
\|[P, f(X)]\|_3 &\leq \sum_{b\in\mathbb{Z}^2}\left(\sum_{x\in\mathbb{Z}^2} C^3 e^{-3\mu\|b\|} D^3 \frac{\|b\|^3}{(1+\|x\|)^3}\right)^{\frac{1}{3}} \\
&= CD\sum_{b\in\mathbb{Z}^2} e^{-\mu\|b\|}\|b\|\left(\sum_{x\in\mathbb{Z}^2}\frac{1}{(1+\|x\|)^3}\right)^{\frac{1}{3}} \\
&< \infty.
\end{aligned}$$

□

Lemma 3.12 referenced here is Lemma 9.81 in our own LN:

Lemma 9.81. *For any operator $A \in \mathcal{B}(\mathcal{H})$, an ONB $\{\delta_x\}_{x\in\mathbb{Z}}$ of $\mathcal{H}$, and*

$$A_{xy} := \langle \delta_x, A\delta_y \rangle \qquad (x, y \in \mathbb{Z})$$

we have the estimate

$$\|A\|_p \leq \sum_{k\in\mathbb{Z}} \left(\sum_{x\in\mathbb{Z}} |A_{x+k,x}|^p \right)^{\frac{1}{p}} .$$

where $\|A\|_p \equiv (\operatorname{tr}(|A|^p))^{\frac{1}{p}}$ is the Schatten-p norm.

Proof. Let us decompose A to its diagonals as

$$A = \sum_{k\in\mathbb{Z}} A^{(k)}$$

defined via $\left(A^{(k)}\right)_{xy} \equiv A_{xy}\delta_{x-y,k}$ for all $k \in \mathbb{Z}$. Since $\|\cdot\|_p$ is a norm, applying the triangle inequality we find

$$\|A\|_p \leq \sum_{k\in\mathbb{Z}} \left\|A^{(k)}\right\|_p .$$

But now,

$$\begin{aligned} \left\|A^{(k)}\right\|_p &= \left(\operatorname{tr}\left(\left|A^{(k)}\right|^p\right)\right)^{\frac{1}{p}} \\ &= \left(\operatorname{tr}\left(\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}} \\ &= \left(\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}^{\frac{p}{2}}\right)^{\frac{1}{p}} \\ &= \sqrt{\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}} . \end{aligned}$$

But note that

$$\begin{aligned} \left(\left|A^{(k)}\right|^2\right)_{xy} &\equiv \left(\left(A^{(k)}\right)^* A^{(k)}\right)_{xy} \\ &= \sum_{z\in\mathbb{Z}} \left(\left(A^{(k)}\right)^*\right)_{xz} \left(A^{(k)}\right)_{zy} \\ &= \sum_{z\in\mathbb{Z}} \left(A_{zx}\delta_{z-x,k}\right)^* A_{zy}\delta_{z-y,k} \\ &= \delta_{x,y} \sum_{z\in\mathbb{Z}} \left(A_{zx}\delta_{z-x,k}\right)^* A_{zy}\delta_{z-y,k} \\ &= \delta_{x,y} \left|A_{x+k,x}\right|^2 . \end{aligned}$$

Since $\left|A^{(k)}\right|^2$ is a-posteriori a diagonal operator, it is easy to calculate its Schatten-$\frac{p}{2}$ norm, since it is easy to take its powers. Indeed,

$$\left[\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right]_{xy} = \delta_{x,y} \left|A_{x+k,x}\right|^p$$

and so

$$\begin{aligned} \left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}^{\frac{p}{2}} &= \operatorname{tr}\left(\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right) \\ &= \sum_{x\in\mathbb{Z}} \left[\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right]_{xx} \\ &= \sum_{x\in\mathbb{Z}} \left|A_{x+k,x}\right|^p . \end{aligned}$$

Collecting everything together we find

$$\begin{aligned} \|A\|_p &\leq \sum_{k\in\mathbb{Z}} \sqrt{\left(\sum_{x\in\mathbb{Z}} |A_{x+k,x}|^p\right)^{\frac{2}{p}}} \\ &\leq \sum_{k\in\mathbb{Z}} \left(\sum_{x\in\mathbb{Z}} |A_{x+k,x}|^p\right)^{\frac{1}{p}} . \end{aligned}$$

□

$\boxed{Q6}$ This is Thm. 10.7 in our own LN (which has been promoted now from a sketch to actual proof).

$\boxed{Q7}$ Let $-\Delta \equiv 2\mathbb{1} - R - R^*$ on $\ell^2(\mathbb{Z})$ w/ R the bilat. right shift.

Then w/
$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$$
$$\psi \mapsto \left(k \mapsto \sum_{n\in\mathbb{Z}}{}' e^{-ikn}\psi_n\right)$$

we get $\mathcal{F}(-\Delta)\mathcal{F}^* = M_a$ w/

$$\varepsilon(k) = 2 - 2\cos(k) \qquad k \in [-\pi, \pi].$$

This diagonalizes the Laplacian as follows: let $f : \mathbb{R} \to \mathbb{C}$ be bdd. & msrbl. Then

$\mathcal{F}$ is unitary
$$\langle \varphi, f(-\Delta)\psi\rangle = \langle \mathcal{F}\varphi, \mathcal{F} f(-\Delta)\psi\rangle$$
$$= \langle \mathcal{F}\varphi, \mathcal{F} f(-\Delta)\mathcal{F}^*\mathcal{F}\psi\rangle$$

$$= \frac{1}{2\pi} \int_{k=-\pi}^{\pi} dk \ \overline{(\mathcal{F}\varphi)(k)} \ f(\varepsilon(k)) \ (\mathcal{F}\psi)(k)$$

But not quite as we'd like since we want a mult. op. by $E \mapsto E$.

One concrete possibility is a change of var $k \mapsto E$.

$$E := \varepsilon(k) \Leftrightarrow k = \varepsilon^{-1}(E) = \pm \arccos(1 - \tfrac{1}{2}E)$$

Then $dk = \frac{1}{\sqrt{E(4-E)}} dE$.

We then have $(\hat{\varphi} := \mathcal{F}\varphi, \ \hat{\psi} := \mathcal{F}\psi)$:

$$\hat{\varphi}(k) = \hat{\varphi}_1(k) + \hat{\varphi}_2(k) \quad \text{w/}$$

$$\hat{\varphi}_{1,2}(k) \equiv \frac{1}{2}\left(\hat{\varphi}(k) \pm \hat{\varphi}(-k)\right).$$

Since $k \mapsto \varepsilon(k)$ is even, so is $f \circ \varepsilon$.

$\Rightarrow$ The cross terms drop out and we get

$$\langle \varphi, f(-\Delta) \psi \rangle = \sum_{i=1,2} \frac{1}{2\pi} \int_{k=-\pi}^{\pi} \overline{\hat{\varphi}_i(k)} \ f(\varepsilon(k)) \ \hat{\psi}_i(k)$$

Now, since $\overline{\hat{\varphi}_i(k)}\hat{\psi}_i(k)$ is even, we may write

$$\langle\varphi, f(-\Delta)\psi\rangle = \sum_{i=1,2} \frac{1}{\pi}\int_{k=0}^{\pi} \overline{\hat{\varphi}_i(k)}\, f(\mathcal{E}(k))\,\hat{\psi}_i(k)$$

On $[0,\pi]$ the change of var. $k\mapsto E$ makes sense so we get

$$\langle\varphi, f(-\Delta)\psi\rangle = \sum_{i=1,2}\int_{E=0}^{4} \overline{\hat{\varphi}_i(\arccos(1-\tfrac{1}{2}E))}\,\hat{\psi}_i(\arccos(1-\tfrac{1}{2}E))\,\frac{1}{\pi\sqrt{E(4-E)}}\,dE$$

We now identify $E\mapsto \frac{1}{\pi}\frac{1}{\sqrt{E(4-E)}}$ as the (Radon-Nikodym derivative w.r.t. the Leb. msr.) of the spectral msr. of $-\Delta$ within each cyclic s/sp. What are these s/spaces? Well, $\mathcal{F}$ preserves parity, so these are the even/odd wave f^n's on $\ell^2(\mathbb{Z})$.

So $\ell^2(\mathbb{Z}) = \mathcal{H}_e \oplus \mathcal{H}_o$ w/

$$\mathcal{H}_{e/o} \equiv \{\psi\in\ell^2(\mathbb{Z}) \mid \psi_{-n} = \pm\psi_n \quad \forall n\in\mathbb{Z}\}.$$

Indeed, these are closed v/s/sp. which are $\perp$.

Then, let μ be the msr. on $\mathbb{R}$ def. by

$$\frac{d\mu}{d\lambda}(E) := \frac{1}{\pi} \frac{1}{\sqrt{E(4-E)}} \chi_{[0,4]}(E)$$

λ: Leb. msr.

$$L^2(\mathbb{R},\mu) \equiv \left\{ \psi : \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{\mathbb{R}} |\psi|^2 d\mu < \infty \right\}$$

$$\begin{aligned} U_i : \mathcal{H}_i &\longrightarrow L^2(\mathbb{R},\mu) \\ \psi &\longmapsto \hat{\psi} \circ \arccos(1 - \tfrac{1}{2}\,\bullet) \, . \end{aligned}$$

By the above this map is well-def. and unitary.

$U_i(-\Delta)U_i^*$ is mul. by $E \mapsto E$.

$$U := U_1 \oplus U_2 \, .$$

This may also be done more systematically by showing that δ_0 and $\delta_1 - \delta_{-1}$

are cyclic for $-\Delta$. In fact,

$$\mathcal{H}_1 = \operatorname{span}\{(-\Delta)^n \delta_0 \mid n \in \mathbb{N}\})$$

$$\mathcal{H}_2 = \operatorname{span}\{(-\Delta)^n (\delta_1 - \delta_{-1}) \mid n \in \mathbb{N}\}).$$

Then calc. the spec. msr. of these vectors. It will be μ.

$\boxed{Q8}$ On $\mathcal{H} := L^2([0,1] \to \mathbb{C})$, define K via

$$(K\psi)(x) := \int_{y=x}^{1} \int_{z=0}^{y} \psi(z)\,dz \qquad (x \in [0,1],\ \psi \in \mathcal{H})$$

<u>Claim</u>: $K \in \mathcal{B}(\mathcal{H} \to \mathcal{H})$

<u>Proof</u>: Define $V : \mathcal{H} \to \mathcal{H}$ via

$$(V\psi)(x) := \int_{y=0}^{x} \psi(y)\,dy \qquad (x \in [0,1],\ \psi \in \mathcal{H})$$

See HW8Q7(3)

<u>Claim</u>: $V \in \mathcal{B}(\mathcal{H})$

<u>Proof</u>:
$$\|V\psi\|_{L^2}^2 \equiv \int_{x=0}^{1} \left| \int_{y=0}^{x} \psi(y)\,dy \right|^2 dx$$
$$\leq \int_{x=0}^{1} \left(\int_{y=0}^{x} |\psi(y)|\,dy \right)^2 dx$$
$$\leq \int_{x=0}^{1} \left(\int_{y=0}^{1} |\psi(y)|\,dy \right)^2 dx$$
$$= \left(\|\psi\|_{L^1} \right)^2 .$$

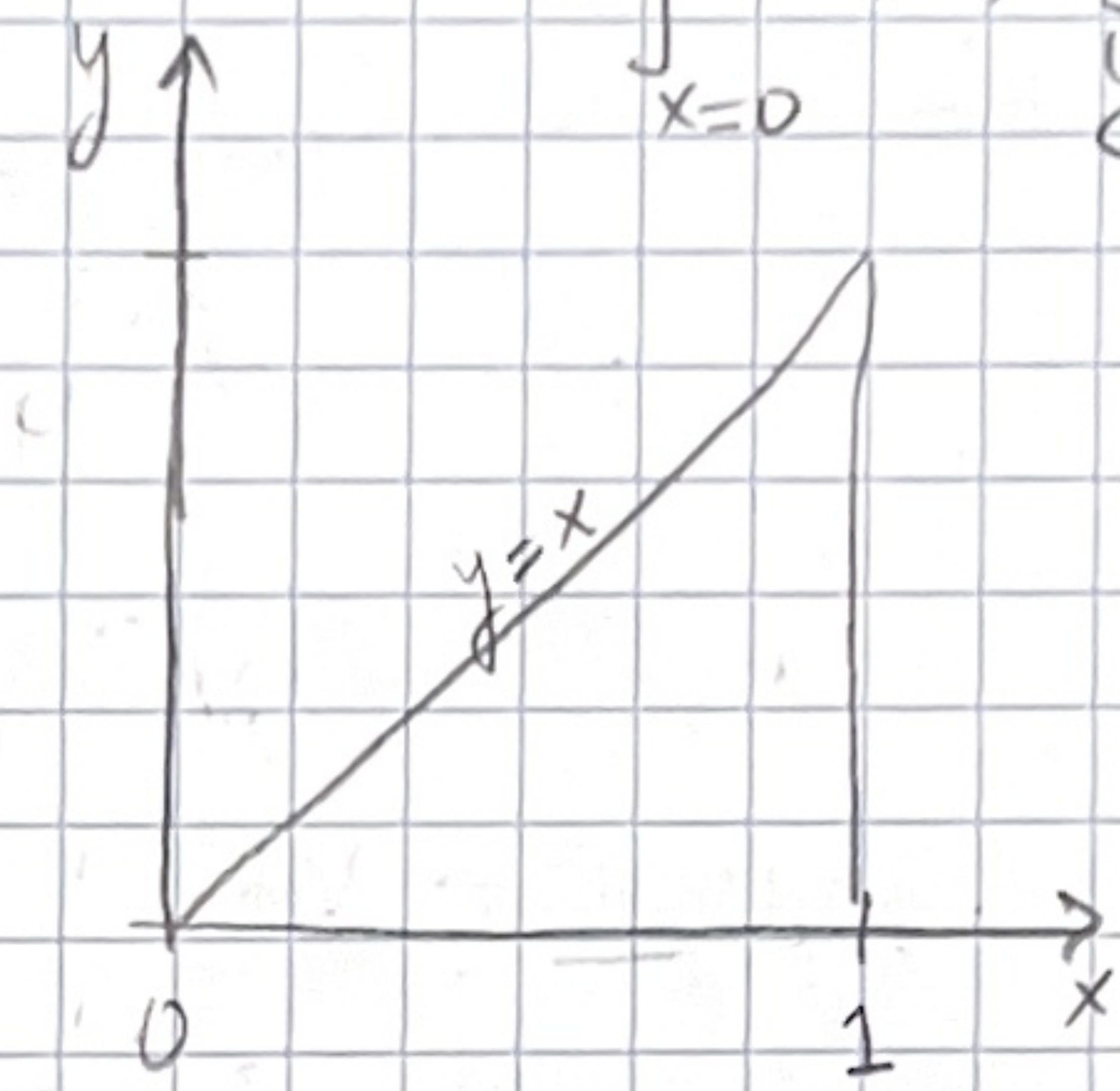
But $\|\psi\|_{L^1} \equiv \int_0^1 |\psi| = \int_0^1 |\psi| \cdot 1 \overset{\text{C.S.}}{\leq} \|\psi\|_{L^2}$

$\Rightarrow \|V\|_{\mathcal{B}(L^2)} \leq 1.$ Linearity is clear. $\blacksquare$

$\boxed{2}$

Claim: $(V^*\psi)(x) = \int_{y=x}^{1} \psi(y)\, dy \qquad (x\in[0,1],\ \psi\in\mathcal{H})$

Proof:
$$\langle V^*\psi, \varphi\rangle \equiv \langle \psi, V\varphi\rangle = \int_{x=0}^{1} \overline{\psi(x)}\,(V\varphi)(x)\, dx$$
$$= \int_{x=0}^{1} \overline{\psi(x)} \int_{y=0}^{x} \varphi(y)\, dy\, dx$$



Fubini

$$\overset{\text{Fubini}}{=} \int_{(x,y)\in\Delta} \overline{\psi(x)}\,\varphi(y)\, dx\, dy$$
$$= \int_{y=0}^{1} dy \int_{x=y}^{1} dx\, \overline{\psi(x)}\,\varphi(y)$$
$$\equiv \int_{y=0}^{1} dy\, \overline{(V^*\psi)(y)}\,\varphi(y).$$
∎

Automatically due to $\|A^*\| = \|A\|$ we get $\|V^*\| \leq 1$.

Claim: $K = |V|^2$

Proof: $(V^* V\psi)(x) \equiv \int_{y=x}^{1} (V\psi)(y)\, dy$

$$= \int_{y=x}^{1} \left(\int_{z=0}^{y} \psi(z)\,dz \right) dy. \qquad \blacksquare$$

$\Rightarrow$ K is clearly linear and bdd. $\blacksquare$

(a) Since $K = |V|^2$, it is clearly self-adjoint.

(b) Claim: K is cpt.

Proof 1: We note that the integral kernel of

V is $V(x,y) = \chi_{[0,x]}(y)$

V^* is $V^*(x,y) = \chi_{[x,1]}(y)$

Hence, e.g.

$$\|K\|_1 \equiv \|V^* V\|_1 \le \underbrace{\|V^*\|_{\mathcal{B}}}_{\le 1} \underbrace{\|V\|_1}_{\le \int_{x,y=0}^{1} V(x,y)\,dx\,dy}$$

$$\le \int_{x,y=0}^{1} \chi_{[0,x]}(y)\,dx\,dy$$

$$= \frac{1}{2} < \infty.$$

Since K is trace-class, it is cpt. (see e.g. Prop. 9.73 in lecture notes). $\blacksquare$

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Alt. proof w/o trace: (Using Ascoli)

Let $\{\psi_n\}_n \subseteq B_M(0_{\mathcal{H}})$ be some bdd. seq. By Lemma 9.34 in LN, if we can show $\{V\psi_n\}_n \subseteq \mathcal{H}$ has a convergent subseq. then V would be cpt. But that would imply cpt.-ness of K as $K(\mathcal{H})$ is a two-sided ideal.

Claim: $\{V\psi_n\}_n$ are pointwise bdd.

Proof: $\left|\int_0^x \psi_n\right| \leq \int_0^1 |\psi_n| \leq \|\psi_n\|_{L^2} < M$

Claim: $\{V\psi_n\}_n$ is equicontinuous.

Proof: $|(V\psi_n)(x) - (V\psi_n)(y)| = \left|\int_0^x \psi_n - \int_0^y \psi_n\right|$

$$= \left|\int_x^y \psi_n\right| = |\langle \chi_{[x,y]}, \psi_n\rangle|$$

$$\leq \underbrace{\|\chi_{[x,y]}\|_{L^2}}_{\sqrt{|x-y|}} \underbrace{\|\psi_n\|_{L^2}}_{\leq M} \leq \sqrt{|x-y|}\,M$$

$\Rightarrow$ By Ascoli's Thm. (Munkres Thm. 45.4) $\{V\psi_n\}_n$ has cpt. closure in $C([0,1] \to \mathbb{C})$ where the latter is taken with the uniform top.

Hence by the seq. char. of cpt. sets, $\{V\psi_n\}_n$ has a conv. subseq. ∎

Alt. proof.: (using fin. rank)

$$(V\psi)(x) \equiv \int_0^x \psi$$

$$\equiv \int_0^1 \chi_{[0,x]}\psi$$

$$= \langle \chi_{[0,x]}, \psi \rangle$$

$$= \int_0^1 \langle \chi_{[0,y]}, \psi \rangle \delta(x-y)\, dy$$

Riemann sum approx., δ-f^n approx.

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \frac{1}{N} \times \langle \chi_{[0,\frac{j}{N}]}, \psi \rangle \times N\chi_{[\frac{j}{N},\frac{j+1}{N}]}(x)$$

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \langle \chi_{[0,\frac{j}{N}]}, \psi \rangle \chi_{[\frac{j}{N},\frac{j+1}{N}]}(x)$$

But $\{\chi_{[0,j/N]}\}_{j=1}^N$ and $\{\chi_{[j/N,\frac{j+1}{N}]}\}_{j=1}^N$

are two sets in L^2, and the conv. is actually in op. norm:

$$V_N := \sum_{j=1}^{N} \chi_{[j/N,\frac{j+1}{N}]} \otimes \chi_{[0,j/N]}^*$$

$$\|(V-V_N)\psi\|_{L^2} \equiv \int_{x=0}^{1} |((V-V_N)\psi)(x)|^2\, dx$$

$$((V-V_N)\psi)(x) = \int_0^x \psi - \sum_{j=1}^{N} \langle \chi_{[0,j/N]}, \psi \rangle \chi_{[j/N,\frac{j+1}{N}]}(x)$$

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$= \int_0^x \psi - \langle \chi_{[0, j_*/N]}, \psi \rangle$

where $j_* = 1, \ldots, N$ is the unique index so that $x \in [\frac{j_*}{N}, \frac{j_*+1}{N}]$.

$$= \int_{j_*/N}^{x} \psi$$

$$\Rightarrow |((V - V_N)\psi)(x)| = |\langle \chi_{[j_*/N, x]}, \psi \rangle|$$

$$\leq \sqrt{x - j_*/N}\, \|\psi\|_{L^2}$$

$$\Rightarrow \|(V - V_N)\psi\|_{L^2}^2 \leq \int_{x=0}^{1} \underbrace{(x - j_*/N)}_{\leq \frac{1}{N}} \|\psi\|_{L^2}^2 \, dx$$

$$\leq \frac{\|\psi\|^2}{N}$$

$$\Rightarrow \boxed{\|V - V_N\| \leq \frac{1}{\sqrt{N}}} .$$

■

(c) Want to determine $\sigma(K)$. Since it is cpt., we know by Riesz–Schauder that $\sigma_{ess}(K) = \{0\}$ (Thm. 9.60 in LN) so we only need to determine fin. deg. eigenvalues which are real and non-zero.

I.e. we want to find $\psi \in L^2 \setminus \{0\}$

s.t. $K\psi = \lambda\psi$ $\exists$ $\lambda \in \mathbb{R} \setminus \{0\}$.

Moreover, $K = |V|^2$, so $K \geq 0$ and hence we may assume $\lambda > 0$.

Intuition: $V\psi \equiv \int_0^\cdot \psi$ is like the inverse of the momentum op. on $L^2([0,1])$, so by spectral mapping we expect K to have spec. which is the inverse of Dirichlet Laplacian on $[0,1]$.

$$\int_{y=x}^{1} dy \int_{z=0}^{y} dz\, \psi(z) \overset{!}{=} \lambda \psi(x) \qquad x \in [0,1]$$

$$\Leftrightarrow \quad \psi(x) = \frac{1}{\lambda} \int_{y=x}^{1} dy\, (V\psi)(y)$$

$\Rightarrow$ ψ is abs. conti.

Differentiate eigeneq-n to get

$$\psi'(x) = \frac{-1}{\lambda} (V\psi)(x)$$

This shows ψ' is diff. so diff. again we get $\psi''(x) = -\frac{1}{\lambda}\psi(x)$

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hence $\psi(x) = A\exp\left(i\frac{1}{\sqrt{\lambda}}x\right) + B\exp\left(-i\frac{1}{\sqrt{\lambda}}x\right)$

for some $A, B \in \mathbb{C}$.

B.C. are determined from the eq-n itself:

$$\psi'(x) = -\frac{1}{\lambda}\int_0^x \psi \overset{x=0}{\Longrightarrow} \boxed{\psi'(0) = 0}$$

$$\psi(x) = \frac{1}{\lambda}\int_{y=x}^1 dy\,(V\psi)(y) \overset{x=1}{\Longrightarrow} \boxed{\psi(1) = 0}$$

This yields: $\begin{cases} Ae^{i\frac{1}{\sqrt{\lambda}}} + Be^{-i\frac{1}{\sqrt{\lambda}}} \overset{!}{=} 0 \\ i\frac{1}{\sqrt{\lambda}}A - i\frac{1}{\sqrt{\lambda}}B \overset{!}{=} 0 \end{cases}$

$$\Rightarrow \boxed{A = B}$$

and $\cos\left(\frac{1}{\sqrt{\lambda}}\right) = 0$

$$\Leftrightarrow \frac{1}{\sqrt{\lambda}} = \left(n + \frac{1}{2}\right)\pi \qquad (n \in \mathbb{Z})$$

But $\lambda > 0$ so $n \in \mathbb{N}$.

We find

$$\boxed{\sigma(K) = \underbrace{\{0\}}_{\text{ess. spec.}} \cup \underbrace{\left\{ \frac{1}{(n+\frac{1}{2})^2\pi^2} \;\middle|\; n \in \mathbb{N}_{\geq 0} \right\}}_{\text{point spec.}}}.$$

$\boxed{Q9}$

See Thm. 8.15 & Lemma 9.5 in LN

Claim: Let $A \in \mathcal{B}(\mathcal{H})$ be given. Then TFAE:

i. $\langle \psi, A\psi \rangle \geq 0 \qquad (\psi \in \mathcal{H})$

ii. $A = A^* \quad \wedge \quad \sigma(A) \subseteq [0, \infty)$

iii. $A = |B|^2 \quad \exists \; B \in \mathcal{B}(\mathcal{H})$.

Proof: $\boxed{i \Rightarrow ii}$

Write $$A_R = \mathrm{Re}\{A\} \equiv \tfrac{1}{2}(A + A^*)$$
$$A_I = \mathrm{Im}\{A\} \equiv \tfrac{1}{2i}(A - A^*).$$

Then $A = A_R + iA_I$, and

$\langle \psi, A\psi \rangle = \langle \psi, A_R\psi \rangle + i\langle \psi, A_I\psi \rangle$

For any $\underline{B = B^*}$,

$$\langle \psi, B\psi \rangle \equiv \langle B\psi, \psi \rangle = \langle \psi, B^*\psi \rangle = \langle \psi B \psi \rangle$$

$\Rightarrow \langle \psi, A_R\psi \rangle, \langle \psi, A_I\psi \rangle \in \mathbb{R}$.

But $\langle \psi, A\psi \rangle \geq 0$ for $\underline{\text{any}}$ ψ.

$\Rightarrow A_I = 0 \Leftrightarrow A = A^*$. ✓

By Weyl's criterion (Thm. 9.22), iff

$\lambda \in \sigma(A) \; \exists \; \{\varphi_n\}_n \subseteq \partial B_1(0_{\mathcal{H}})$:

$$\lim_n \|(A - \lambda \mathbb{1})\varphi_n\| = 0.$$

$\Leftrightarrow \forall \varepsilon > 0 \; \exists \; N_\varepsilon \in \mathbb{N}$; if $n \geq N_\varepsilon$,

$$\|(A - \lambda\mathbb{1})\varphi_n\| < \varepsilon.$$

Thus $\lambda = \lambda \|\varphi_n\|^2$

$$= \langle \varphi_n, \lambda \varphi_n \rangle$$

$$= -\langle \varphi_n, (A - \lambda \mathbb{1})\varphi_n \rangle + \langle \varphi_n, A\varphi_n \rangle$$

$$\geq \langle \varphi_n, A\varphi_n \rangle - \|(A - \lambda \mathbb{1})\varphi_n\|$$

$$\geq \underbrace{\langle \varphi_n, A\varphi_n \rangle}_{\geq 0 \text{ by hypothesis}} - \varepsilon$$

$$\geq -\varepsilon,$$

Since ε was arbitrary, we conclude $\lambda \geq 0$.

$\Rightarrow \quad \sigma(A) \subseteq [0, \infty)$.

$\boxed{ii \Rightarrow iii}$

Using the spectral thm.,

$$A = \int_{\lambda \in \mathbb{R}} \lambda \; dP(\lambda)$$

proj.-valued spec. msr. of A

Since $\sigma(A) \subseteq [0, \infty)$, P is supported only on $[0, \infty)$, so we may write

$$\sqrt{A} = \int_{\lambda=0}^{\infty} \sqrt{\lambda} \; dP(\lambda)$$

Let U be <u>any</u> unitary and define $B := U\sqrt{A}$. Then

$$|B|^2 = \sqrt{A}\,U^* U \sqrt{A}$$
$$= (\sqrt{A})^2 = A.$$

$\boxed{iii \Rightarrow i}$

Let $\psi \in \mathcal{H}$. Then

$$\langle \psi, A\psi \rangle = \langle \psi, |B|^2 \psi \rangle$$
$$= \langle B\psi, B\psi \rangle$$
$$= \|B\psi\|^2 \geq 0.$$

$\blacksquare$

(b) This is Thm. 10.20 in LN.

Note: See the proof of the Krammers-Kronig relation in my MAT330 LN.

(c) Claim: Let $A \in \mathcal{B}(\mathcal{H})$ be normal and $\psi \in \mathcal{H}$ be cyclic for A:

$$\overline{\{A^n \psi \mid n \in \mathbb{N}_{\geq 0}\}} = \mathcal{H}.$$

Then ψ is cyclic for A^*.

Proof: Let $\varphi \in \mathcal{H}$ and $\varepsilon > 0$. Then

$$\|\varphi - \sum_{n=0}^{N} a_n A^n \psi\| < \varepsilon/2$$

$\exists\, N \in \mathbb{N}, \{a_n\}_{n=0}^{N} \subseteq \mathbb{C}$, since ψ is cyclic for A

Applying cyclicity of ψ again on the vector $\left(\sum_{n=0}^{N} a_n A^n\right)^* \psi$, we find

$$\left\| \left(\sum_{n=0}^{N} a_n A^n\right)^* \psi - \sum_{n=0}^{M} b_n A^n \psi \right\| < \frac{\varepsilon}{2}$$

for some $M \in \mathbb{N}$, $\{b_n\}_{n=0}^{M} \subseteq \mathbb{C}$.

Note $\|B\psi\| = \|B^*\psi\|$ if $|B|^2 = |B^*|^2$:

$$\|B\psi\|^2 = \langle \psi, |B|^2 \psi \rangle$$
$$= \langle \psi, |B^*|^2 \psi \rangle$$
$$= \|B^*\psi\|^2 \quad ✓$$

<u>Claim</u>: $B := \left(\sum_{n=0}^{N} a_n A^n\right)^* - \sum_{n=0}^{M} b_n A^n$ is normal.

<u>Proof</u>: $B = \sum_n a_n (A^*)^n - b_n A^n$

$$B^* B = \sum_{n,m} \underbrace{\left(a_n (A^*)^n - b_n A^n\right)^*}_{\overline{a_n} A^n - \overline{b_n} (A^*)^n} \left(a_m (A^*)^m - b_m A^m\right)$$

$$= \sum_{n,m} \overline{a_n} a_m A^n (A^*)^m - \dots$$

$$= \dots = BB^* \quad \text{as } [A, A^*] = 0.$$

$$\Rightarrow \left\| \sum_{n=0}^{N} a_n A^n \psi - \sum_{n=0}^{M} \overline{b_n} (A^*)^n \psi \right\| < \frac{\varepsilon}{2}.$$

$$\Rightarrow \left\| \psi - \sum_{n=0}^{M} \overline{b_n} (A^*)^n \psi \right\| < \varepsilon.$$

Since ψ may be approx. by an arbitrary poly. of A^*, ψ is a cyclic vector of A^*. ∎

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So let

$$P := \chi_{\{+1\}}(A)$$

$$Q := \chi_{\{-1\}}(A)$$

via the msrbl. f^n-al calc.
(or just via the Riesz proj. formula of the holo. f^n-al calc.).

Note these operators may happen to be the zero proj., e.g., if $A = \pm \mathbb{1}$. Anyway, then

$$A = \int_{\lambda \in \mathbb{R}} \lambda \, d\chi_{(-\infty,\lambda)}(A)$$

$$= P - Q \quad .$$

∎

Since we also have

$$\mathbb{1} = \int_{\lambda \in \mathbb{R}} d\chi_{(-\infty,\lambda)}(A)$$

$$= P + Q$$

then we must have $\mathcal{H} = \underbrace{(P\mathcal{H})}_{=:\mathcal{H}_+} \oplus \underbrace{(Q\mathcal{H})}_{=:\mathcal{H}_-}$.

Q10 Let $A = A^* \in \mathcal{B}(\mathcal{H})$ and $\chi_{\cdot}(A)$ the proj.-val. msr. of A.

Claim: $\sigma(A) = \left\{ \lambda \in \mathbb{R} \;\middle|\; \forall \varepsilon > 0,\ \chi_{B_\varepsilon(\lambda)}(A) \neq 0 \right\}$

Proof: We will show $\rho(A) = \{\dots\}^c$.

$\boxed{\subseteq}$ For $\mu_{A,\psi}$ the spec. msr. of (A,ψ), we know $\mathrm{Supp}(\mu_{A,\psi}) \subseteq \sigma(A)$.

So if $\lambda \in \rho(A)$, $\mu_{A,\psi}(B_\varepsilon(\lambda)) = 0 \; \exists \varepsilon > 0$.

But ψ is arbit and

$$\mu_{A,\psi}(B_\varepsilon(\lambda)) = \langle \psi, \chi_{B_\varepsilon(\lambda)}(A)\psi \rangle = 0.$$

Hence $\chi_{B_\varepsilon(\lambda)}(A) = 0$ as this is a S.A. proj.

$\boxed{\supseteq}$ Let $\lambda \in \{\dots\}^c$. Then $\exists\ \varepsilon > 0$:
$\forall \psi, \varphi \in \mathcal{H}$, $\langle \psi, \chi_{B_\varepsilon(\lambda)}(A)\varphi \rangle = 0$.

I.e.,

$$\langle \psi, \varphi \rangle = \langle \psi, \left(\chi_{B_\varepsilon(\lambda)}(A) + [\chi_{B_\varepsilon(\lambda)}(A)]^\perp \right) \varphi \rangle$$

by hypo.

$$\stackrel{\text{by hypo.}}{=} \langle \psi, \chi_{B_\varepsilon(\lambda)}(A)^\perp \varphi \rangle$$

$$\equiv \langle \psi, \chi_{B_\varepsilon(\lambda)^c}(A) \varphi \rangle .$$

Now, if $f(x) := \begin{cases} \frac{1}{x-\lambda} & x \in B_\varepsilon(\lambda)^c \\ 0 & \text{else} \end{cases}$

and $g(x) := x - \lambda$ we get

$$\langle \psi, f(A) g(A) \varphi \rangle = \langle \psi, (fg)(A) \varphi \rangle$$

$$= \int_{\lambda \in \mathbb{R}} (fg)(\lambda) \, d\mu_{A,\psi,\varphi}(\lambda)$$

$$= \int_{\lambda \in \mathbb{R}} f(\lambda) g(\lambda) \, d\mu_{A,\psi,\varphi}(\lambda)$$

$\operatorname{supp}(f) \subseteq B_\varepsilon(\lambda)^c$

$$\stackrel{\operatorname{supp}(f) \subseteq B_\varepsilon(\lambda)^c}{=} \int_{\lambda \in B_\varepsilon(\lambda)^c} f(\lambda) g(\lambda) \, d\mu_{A,\psi,\varphi}(\lambda)$$

$$\underset{\text{on } B_\varepsilon(\lambda)^c}{\overset{fg=1}{\geq}} \int_{\lambda \in B_\varepsilon(\lambda)^c} d\mu_{A,\psi,\varphi}(\lambda)$$

$$= \langle \psi, \chi_{B_\varepsilon(\lambda)^c}(A)\, \varphi \rangle$$

$$= \langle \psi, \varphi \rangle$$

Since ψ, φ were arbitrary,

$g(A) \equiv A - \lambda \mathbb{1}$ has an inverse.

$\Leftrightarrow \lambda \in \rho(A)$.

$\blacksquare$

$\boxed{Q7}$ Let $\mathcal{H}$ be a sep. Hil. sp.

Claim: The only op-norm-closed $*$-ideals in $\mathcal{B}(\mathcal{H})$ are $\{0\}$, $\mathcal{K}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$.

Proof: Let $\mathcal{I} \subseteq \mathcal{B}(\mathcal{H})$ be some non-triv. $*$-closed ideal.

Claim: $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{I}$.

Proof: Let P be a rank-1 proj.
Then $\forall A \in \mathcal{I} \setminus \{0\}$, $PA \in \mathcal{I}$ is a rank-1 op.

$$PA = \varphi \otimes \psi^* \quad \exists \; \varphi, \psi \in \mathcal{H}.$$

By star-closedness, $\psi \otimes \varphi^* \in \mathcal{I}$ too.
From there by composing w/

$\varphi \mapsto \tilde{\varphi}$
$\psi \mapsto \tilde{\psi}$

we get to any other rank-1 op., and by lin. comb. to any fin. rank.
Norm closed $\Rightarrow$ cpt. op. $\blacksquare$

Now, if $A \in \mathcal{I} \setminus \mathcal{K}(\mathcal{H})$, W.T.S. $\mathbb{1} \in \mathcal{I}$.

Since A is NOT cpt., it is impossible that both $\mathrm{Re}\{A\}, \mathrm{Im}\{A\}$ are cpt., so by Thm. 9.60, $\sigma_{ess}(B) \neq \{0\}$

for some $B = B^* \in \mathcal{I}$.

This implies that B is a Fredholm op., so by Atkinson's thm.

(Thm. 9.51) that $\exists\ G \in \mathcal{F}$

s.t. $\begin{matrix} \mathbb{1} - BG \\ \mathbb{1} - GB \end{matrix} \in \mathcal{K}(\mathcal{H})$

But since $\mathcal{I}$ is an ideal,

that means $BG, GB \in \mathcal{I}$, i.e.,

$$\mathbb{1} - K_1,\ \mathbb{1} - K_2 \in \mathcal{I}$$

for some cpt. K_1, K_2. But $\mathcal{K} \subseteq \mathcal{I}$,

so $\mathbb{1} \in \mathcal{I} \iff \mathcal{I} = \mathcal{B}(\mathcal{H})$.

Q1

(a) Let X, Y be two normed spaces, $A: X \to Y$ linear.

Suppose whenever $\{x_n\}_n \subseteq X$ converges weakly to zero, $\{Ax_n\}_n \subseteq Y$ converges weakly to zero.

Claim: A is bounded.

Proof: Claim: X^*, Y^* are Banach spaces.

<u>Proof</u>: $X^* \equiv \{\lambda : X \to \mathbb{C}$ linear & bdd$\}$

w/ norm

$\|\lambda\| \equiv \sup(\{|\lambda \psi| \mid \|\psi\| = 1\})$

W.T.S. X^* is complete.

Let $\{\lambda_n\}_n \subseteq X^*$ be Cauchy.

Then $\forall x \in X$,

$|\lambda_n(x) - \lambda_m(x)| \leq \underbrace{\|\lambda_n - \lambda_m\|}_{\text{small}} \|x\|$

$\Rightarrow \forall x \in X$, $\{\lambda_n(x)\}_n \subseteq \mathbb{C}$ is Cauchy, so by completeness of $\mathbb{C}$ converges to some $\lambda(x) \in \mathbb{C}$.

<u>Claim</u>: $\lambda \in X^*$

<u>Proof</u>: By linearity of limit,

$\lambda : X \to \mathbb{C}$ is linear.

It is bounded too,

$|\lambda(x)| = \lim_n |\lambda_n(x)|$

But $\{\lambda_n\}_n$ is Cauchy,

so it is bdd

$\Rightarrow |\lambda(x)| \leq C$

$\Rightarrow \|\lambda\| \leq C$. ∎

Clearly $\lambda_n \to \lambda$ in op. norm $\blacksquare$

<u>Claim</u>: If $\{\psi_n\}_n \to \psi$ weakly in a normed space X then $\{\psi_n\}_n$ is norm bounded.

<u>Proof</u>: By Lemma 5.11,

$\lambda(\psi_n) \to \lambda(\psi) \quad \forall \, \lambda \in X^*$.

The injection $J : X \to X^{**}$ yields a seq. $\{J(\psi_n)\}_n$ within $\mathcal{B}(X^* \to \mathbb{C})$. Since X^* is complete, we have by Thm 3.28 (Banach-Steinhaus) we have that

$$\sup_n \| J(\psi_n) \| < \infty.$$

But J is an isometry. $\blacksquare$

Now assume A is unbounded. Then $\exists \, \{\psi_n\}_n \subseteq X : \|\psi_n\| \leq 1$ and $\|A\psi_n\| \to \infty$. So $\{\frac{1}{n}\psi_n\}_n$ converges in norm and hence weakly to zero while $\|A \frac{1}{n}\psi_n\| = \frac{1}{n}\|A\psi_n\|$

could be made to converge to ∞. E.g. if we pick

$$\|A\psi_n\| \geq e^n$$

then $\|A\frac{1}{n}\psi_n\| \geq \frac{1}{n}e^n \to \infty$.

But by hypothesis $\{A\frac{1}{n}\psi_n\}_n \to 0$ weakly and by the claim above $\{A\frac{1}{n}\psi_n\}_n$ is supposed to be norm bdd. $\Rightarrow \boxed{\perp}$. ∎

(b) Let X, Y, Z be Banach sp.

$$\left.\begin{array}{l} A: X \to Y \\ J: Y \to Z \end{array}\right\} \text{linear}$$

Assume J is bdd and injective.

Assume JA is bdd.

<u>Claim</u>: A is bdd.

<u>Proof</u>: By the closed graph thm. (Thm 3.37), suffice to show

$$\Gamma(A) \in \text{Closed}(X \times Y).$$

Let $\{\psi_n\}_n \subseteq X$; $\psi_n \to \psi \ \exists \psi \in X$

$A\psi_n \to \varphi \ \exists \varphi \in Y$.

$\boxed{\textbf{Q12}}$ Let $A = U|A|$ be the polar decomp.

$$f_n(x) := \begin{cases} \frac{1}{x} & x \geq \frac{1}{n} \\ n & x \leq \frac{1}{n} \end{cases} \quad (x \geq 0)$$

Claim: $U = \underset{n\to\infty}{s\text{-}\lim}\, A f_n(|A|)$

Proof: $\iff U - \underset{n\to\infty}{s\text{-}\lim}\, U|A| f_n(|A|) = 0$

$\iff U \underset{n\to\infty}{s\text{-}\lim} \left(\mathbb{1} - |A| f_n(|A|) \right) = 0$

$\iff U \left(\underset{n\to\infty}{s\text{-}\lim}\, g_n(|A|) \right) \overset{\circledast}{=} 0$

w/ $g_n(x) := 1 - x f_n(x) \quad \forall\ x \geq 0.$

I.e., $g_n(x) = \begin{cases} 0 & x \geq \frac{1}{n} \\ 1 - nx & 0 \leq x \leq \frac{1}{n} \end{cases}$

g_n is Borel msrbl. & bdd. w/ $\|g_n\|_\infty = 1$. Moreover, $g_n \to \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

the limit being L^2-equiv. to the zero f^n.

Hence by Thm. 10.66 in L.N.,

$\underset{n \to \infty}{\text{s-lim}}\, g_n(|A|) = \chi_{\{0\}}(|A|)$. But by the polar decomp., $\text{im}(\chi_{\{0\}}(|A|)) = \ker(U) \Rightarrow (*)$. ✓

$\boxed{Q3}$ Claim: If $A \in \mathcal{B}(\mathcal{H})$ is normal then

$r(A) = \|A\|$.

Proof: By the functional calculus,

$$\|A\| = \left\| \int_{\lambda \in \mathbb{C}} \lambda \; dP_A(\lambda) \right\|$$

proj.-val. msr. of A

Q13 Let $A, B \in \mathcal{B}(\mathcal{H})$ be s.A. ; $[A, B] = 0$.
Then $[R_A(z), R_B(w)] = 0$ for

$$R_A(z) \equiv (A - z\mathbb{1})^{-1} \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

Via Stone's thm. we may recover the projection-valued measures dP_A as

$$\frac{1}{2}\left(\chi_{[a,b]}(A) + \chi_{(a,b)}(A)\right) = \text{s-}\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_a^b \operatorname{Im}\{R_A(E+i\varepsilon)\} dE.$$

Moreover, this formula shows

$$[dP_A, dP_B] = 0.$$

This allows us to define a msr.

$$Q_{AB}(S_1 \times S_2) := P_A(S_1)\, P_B(S_2) \qquad (S_1, S_2 \subseteq \mathbb{R})$$

on "cylinder" sets from which we may extend to msrbl. sets of $\mathbb{R}^2$.

Thus we now define, $\forall$ Borel bdd.

$$f : \mathbb{R}^2 \to \mathbb{C}$$

the operator

$$f(A,B) := \int_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} f(\lambda_1, \lambda_2)\, dQ_{AB}(\lambda_1, \lambda_2).$$

In particular, to get the unitary define, $\forall\, \psi \in \mathcal{H}$

$$\mathcal{H}_\psi := \overline{\{ f(A,B)\psi \mid f : \mathbb{R}^2 \to \mathbb{C} \text{ msrbl. bdd.} \}}$$

and $U : \mathcal{H}_\psi \longrightarrow L^2(dQ_{AB}\psi)$

$$\begin{aligned} \psi &\longmapsto \underline{1} \\ A\psi &\longmapsto \lambda \mapsto \lambda_1 \\ B\psi &\longmapsto \lambda \mapsto \lambda_2 \end{aligned}$$

and if $\mathcal{H}_\varphi \neq \mathcal{H}_1$ continue in this way.

For more details, see Feldman e.g.

(his notes are attached here, slightly different approach ...)

Spectral Theorem for Commuting Normal Operators

Throughout these notes $\mathcal{H}$ is a Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of all bounded linear operators with domain $\mathcal{H}$ and taking values in $\mathcal{H}$. First recall

Definition 1 (Normal Operator) An operator $A \in \mathcal{L}(\mathcal{H})$ is called *normal* if $A^*A = AA^*$. That is, if A commutes with its adjoint.

Remark 2 (Normal Operators)

(a) A self–adjoint operator $A \in \mathcal{L}(\mathcal{H})$ obeys $A = A^*$ and hence is normal.

(b) A unitary operator $U \in \mathcal{L}(\mathcal{H})$ obeys $UU^* = U^*U = \mathbb{1}$ and hence is normal.

(c) Any operator $A \in \mathcal{L}(\mathcal{H})$ can be written in the form $A = \mathrm{Re}\, A + i\, \mathrm{Im}\, A$ with, by definition, $\mathrm{Re}\, A = \frac{1}{2}(A + A^*)$ and $\mathrm{Im}\, A = \frac{1}{2i}(A - A^*)$. Both $\mathrm{Re}\, A$ and $\mathrm{Im}\, A$ are self–adjoint. The operator A is normal if and only if $\mathrm{Re}\, A$ and $\mathrm{Im}\, A$ commute.

In these notes we prove

Theorem 3 (Spectral Theorem for Commuting Bounded Normal Operators) *Let $n \in \mathbb{N}$ and let $\{A_1, A_2, \cdots, A_n\} \subset \mathcal{L}(\mathcal{H})$ be a finite set of commuting, normal, bounded operators. Then there exist*

- *a measure space $\langle \mathcal{M}, \Sigma, \mu \rangle$ and*
- *n bounded measurable functions $a_i : \mathcal{M} \to \mathbb{C}$, $1 \le i \le n$ and*
- *a unitary operator $U : \mathcal{H} \to L^2(\mathcal{M}, \Sigma, \mu)$*

such that

$$\big(UA_iU^{-1}\varphi\big)(m) = a_i(m)\,\varphi(m)$$

for all $\varphi \in L^2(M, \Sigma, \mu)$ and all $1 \le i \le n$. If $\mathcal{H}$ is separable, μ can be chosen to be a finite measure.

Proof: *Step 0 (Reduction to self–adjoint operators):*
By Fuglede's theorem (proven below), if the normal operators $\{A_1, A_2, \cdots, A_n\}$ commute, then so do all of the operators $\{A_1, A_2, \cdots, A_n, A_1^*, A_2^*, \cdots, A_n^*\}$. Consequently we may restrict our attention to commuting, self–adjoint, bounded operators simply by replacing $\{A_1, A_2, \cdots, A_n\}$ with $\{\mathrm{Re}\, A_1, \mathrm{Im}\, A_1, \mathrm{Re}\, A_2, \mathrm{Im}\, A_2, \cdots, \mathrm{Re}\, A_n, \mathrm{Im}\, A_n\}$. So from now on assume that $\{A_1, A_2, \cdots, A_n\} \subset \mathcal{L}(\mathcal{H})$ is a finite set of commuting, self–adjoint, bounded operators.

Step 1 ($f(A_1, \cdots, A_n)$ for some simple functions f):
Set, for $1 \le i \le n$, $I_i = [-\|A_i\|, \|A_i\|]$ and then set $I = I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$. Define the set of "rectangles" in I to be

$$\mathcal{R} = \left\{ \ B_1 \times B_2 \times \cdots \times B_n \subset I \ \middle| \ B_i \subset I_i, \text{ Borel, for each } 1 \le i \le n \ \right\}$$

There are quotation marks around "rectangles" because the sides of the "rectangles" are Borel sets rather than intervals. We are about to define $f(A_1, \cdots, A_n)$ for all simple functions $f : I \to \mathbb{C}$ that have the special form specified in

$$\mathcal{S} = \left\{ \ f(x) = \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(x) \ \middle| \ \alpha_j \in \mathbb{C}, \ R_j \in \mathcal{R}, \ 1 \le j \le m \right\}$$

We have already defined, in the functional calculus version of the spectral theorem (Theorem 27 in the notes [spectralReview.pdf]), $\chi_{B_i}(A_i)$ for each Borel $B_i \subset I_i$ and $1 \le i \le n$. We also already know the following.

- ◦ $\chi_{B_i}(A_i)$ is an orthogonal projection. (This is an immediate consequence of [spectralReview.pdf, Theorem 27.a].)
- ◦ $\chi_{B_i}(A_i)$ and $\chi_{B_j}(A_j)$ commute for all measurable $B_i \subset I_i$, $B_j \subset I_j$, $1 \le i, j \le n$. (This is an immediate consequence of [spectralReview.pdf, Theorem 27.g].)
- ◦ If the measurable sets $B_i, B_i' \subset I_i$ are disjoint, then $\chi_{B_i}(A_i)\chi_{B_i'}(A_i) = 0$. (This is an immediate consequence of [spectralReview.pdf, Theorem 27.a,b].)

We define, for each $R = B_1 \times B_2 \times \cdots \times B_n \in \mathcal{R}$

$$\chi_R(A_1, \cdots, A_n) = \prod_{j=1}^{n} \chi_{B_i}(A_i)$$

and for each $f = \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(x) \in \mathcal{S}$

$$f(A_1, \cdots, A_n) = \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(A_1, \cdots, A_n)$$

From the above bullets

- ◦ $\chi_R(A_1, \cdots, A_n)$ is an orthogonal projection for each rectangle $R \in \mathcal{R}$.
- ◦ If the rectangles $R, R' \in \mathcal{R}$ are disjoint, then $\chi_R(A_1, \cdots, A_n) \, \chi_{R'}(A_1, \cdots, A_n) = 0$.

Here is the main property that we need of the operators $f(A_1, \cdots, A_n)$, $f \in \mathcal{S}$.

Lemma 4 *If $f \in \mathcal{S}$ then*

$$\|f(A_1, \cdots, A_n)\| \le \sup_{x \in I} |f(x)|$$

Proof. Let $f \in \mathcal{S}$. We may always write f in the form $f = \sum_{j=1}^m \alpha_j\, \chi_{R_j}(x)$ with all of the R_j's disjoint (by possibly subdividing some of the R_j's) and with $\bigcup_{j=1}^n R_j = I$ (by possibly having some of the α_j's zero). Then every $x \in I$ is an element of exactly one R_j and the range of f is exactly $\{\ \alpha_j\ \big|\ 1 \le j \le m\ \}$. So

$$\sup_{x \in I} |f(x)| = \max\{|\alpha_j| \mid 1 \le j \le m\}$$

Now the $\chi_{R_j}(A_1, \cdots, A_n)$'s project onto mutually orthogonal subspaces of $\mathcal{H}$ and, since $\bigcup_{j=1}^n R_j = I$, we have $\sum_{j=1}^m \chi_{R_j}(A_1, \cdots, A_n) = \mathbb{1}$. So, for every $\mathbf{v} \in \mathcal{H}$,

$$\begin{aligned} \mathbf{v} &= \sum_{j=1}^m \chi_{R_j}(A_1, \cdots, A_n)\mathbf{v} \\ \Longrightarrow\ \|\mathbf{v}\|^2 &= \sum_{j=1}^m \|\chi_{R_j}(A_1, \cdots, A_n)\mathbf{v}\|^2 \end{aligned}$$

and

$$\begin{aligned} f(A_1, \cdots, A_n)\mathbf{v} &= \sum_{j=1}^m \alpha_j\, \chi_{R_j}(A_1, \cdots, A_n)\mathbf{v} \\ \Longrightarrow\ \|f(A_1, \cdots, A_n)\mathbf{v}\|^2 &= \sum_{j=1}^m |\alpha_j|^2\, \|\chi_{R_j}(A_1, \cdots, A_n)\mathbf{v}\|^2 \\ &\le \max\{|\alpha_j| \mid 1 \le j \le m\}^2 \sum_{j=1}^m \|\chi_{R_j}(A_1, \cdots, A_n)\mathbf{v}\|^2 \\ &= \max\{|\alpha_j| \mid 1 \le j \le m\}^2 \|\mathbf{v}\|^2 \end{aligned}$$

■

The rest of the proof is identical to the corresponding parts of the proof of the multiplication operator version of the spectral theorem. Here is a very coarse outline of the remaining steps in the proof.

Step 2 ($f(A_1, \cdots, A_n)$ for continuous functions f):
By the Stone–Weierstrass Theorem, every continuous function $f : I \to \mathbb{C}$, is a uniform limit of a sequence $\{f_\ell\}_{\ell \in \mathbb{N}}$ of simple functions in $\mathcal{S}$. So we can define

$$f(A_1, \cdots, A_n) = \lim_{\ell \to \infty} f_\ell(A_1, \cdots, A_n) \in \mathcal{L}(\mathcal{H})$$

By Lemma 4 in Step 1, the right hand side converges in norm. Consequently the map $f \in C(I) \mapsto f(A_1, \cdots, A_n) \in \mathcal{L}(\mathcal{H})$ is

- ∘ continuous and
- ∘ linear and obeys
- ∘ $(fg)(A_1, \cdots, A_n) = f(A_1, \cdots, A_n)\, g(A_1, \cdots, A_n)$ and
- ∘ $f(A_1, \cdots, A_n)^* = (\bar{f})(A_1, \cdots, A_n)$.

Step 3 (Construction of $\mu_{\mathbf{v}}$):
Let $\mathbf{0} \neq \mathbf{v} \in \mathcal{H}$. Then

$$\ell_{\mathbf{v}}(f) = \langle \mathbf{v}\, ,\, f(A_1, \cdots, A_n)\, \mathbf{v} \rangle_{\mathcal{H}}$$

is a positive linear functional on $C(I)$. So, by the Riesz–Markov Theorem, there is a unique, fnite, regular Borel measure $\mu_{\mathbf{v}}$ on I such that

$$\langle \mathbf{v}\, ,\, f(A_1, \cdots, A_n)\, \mathbf{v} \rangle_{\mathcal{H}} = \int_I f(x)\, d\mu_{\mathbf{v}}(x)$$

for all $f \in C(I)$.

Step 4 (Construction of $\mathcal{H}_{\mathbf{v}}$ and $U_{\mathbf{v}}$):
Let $\mathbf{0} \neq \mathbf{v} \in \mathcal{H}$ and set

$$\mathcal{H}_{\mathbf{v}} = \overline{\left\{\ f(A_1, \cdots, A_n)\, \mathbf{v}\ \middle|\ f \in C(I)\ \right\}}$$

Lemma 5 *There is a unique unitary operator* $U_{\mathbf{v}} : \mathcal{H}_{\mathbf{v}} \to L^2(\mu_{\mathbf{v}})$ such that

$$U_{\mathbf{v}}\mathbf{v} = 1$$
$$(U_{\mathbf{v}} A_i U_{\mathbf{v}}^{-1}) f(x) = x_i\, f(x) \qquad 1 \le i \le n$$

Proof. Set

$$\mathcal{D}_{\mathbf{v}} = \left\{\ f(A_1, \cdots, A_n)\, \mathbf{v}\ \middle|\ f \in C(I)\ \right\}$$

and define $\tilde{U}_{\mathbf{v}} : \mathcal{D}_{\mathbf{v}} \to L^2(\mu_{\mathbf{v}})$ by

$$\big(\tilde{U}_{\mathbf{v}} f(A_1, \cdots, A_n)\, \mathbf{v}\big)(x) = f(x)$$

This operator is

- ∘ well–defined
- ∘ linear
- ∘ inner product preserving

As $\mathcal{D}_{\mathbf{v}}$ is dense in $\mathcal{H}_{\mathbf{v}}$, we can use the BLT theorem to define $U_{\mathbf{v}}$ as the continuous extension of $\tilde{U}_{\mathbf{v}}$ to $\mathcal{H}_{\mathbf{v}}$. Then $U_{\mathbf{v}}$ has the required properties and is indeed uniquely determined by those properties.

Step 5 (Completion of the proof by Zornification):
If $\mathcal{H}_{\mathbf{v}} = \mathcal{H}$, we are done. If not Zornify. ■

Theorem 6 *Let $A, T \in \mathcal{L}(\mathcal{H})$. If A is normal and T commutes with A, then T commutes with A^*.*

Proof: By induction $A^nT = TA^n$ for all $0 \le n \in \mathbb{Z}$. As the exponential series $e^{\bar{\lambda}A} = \sum_{n=0}^{\infty} \frac{1}{n!}(\bar{\lambda}A)^n$ converges in norm, we have

$$e^{\bar{\lambda}A}T = Te^{\bar{\lambda}A} \implies e^{\bar{\lambda}A}Te^{-\bar{\lambda}A} = T \implies e^{-\lambda A^*}e^{\bar{\lambda}A}Te^{-\bar{\lambda}A}e^{\lambda A^*} = e^{-\lambda A^*}Te^{\lambda A^*}$$

for all $\lambda \in \mathbb{C}$. As A is normal, we have that $e^{-\lambda A^*}e^{\bar{\lambda}A} = e^{-\lambda A^* + \bar{\lambda}A}$ and furthermore that $U(\lambda) = e^{-\lambda A^* + \bar{\lambda}A}$ obeys $U(\lambda)^* = U(-\lambda) = U(\lambda)^{-1}$. Thus $U(\lambda)$ is unitary and is hence of norm 1. So

$$\|e^{-\lambda A^*}Te^{\lambda A^*}\| = \|U(\lambda)\, T\, U(-\lambda)\| \le \|T\|$$

This shows that the analytic operator valued function $e^{-\lambda A^*}Te^{\lambda A^*}$ is bounded uniformly on all of $\mathbb{C}$. So $e^{-\lambda A^*}Te^{\lambda A^*}$ has to be independent of λ and

$$e^{-\lambda A^*}Te^{\lambda A^*} = e^{-\lambda A^*}Te^{\lambda A^*}\Big|_{\lambda=0} = T$$

for all λ. Differentiating with respect to λ and then setting $\lambda = 0$ gives

$$-A^*T + TA^* = 0$$

as desired. ∎