Functional Analysis
Princeton University MAT520
December 2024 Take-home Final Exam

December 26, 2024

Instructions: This is a take-home exam, meaning you are expected to do it "at home" by yourself in a quiet environment with no other person aiding you (this is not a group assignment). You may very well use any source material which is "not interactive" such as books / notes / lecture notes / etc. But you may *not* communicate with other people, nor post any questions online. Pretend you were in a classroom and could bring with you whichever *non-electronic offline* reference material you needed. Obviously I have no way of policing that, so we shall rely on Princeton's honor code.

- I am not very interested in enforcing any time frames about this exam, but according to the registrar you must have your exam in by the end of exams week, which is the end of Dec 20th.
- If you somehow can't submit the assignment through Gradescope just send it to my email (shapiro@math.princeton.edu).
- During your exam it is possible to email us (shapiro@math.princeton.edu or juihui@princeton.edu) if there is any doubt about the phrasing of the questions (but don't expect to get help or hints, in the interest of fairness).
- Each question is worth $\frac{100}{12}$ points.

1. Prove Kuiper's theorem: Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and let $A \in \mathcal{B}(\mathcal{H})$ be invertible. Show that there is an operator-norm-continuous map $\gamma : [0,1] \to \mathcal{B}(\mathcal{H})$ such that: (1) $\gamma(0) = A$, (2) $\gamma(1) = \mathbb{1}$ and (3) $\gamma(t)$ is invertible for all $t \in [0,1]$.

2. Let Z be essentially unitary ($|Z|^2 - \mathbb{1}$ and $|Z^*|^2 - \mathbb{1}$ are both compact) and Fredholm of index zero. Show that there exists a unitary operator U such that $Z - U$ is compact.

3. Let $\mathcal{A}$ be a C-star algebra. If u is a unitary in $\mathcal{A}$ such that $\sigma(u) \neq \mathbb{S}^1$, show there is a continuous path within the unitaries of $\mathcal{A}$ which connects u with $\mathbb{1}_{\mathcal{A}}$.

4. In $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ a separable Hilbert space, let Λ be a non-trivial projection, i.e., a projection for which $\mathrm{im}\Lambda, \ker\Lambda$ are both infinite dimensional. Show that if U is a unitary operator which essentially commutes with Λ (i.e., $[\Lambda, U] \in \mathcal{K}$) and such that $\mathrm{index}\left(\Lambda U \Lambda + \Lambda^\perp\right) = 0$ then there is a continuous path within unitaries which essentially commute with Λ, starting at U and ending at $\mathbb{1}$. *Hint*: Prove that U may be factorized as $U = AB$ where A, B are two unitaries which essentially commute with Λ such that $\mathbb{1} - A$ is compact (so in particular $\sigma(A) \neq \mathbb{S}^1$) and $[B, \Lambda] = 0$.

5. On $\ell^2\left(\mathbb{Z}^2\right)$, let $L := \mathrm{e}^{\mathrm{i}\arg(X_1 + \mathrm{i}X_2)}$ where X_1, X_2 are the two position operators on $\ell^2\left(\mathbb{Z}^2\right)$, and let P be an orthogonal projection such that there exists some $\nu > 0$ with which
$$|\langle \delta_x, P\delta_y \rangle| \leq \frac{1}{\nu}\exp\left(-\nu\|x - y\|\right) \qquad \left(x, y \in \mathbb{Z}^2\right) .$$
Show that $[P, L]$ is Schatten-3.

6. With $\mathbb{C}_+$ the open upper complex plane, show that if $f : \mathbb{C}_+ \to \mathbb{C}_+$ is analytic, and there exists some $M < \infty$ with which
$$|f(z)| \quad \leq \quad \frac{M}{|\mathbb{Im}\{z\}|} \qquad (z \in \mathbb{C}_+)$$
then there exits a unique positive Borel measure μ_f on $\mathbb{R}$ such that f is the Borel transform of μ_f, i.e., such that
$$f(z) = \int_{x \in \mathbb{R}} \frac{1}{x - z} \mathrm{d}\mu_f(x) \qquad (z \in \mathbb{C}_+) .$$

7. On $\ell^2(\mathbb{Z})$, let R be the bilaterial right shift operator

$$(R\psi)_n \quad \equiv \quad \psi_{n-1} \qquad \left(\psi \in \ell^2, n \in \mathbb{Z}\right) .$$

Let the discrete Laplacian $-\Delta$ be given by

$$-\Delta := 2\mathbb{1} - 2\,\mathbb{Re}\{R\} .$$

Find a unitary $U : \ell^2(\mathbb{Z}) \to \oplus_{j=1}^{2} L^2(\mathbb{R}, \mu_j)$ where μ_1, μ_2 are finite Borel measures, such that on each factor, $(U(-\Delta)U^*)_j$ is a multiplication operator by the symbol $E \mapsto E$.

8. Define the operator K on $L^2([0,1])$ via

$$(K\psi)(x) \quad := \quad \int_{y=x}^{1} \left(\int_{0}^{y} \psi(z)\,\mathrm{d}z \right) \mathrm{d}y \qquad \left(x \in [0,1], \psi \in L^2([0,1])\right) .$$

Show that:

(a) K is self-adjoint.
(b) K is compact.
(c) Find the spectrum of K.

9. This question is divided into three independent parts.

(a) Prove (with the spectral theorem) that if $A \in \mathcal{B}(\mathcal{H})$ then the following are equivalent:
   i. For any $\psi \in \mathcal{H}$, $\langle \psi, A\psi \rangle \geq 0$.
   ii. $A = A^*$ and $\sigma(A) \subseteq [0, \infty)$.
   iii. There exists some $B \in \mathcal{B}(\mathcal{H})$ such that $A = |B|^2$.

(b) Prove Stone's formula: if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and

$$\tilde{\chi}_{[a,b]}(\lambda) := \begin{cases} 1 & \lambda \in (a,b) \\ 0 & \lambda \notin [a,b] \\ \frac{1}{2} & \lambda \in \{a, b\} \end{cases} \qquad (\lambda \in \mathbb{R})$$

then

$$\tilde{\chi}_{[a,b]}(A) = \operatorname*{s-lim}_{\varepsilon \to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\lambda=a}^{b} \left[(A - (\lambda + \mathrm{i}\varepsilon)\mathbb{1})^{-1} - (A - (\lambda - \mathrm{i}\varepsilon)\mathbb{1})^{-1} \right] \mathrm{d}\lambda .$$

(c) Show that if $A \in \mathcal{B}(\mathcal{H})$ is normal and $\psi \in \mathcal{H}$ is a cyclic vector for A then it is also cyclic for A^*.

10. Prove that for $A \in \mathcal{B}(\mathcal{H})$ self-adjoint and $\chi_{\cdot}(A)$ the projection-valued measure of A, we have

$$\lambda \in \sigma(A) \Longleftrightarrow \left[\chi_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A) \neq 0 \qquad (\varepsilon > 0)\right] \qquad (\lambda \in \mathbb{R}) .$$

11. The following question has two independent parts:

(a) Let X, Y be normed spaces and $A : X \to Y$ linear. Suppose that whenever $\{\psi_n\}_n \subseteq X$ converges weakly to zero, $\{A\psi_n\} \subseteq Y$ converges weakly to zero. Show that A is bounded.
(b) Let X, Y, Z be Banach spaces. Let $A : X \to Y$ and $J : Y \to Z$ be linear. Suppose that J is bounded and injective and JA is bounded. Show that A is bounded.

12. Let $A = U|A|$ be the polar decomposition of an operator $A \in \mathcal{B}(\mathcal{H})$. Let

$$f_n(x) := \begin{cases} \frac{1}{x} & x \geq \frac{1}{n} \\ n & x \leq \frac{1}{n} \end{cases} \qquad (x \geq 0) .$$

Prove that

$$U = \operatorname*{s-lim}_{n \to \infty} A f_n(|A|) .$$

Do the same with

$$f_n(x) \quad := \quad \frac{1}{x + \frac{1}{n}} \qquad (x \geq 0) .$$

13. [extra credit] Show that if $A, B \in \mathcal{B}(\mathcal{H})$ are two self-adjoint operators such that $[A, B] = 0$ then there exists a finite measure space (M, μ) and a unitary

$$U : \mathcal{H} \to L^2(M, \mu)$$

such that there are two bounded Borel functions $f, g : M \to \mathbb{R}$ which obey

$$\begin{aligned}(UAU^*\psi)(m) &= f(m)\,\psi(m) \\ (UBU^*\psi)(m) &= g(m)\,\psi(m)\end{aligned}$$

for all $m \in M$ and $\psi \in L^2(M, \mu)$.