Abstract
These lecture notes correspond to a course given in the Fall semester of 2023 in the math department of Princeton University.

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Syllabus

- Main source of material for the lectures: this very document (to be published and weekly updated on the course website–please do not print before the course is finished and the label “final version” appears at the top).

- Official course textbook: No one, main official text will be used but in preparing these notes; I will probably make heavy use of [Rud91] as well as [RS80] and [BB89].

- Other books one may consult are [HN01, Sch01, Bre10, Dou98, LL01, Lax14, Con19, BS18, Yos12, Sim15, Sim10, HS12].

- Two lectures per week: Tue and Thur, 1:30 pm – 2:50 pm in Fine Hall 601.

- People involved:
  - Instructor: Jacob Shapiro jacobShapiro@princeton.edu
    Office hours: Fine 603, Thursdays 3:30pm-4:30pm (starting Sep 14th 2023), or, by appointment.
  - Assistant: ∅

- HW to be submitted weekly on Friday evening either via email to shapiro@math.princeton.edu or in hard copy to my mailbox (at my door on Fine 603) or on Gradescope.
  Submission by Sunday evening will not harm your grade but is not recommended (i.e. you get an automatic extension of two days always).
  HW may be worked together in groups but needs to be written down and submitted separately for each student.
  10% automatic extra credit on HW (up to a maximum of 100%) if you write legibly and coherently.
  You may use LaTeX or LyX to submit if you like but it is not a requirement.

- Grade: 40% HW, 20% Midterm, 40% (take home) Final.

- Attendance policy: 5% extra credit to students who attend lectures regularly and ask questions or point out problems.

- Anonymous Ed discussion enabled. Use it to ask questions or to raise issues (technical or academic) with the course.

- If you alert me about typos and mistakes in this manuscript (unrelated to the sections marked [todo]) I’ll grant you 1 extra point to your final grade (whose maximum is 100 points = A+). The total maximal extra credits due to finding typos is 5 points. In doing so, please refer to a version of the document by the date of typesetting.
  - Thanks goes to: Ethan Hall.

Semester plan

List of (big) theorems and topics aimed at being included:

- The Banach-Steinhaus theorem, open-mapping theorem, closed-graph theorem.
- Hahn-Banach theorem and convexity.
- Weak topologies and Banach-Alaoglu theorem.
- Duality in Banach spaces.
- Polar decomposition.
- Bounded operators on Hilbert space: spectra.
- Spectral theorem for bounded self-adjoint operators and the functional calculus.
- Spectral theorem for unbounded self-adjoint operators, the Hille-Yosida theorem, semigroups.
- The trace ideals, Schatten classes, etc.
- Fredholm theory: Atkinson’s, Dieudonné’s, Fedosov, Atiyah-Jänich, Kuiper, etc.
- Mathematical quantum mechanics.
1 Soft introduction

Functional analysis is the branch of mathematics that is obtained when one marries together topology (or point set topology) and linear algebra, both of which are assumed the reader is very well familiar with (as well as measure theory). The first order of business, is why you should want to combine these two?

Remark 1.1. For notational simplicity we will almost always assume that my vector spaces are over $\mathbb{C}$, sometimes some statements may be recast for $\mathbb{R}$, certainly some care should be taken for general fields. Hence from now on please read the phrase “vector space” as “$\mathbb{C}$-vector-space”.

In a vague sense (and there will be more about this in the homework) all finite dimensional vector spaces are “equivalent” and hence the question of their topology only becomes interesting when the dimension of the vector space goes to infinity. To be a bit more explicit, in \( n \in \mathbb{N} \) dimensions, there is only one vector space, and we denote it by \( \mathbb{C}^n \). We measure distances between vectors in Euclidean fashion, which induces a topology, and we have a notion of a standard basis \( \{ e_j \}_{j=1}^n \) which gives us a way to concretely write down matrices and vectors as tables and columns. This class is about what happens when \( n \) goes to infinity. Then, one has to make some additional choices which concern the interplay between topology (and more concretely, analysis) and the vector space structure, and one has to contend with topological and analytical questions of convergence that essentially build the heart of what is functional analysis. These types of infinite dimensional vector spaces usually arise in applications as spaces of functions, which is the reason for the name of the field “functional analysis”: we will do analysis on functions, whereas so far we have done analysis on numbers.

Enter the concept of topological vector spaces where \( \mathbb{C}^n \). Surely on \( \mathbb{C} \) we may furnish two different topologies \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) such that \((\mathbb{C}, \mathcal{T}_1)\) and \((\mathbb{C}, \mathcal{T}_2)\) are not homeomorphic\(^1\).

**Example 1.2.** If we define \( \mathcal{T}_1 \) to be the Euclidean topology on \( \mathbb{C} \) (associated with the Euclidean distance \( \mathbb{C} \ni z \mapsto |z| \)) and \( \mathcal{T}_2 \) to be the topology associated to the French metro metric

\[
d(z,w) := \begin{cases} |z-w| & z = \alpha w \exists \alpha \in \mathbb{R} \\ |z| + |w| & \text{else} \end{cases} \quad (z,w \in \mathbb{C}) .
\]

(one first shows this is indeed a metric). Then \((\mathbb{C}, \mathcal{T}_1)\) and \((\mathbb{C}, \mathcal{T}_2)\) are not homeomorphic.

**Proof.** Fix any \( z \in \mathbb{C} \setminus \{ 0 \} \) and let \( r < |z| \). Then in the French metro metric, \( B_r(z) \) is an open line segment of length \( 2r \) on the ray defined by \( z \), and centered at \( z \). That ball is by definition, an open ball in the topology \( \mathcal{T}_2 \). If there were a continuous map \( f : (\mathbb{C}, \mathcal{T}_1) \to (\mathbb{C}, \mathcal{T}_2) \), it better be the case that \( f^{-1}(B_r(z)) \) is open in the Euclidean topology, but we know it is not. \( \square \)

Of course one could also take for \( \mathcal{T}_2 \) the trivial topology (where all singletons are open) to get a counter-example, albeit a bit too silly since it is not even Hausdorff.

This example worked in \( \mathbb{C} \), which is one dimensional, and we have seen two ways to furnish \( \mathbb{C} \) with inequivalent topologies. So what’s all this talk about only needing functional analysis in infinite dimensions?

Enter the concept of topological vector spaces.

Whenever we are doing mathematics with two (or more) mathematical structures (i.e., categories) we need to make sure they are compatible. A familiar example should be that of a Lie group, which is a group which is also a manifold, but moreover and crucially, the two structures are compatible in the sense that the group operations respect the manifold structure: they must be smooth functions on the manifold. We want to achieve the same thing when we combine vector spaces with topological spaces.

In category theory, to preserve a structure means to be a morphism in a given category. So in the category of topological spaces, it is the continuous functions which are “topological space morphisms”.

**Definition 1.3** (Topological vector spaces). Let \( X \) be a vector space furnished with a \( T_1 \)-topology \( \text{Open}(X) \). \( X \) is a topological vector space iff the two vector space operations, vector addition \( + : X^2 \to X \), \((u,v) \mapsto u + v \) and scalar multiplication \( m : \mathbb{C} \times X \to X \), \((\alpha,v) \mapsto \alpha v \), are continuous with respect to \( \text{Open}(X) \) (and the standard topology on \( \mathbb{C} \))

**Remark 1.4.** Recall that for a topology to be \( T_1 \) means that all singletons are closed. Presumably it is not very useful to talk about topological vector spaces where \( \text{Open}(V) \) is not \( T_1 \) [Rud91].

**Claim 1.5.** In Example 1.2, \((\mathbb{C}, \mathcal{T}_1)\) is a topological vector space (indeed \( \mathbb{C}^n \) with its standard topology is, for any \( n \in \mathbb{N} \)) whereas \((\mathbb{C}, \mathcal{T}_2)\) is not.

\(^1\)homeomorphism is the appropriate notion of equivalence in the category of topological spaces
We have already seen above in Definition 1.3 the definition of a TVS. Let us get our hands dirty with a few examples and properties.

**Example 1.7.** Consider the space $\ell^p (\mathbb{N} \to \mathbb{C})$ of $p$-summable sequences, i.e.,

$$\ell^p (\mathbb{N}) \equiv \left\{ a : \mathbb{N} \to \mathbb{C} \mid \sum_{n \in \mathbb{N}} |a(n)|^p < \infty \right\}.$$  

It is a vector space (with pointwise addition and scalar multiplication), and we furnish it with a topology which is associated with a metric, which is associated with the norm

$$\ell^p (\mathbb{N}) \ni a \mapsto \|a\|_p \equiv \left( \sum_{n \in \mathbb{N}} |a(n)|^p \right)^{\frac{1}{p}}.$$  

We observe that for different values of $p$, the space itself (as a vector space) is very similar and the main thing that changes is the topology! One may verify $\ell^p (\mathbb{N})$ is a TVS, and moreover, it is infinite dimensional. We claim that $\ell^p (\mathbb{N})$ is not homeomorphic to $\ell^{p'} (\mathbb{N})$ if $p \neq p'$. Not that only if $p \geq 1$ are these spaces actually Banach.

**Proof.** It is clear that $\ell^p (\mathbb{N})$ is indeed a vector space, so one merely has to show it is a TVS. This follows similarly to (1.1). To see that $\ell^p (\mathbb{N})$ is infinite dimensional we may observe it has a basis $\{ \delta_n \} \subseteq \ell^p (\mathbb{N})$ where $\delta_n (m) \equiv \delta_{nm}$.

The proof that $\ell^p$ is not isomorphic to $\ell^{p'}$ is somewhat delicate. We will contend ourselves to the fact that if $p < 1$ and $p' > 1$ then these can’t be isomorphic, as the former is not metrizable and the latter is.

2. **Topological vector spaces and some abstract theory**

We have already seen above in Definition 1.3 the definition of a TVS. Let us get our hands dirty with a few examples and properties.

**Example 2.1** (Some TVS which cannot be realized via norms). The following spaces are examples of topological vector spaces which cannot be realized as Banach spaces—a complete normed vector space—an important structure we’ll get to later.

1. For any $\Omega \in \text{Open} (\mathbb{R}^n)$, $C (\Omega \to \mathbb{C})$ (often times we’ll write $C (\Omega)$ when the codomain is implicitly understood as $\mathbb{C}$ if not indicated) is the space of all continuous functions $\Omega \to \mathbb{C}$. It is an infinite dimensional TVS. [TODO: define the topology on this].

2. For any $\Omega \in \text{Open} (\mathbb{C})$, $H (\Omega)$ is the space all holomorphic functions $\Omega \to \mathbb{C}$. [TODO: define the topology on this].
Definition 2.2 (Bounded, balanced and absorbing sets). A subset $S$ of a TVS $X$ is said to be bounded if for every neighborhood $N$ of $0$ in $X$ there is some $t > 0$ such that
$$S \subseteq sN \quad (s > t).$$

$S$ is said to be balanced iff $\alpha S \subseteq S$ for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Think of balanced like “star-shaped”, which need not be convex.

$S$ is said to be absorbing iff for any $x \in X$, there is some $t > 0$ with $x \in tS$ for some $t$ sufficiently large.

Remark 2.3 (Caution!). This notion of boundedness need not necessarily agree with the usual notion of boundedness in metric spaces (that where there is a sufficiently large $R > 0$ such that the set is covered by $B_R(0)$), actually (see Rudin 1.29). When the metric is induced by a norm, however, the two notions will agree [TODO].

Remark 2.4 (Caution!). It might happen that $A + A \neq 2A$. Example: $A = \{0, 1\}$ then $A + A = \{0, 1, 2\}$ whereas $2A = \{0, 2\}$.

Recall that a local base at a point $\psi \in X$ is a collection of neighborhoods $\gamma$ (open sets which contain $\psi$) such that every neighborhood of $\psi$ contains an element of $\gamma$.

Now, in a TVS $X$, for any fixed $\psi \in X$ we define the translation map
$$T_\psi : X \to X$$
$$\varphi \mapsto \psi + \varphi$$
whose inverse is $T_{-\psi}$ and for any fixed scalar $\lambda \in \mathbb{C}$, we define the multiplication map
$$M_\lambda : X \to X$$
$$\varphi \mapsto \lambda \varphi$$
whose inverse (if $\lambda \neq 0$) is $M_\lambda^{-1}$ with which it is clear that $T_\psi$ and $M_\lambda$ (for $\lambda \neq 0$) are TVS isomorphisms from $V \to V$.

This means that the topology $\text{Open}(X)$ is translation invariant, i.e., $S \in \text{Open}(X)$ iff
$$T_{-\psi}^{-1}(S) \equiv \{ \varphi \in V \mid \varphi - \psi \in S \} = S + \psi$$
for any $\psi \in V$. Hence it suffices to determine $\text{Open}(X)$ by studying any local basis, e.g., that which is defined at the origin $0$ of $V$. We may thus characterize $\text{Open}(X)$ by specifying its local basis at zero: a collection $\mathcal{B}$ of neighborhoods of $0$ such that every neighborhood of $0$ contains a member of $\mathcal{B}$, and $\text{Open}(X)$ is comprised of unions of translates of members of $\mathcal{B}$.

Thus, given a local basis at zero $\mathcal{B}$ for a TVS, we might replace our intuition of “open $\varepsilon$-ball around $x$” for some $\varepsilon > 0$ from metric spaces, with $x + U$ where $U \in \mathcal{B}$.

Definition 2.5. A metric $d$ on $V$ is translation-invariant iff
$$d(\psi + \eta, \varphi + \eta) = d(\psi, \varphi) \quad (\psi, \varphi, \eta \in V).$$

We give the following special terminology for properties of TVS:

1. $X$ is locally convex iff $\exists$ a local base $\mathcal{B}$ of $0$ whose elements are all convex. I.e., if $B \in \mathcal{B}$ then for any $x, y \in B$,
$$tx + (1-t)y$$
lies entirely within $B$ for any $t \in [0, 1]$.

2. $X$ is locally bounded iff $0$ has a bounded neighborhood.

3. $X$ is locally compact iff $0$ has a neighborhood whose closure is compact.

4. $X$ is metrizable iff $\text{Open}(X)$ arises from some metric $d$ and $X$ is an $F$-space iff its topology arises via a complete translation-invariant metric (sometimes called Fréchet space).

5. $X$ is normable iff there is a norm on $X$ which induces a metric which induces $\text{Open}(X)$.

6. $X$ has the Heine-Borel property iff every closed and bounded subset of $X$ is compact.
Lemma 2.6. If $W \in \text{Nbhd}(0_X)$ then $\exists U \in \text{Nbhd}(0_X)$ which is symmetric, i.e., $U = -U$ and such that $U + U \subseteq W$.

Proof. Since addition is continuous and $0 + 0 = 0$, there are $V_1, V_2 \in \text{Nbhd}(0)$ such that $V_1 + V_2 \subseteq W$. Define then

$$U := V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$$

and observe it has the desired properties: it is clearly symmetric, and

$$U + U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2) + V_1 \cap V_2 \cap (-V_1) \cap (-V_2) \subseteq V_1 + V_2 \subseteq W.$$ 

Note that this may be iterated to get a symmetric $U$ with $U + U + U \subseteq W$.

Theorem 2.7 (Separation property). Let $K, C \subseteq X$ where $X$ is a TVS and such that $K$ is compact, $C$ is closed, and $K \cap C = \varnothing$. Then $\exists V \in \text{Nbhd}(0_X)$ such that

$$(K + V) \cap (C + V) = \varnothing.$$ 

We note that $K + V = \bigcup_{x \in K} x + V$ and each $x + V$ is open, so that $K + V$ is in fact open, and contains $K$. Since $C + V$ is open, it is also true that

$$(K + V) \cap (C + V) = \varnothing.$$ 

Indeed, this is equivalent to

$$K + V \subseteq (C + V)^c$$

but the closure is the smallest closed set which contains $K + V$.

- Special case: If we take $K = \{ 0 \}$, we find that in particular that

$$\overline{V} \cap (C + V) = \varnothing.$$ 

Now, for a given $U \in \text{Nbhd}(0)$, $U^c$ is closed and does not contain zero, and hence, there is some $V \in \text{Nbhd}(0)$ with $\overline{V} \cap (U^c + V) = \varnothing$ which implies $\overline{V} \cap U^c = \varnothing$, i.e., $V \subseteq U$.

Proof of Theorem 2.7. Assume WLOG that $K \neq \varnothing$ (otherwise there is nothing to prove, with $\varnothing + V = \varnothing$). Let $x \in K$ then, and $x \notin C$. Then $C^c - x \in \text{Nbhd}(0)$ so that by the above lemma, we have some symmetric $V_x \in \text{Nbhd}(0)$ such that $V_x^c + V_x + V_x \subseteq C^c - x$, i.e.,

$$(x + V_x + V_x + V_x) \cap C = \varnothing.$$ 

Since $V_x$ is symmetric, this implies that

$$(x + V_x + V_x) \cap (C + V_x) = \varnothing.$$ 

Since $K$ is compact, there are finitely many $\{ x_1, \ldots, x_n \}$ such that

$$K \subseteq \bigcup_{j=1}^n (x_j + V_{x_j}).$$ 

Define $V := \bigcap_{j=1}^n V_{x_j}$, so that

$$K + V \subseteq \bigcup_{j=1}^n (x_j + V_{x_j} + V) \subseteq \bigcup_{j=1}^n (x_j + V_{x_j} + V_{x_j})$$

and all of these terms in the union do not intersect $C + V$. 

Lemma 2.8. If $X$ is a TVS and $A \subseteq X$ then $\overline{A} = \bigcap_{U \in \text{Nbhd}(0)} (A + U)$. 

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Proof. By definition, \( x \in A \) iff \( (x + U) \cap A \neq \emptyset \) for every \( U \in \text{Nbhd}(0) \), iff \( x \in A - U \) for every \( U \in \text{Nbhd}(0) \), But \( U \) is a neighborhood of zero iff \( -U \) is. So we find the desired claim. \( \square \)

**Claim 2.9.** If \( E \) is bounded then so is \( \overline{E} \).

**Proof.** By Theorem 2.7, for a given \( V \in \text{Nbhd}(0_X) \), we pick some \( W \in \text{Nbhd}(0_X) \) such that \( \overline{W} \subseteq V \). Since \( E \) is bounded, \( E \subseteq tW \) for all \( t \) sufficiently large. For such \( t \),

\[
\overline{E} \subseteq t\overline{W} \subseteq tV
\]

so that \( \overline{E} \) is bounded too. \( \square \)

**Theorem 2.10.** In a TVS \( X \), (1) For any \( U \in \text{Nbhd}(0_X) \) there exists some \( V \in \text{Nbhd}(0_X) \) which is balanced such that \( V \subseteq U \) and (2) For any \( U \in \text{Nbhd}(0_X) \) which is convex there exists some \( V \in \text{Nbhd}(0_X) \) which is balanced and convex such that \( V \subseteq U \).

**Proof.** Let \( U \in \text{Nbhd}(0_X) \) be given. Thanks to the fact that scalar multiplication is continuous, there is some \( \delta > 0 \) and a \( W \in \text{Nbhd}(0_X) \) such that \( \alpha W \subseteq U \) for all \( 0 < \alpha < \delta \). Set \( V := \bigcup_{\alpha \in (0, \delta)} \alpha W \). Then \( V \) obeys the desired criteria. Indeed, clearly \( V \) is balanced, contained in \( U \) and is a neighborhood of \( 0 \).

Next, suppose that further that \( U \) is convex. Set \( A := \bigcap_{\alpha \in \mathbb{C} : |\alpha| = 1} \alpha U \) and let \( V \) be balanced and constructed as above. \( \square \)

**Theorem 2.11.** If \( X \) is a TVS and \( U \in \text{Nbhd}(0_X) \), then

1. For \( \{ r_n \}_{n \in \mathbb{N}} \subseteq (0, \infty) \) with \( \lim_{n} r_n = \infty \),

\[
X = \bigcup_{n=1}^{\infty} r_n U.
\]

I.e., any neighborhood of zero is absorbing.

2. If \( K \subseteq X \) is compact then \( K \) is bounded.

3. If \( \{ \delta_n \}_{n \in \mathbb{N}} \subseteq (0, \infty) \) are such that \( \delta_j > \delta_{j+1} \) and \( \lim_{n} \delta_n = 0 \), and if furthermore \( U \) is bounded, then \( \{ \delta_n U \}_{n \in \mathbb{N}} \) is a local base for \( X \).

**Proof.** Fix \( x \in X \), and observe the map \( \mathbb{C} \ni \alpha \mapsto \alpha x \) is continuous, and so, \((\cdot x)^{-1}(U) \in \text{Open}(\mathbb{C}) \) and contains the origin (since \( 0x = 0 \in V \)) and so contains \( \frac{1}{r_n} \) for \( n \) sufficiently large. That is, \( \frac{1}{r_n} x \in U \) for \( n \) sufficiently large, which is \( x \in r_n U \).

Next, let \( K \subseteq X \) be compact, and pick some \( W \in \text{Nbhd}(0_X) \) which is balanced and contained in \( U \). By the first part,

\[
K \subseteq \bigcup_{n=1}^{\infty} nW
\]

But this is an open cover, so there is some finite subset \( \{ n_1, \ldots, n_M \} \subseteq \mathbb{N} \) with

\[
K \subseteq \bigcup_{n \in \{ n_1, \ldots, n_M \}} nW \subseteq n_M W
\]

assuming \( n_M \) is the largest one. This last inclusion follows by the balanced assumption on \( W \). Now, if \( t > n_M \), then \( K \subseteq tW \subseteq tU \).

Finally, let \( V \) be a neighborhood of \( 0 \). Since \( U \) is assumed to be bounded, then there exists some \( s > 0 \) such that
Finite-dimensional spaces

2.1 Properties of linear maps between TVS

Claim 2.12. Let $\Lambda : X \to Y$ be a linear map between two TVS $X, Y$. Assume that $\Lambda$ is continuous at 0. Then $\Lambda$ is continuous globally.

Proof. Let $U \in \text{Open}(Y)$. We want to show that $\Lambda^{-1}(U) \in \text{Open}(X)$. It need not be the case that $0_Y \in U$, but there must be some $x \in X$ such that $U + \Lambda x \in \text{Nbhd}(0_Y)$. Since $\Lambda$ is linear, $\Lambda 0_x = 0_Y$, and hence, the fact that $\Lambda$ is continuous at $0_X$ implies that $\Lambda^{-1}(U + \Lambda x) \in \text{Nbhd}(0_x)$. But that means that $\Lambda^{-1}(U + \Lambda x)$ is open, and moreover,

$$
\Lambda^{-1}(U + \Lambda x) \equiv \{ z \in X \mid \Lambda z \in U + \Lambda x \} \\
= \{ z \in X \mid \Lambda (z - x) \in U \} \\
= \{ z \in X \mid z - x \in \Lambda^{-1}(U) \} \\
= \Lambda^{-1}(U) + x.
$$

But translations are homeomorphisms, $\Lambda^{-1}(U)$ is open indeed.

Lemma 2.13. Let $\Lambda : V \to \mathbb{C}$ be a linear functional from a TVS $V$ such that $\ker(\Lambda) \neq V$. Then TFAE:

1. $\Lambda$ is continuous.
2. $\ker(\Lambda) \in \text{Closed}(V)$.
3. $\ker(\Lambda)$ is not dense in $V$.
4. $\exists U \in \text{Nbhd}(0)$ such that $\Lambda|_U : U \to \mathbb{C}$ is bounded.

Proof. Continuity may just as well be characterized by saying the pre-images of closed sets are closed. Since $\ker(\Lambda) \equiv \Lambda^{-1}(\{ 0 \})$ and $\{ 0 \} \in \text{Closed}(\mathbb{C})$, we find that (1) implies (2).

To be dense, we’d have to have $\ker(\Lambda) = V$. But $\ker(\Lambda)$ is already closed, so that would mean $\ker(\Lambda) = V$ which is false by hypothesis. So (2) implies (3).

Let us now assume (3). Then that means $\ker(\Lambda)$ has a complement which has a non-empty interior. I.e., for some $x \in V$,

$$(x + U) \cap \ker(\Lambda) = \emptyset \quad (2.1)$$

where $U \in \text{Nbhd}(0)$. Now, WLOG we may assume that $U$ is balanced via Theorem 2.10. Note that linearity implies that $\Lambda U$ is a balanced subset of $\mathbb{C}$. If $\Lambda U$ happens to be bounded then we have (4). Otherwise, because $\Lambda U$ is balanced and unbonded, $\Lambda U = \mathbb{C}$, in which case, there is some $y \in U$ such that $\Lambda y = -\Lambda x$. But the $\Lambda (x + y) = 0$ so that $x + y \in \ker(\Lambda)$, in contradiction with (2.1).

Now, assume (4). Then $|\Lambda \psi| < M$ for all $\psi \in U$ for some $U \in \text{Nbhd}(0)$ and some $M < \infty$. Now, if $r > 0$, $W := \frac{r}{M} U$, then by linearity, $|\Lambda \psi - \Lambda 0| = |\Lambda \psi| < r$ for any $\psi \in W$, so that $\Lambda$ is continuous at the origin, and is hence continuous globally via Claim 2.12.

2.2 Finite-dimensional spaces

Claim 2.14. If $X$ is a TVS and $f : \mathbb{C}^n \to X$ is linear then $f$ is continuous.

Proof. By linearity, we may write for any $z \in \mathbb{C}^n$ that

$$f(z) = \sum_{j=1}^n z_j f(e_j)$$
where \( \{ e_j \}_{j=1}^n \) is the standard basis of \( \mathbb{C}^n \) and \( z \equiv (z_1, \ldots, z_n) \). Since \( z \mapsto z_j \) is continuous for each \( j \) (by definition of product topology) and addition and scalar multiplication are continuous in \( X \), we find that \( f \) is continuous. \( \square \)

**Claim 2.15.** If \( Y \subseteq X \) is a vector space of a TVS \( X \) such that \( \dim (Y) < \infty \) then \( Y \in \text{Closed} (X) \) and every VS isomorphism \( \mathbb{C}^n \rightarrow Y \) is a TVS isomorphism.

**Proof.** We start by showing the second property. Let \( f : \mathbb{C}^n \rightarrow Y \) be a VS isomorphism. We have just seen that that means \( f \) is continuous. Let us denote by

\[
S := \left\{ z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^2 = 1 \right\}.
\]

Clearly \( S \) is compact as it is topologically \( \mathbb{S}^{2n} \), and hence, with \( f \) being continuous, \( f(S) \) is compact too. Since \( f \) is injective and \( f(0_{\mathbb{C}^n}) = 0 \), and \( 0_{\mathbb{C}^n} \notin S \), we must have \( 0 \notin f(S) \). Hence there is some balanced (by Theorem 2.10) \( V \in \text{Nbhd} (0_X) \) such that \( V \cap f(S) = \emptyset \). Hence

\[
E = f^{-1}(V) = f^{-1}(V \cap Y) \subseteq \mathbb{C}^n
\]

is disjoint from \( S \). Since \( f \) is linear, \( E \) is balanced and connected. Hence \( E \subseteq B_1(0) \) as \( 0 \in E \). So the linear map \( f^{-1} \) takes \( V \cap Y \) into \( B_1(0) \). Hence the linear map \( f^{-1} \) is bounded, and so each of its coordinates is a bounded functional, so that by Lemma 2.13 we learn that \( f^{-1} \) is continuous. Hence \( f \) is a topological homeomorphism, and since we have assumed it is a vector space isomorphism, we conclude altogether it is a TVS isomorphism.

Next, for the first property, we want to show that \( \overline{Y} \subseteq Y \), which is equivalent to \( Y^c \subseteq (\overline{Y})^c \). Let \( x \in Y^c \). Then \( Z := \text{span} (x, Y) \) is also finite dimensional, and by the first part, linearly homeomorphic to \( \mathbb{C}^{n+1} \). Hence, since this property holds in \( \mathbb{C}^{n+1} \approx Z \), we have that \( x \in (\overline{Y})^c \) where now we mean closure and complement in \( Z \) and not in \( X \). By definition of the subspace topology,\n
\[
\text{closure}_Z (Y) = \text{closure}_X (Y) \cap Z
\]

and hence \( (\overline{Y})^c \) with closure and complement in \( Z \) equals

\[
Z \setminus \text{closure}_Z (Y) = Z \setminus (\text{closure}_X (Y) \cap Z) = Z \setminus \text{closure}_X (Y) \subseteq X \setminus \text{closure}_X (Y).
\]

Hence, \( x \notin \text{closure}_X (Y) \) which is what we wanted to show. \( \square \)

**Theorem 2.16.** Every locally compact TVS \( X \) has finite dimension.

**Proof.** Assume that there is some \( V \in \text{Nbhd} (0_X) \) whose closure \( \overline{V} \) is compact. By Theorem 2.11, \( V \) is bounded, and \( \{ 2^{-n}V \}_{n} \) forms a local base (at 0) for \( X \). Since

\[
\bigcup_{x \in X} x + \frac{1}{2}V
\]

is an open cover for \( \overline{V} \), by compactness, there are some \( \{ x_1, \ldots, x_m \} \) such that

\[
\overline{V} \subseteq \bigcup_{x \in \{ x_1, \ldots, x_m \}} x + \frac{1}{2}V
\]

Define \( Y := \text{span} (\{ x_1, \ldots, x_m \}) \), so that \( \dim (Y) \leq m \), and so by the above, \( Y \in \text{Closed} (X) \). Hence \( \overline{V} \subseteq Y + \frac{1}{2}V \). But \( V \subseteq \overline{V} \), so

\[
V \subseteq Y + \frac{1}{2}V.
\]
Using the fact that $\lambda Y = Y$ for any $\lambda \neq 0$ (since $Y$ is a vector subspace) we find

$$\frac{1}{2}V \subseteq Y + \frac{1}{4}V$$

and hence

$$V \subseteq \bigcap_{n=1}^{\infty} \left( Y + 2^{-n}V \right).$$

Now since $\{2^{-n}V\}_n$ is a local base, we use Lemma 2.8 to find $\bigcap_{n=1}^{\infty} \left( Y + 2^{-n}V \right) = \overline{V} = Y$, so $V \subseteq Y$. This implies that $kV \subseteq Y$ for all $k \in \mathbb{N}$ (recall $kY = Y$) So $Y = X$ via Theorem 2.11.

**Theorem 2.17.** If $X$ is a locally bounded TVS with the Heine-Borel property then $X$ is finite-dimensional.

**Proof.** By assumption, there is some $V \in \text{Nbhd}(0_X)$ which is bounded. We claim that $\overline{V}$ is also bounded too (via Claim 2.9). Hence by the Heine-Borel property $\overline{V}$ is compact. Thus $X$ is locally compact, and thus finite dimensional by the above theorem.

2.3 Metrization

**Theorem 2.18.** If $X$ is a TVS with a countable local base at $0_X$ then there is a metric $d : X^2 \to [0, \infty)$ such that:

1. $d$ is compatible with $\text{Open}(X)$.
2. $B_\varepsilon(0_X)$ is balanced for any $\varepsilon > 0$.
3. $d$ is translation-invariant.

**Proof.** See Rudin 1.24.

2.4 Cauchy-Sequences

**Definition 2.19.** A sequence $x : \mathbb{N} \to X$ where $X$ is a TVS (with a local base $\mathcal{B}$) is called Cauchy iff for any $V \in \mathcal{B}$, there exists some $N_V \in \mathbb{N}$ such that

$$x_n - x_m \in V \quad (n, m > N_V).$$

**Claim 2.20.** This definition does not depend on the choice of local base $\mathcal{B}$.

**Claim 2.21.** If $X$ is a TVS with $\text{Open}(X)$ induced by some metric $d$ then sequences are Cauchy iff they are Cauchy in the usual sense from metric spaces.

2.5 Bounded linear maps

**Definition 2.22.** If $X, Y$ are two TVS and $\Lambda : X \to Y$ is linear, $\Lambda$ is called bounded iff $\Lambda$ maps bounded sets into bounded sets.

Note that this is not the same thing as a bounded function, whose range is a bounded set (i.e. it maps the whole domain to a bounded set). But no linear function could ever be bounded in this way.

**Theorem 2.23.** If $X, Y$ are two TVS and $\Lambda : X \to Y$ is linear then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

1. $\Lambda$ is continuous.
2. $\Lambda$ is bounded.
3. If $x : \mathbb{N} \to X$ has $\lim_n x_n = 0$ then $\{\Lambda x_n\}_n$ is bounded.

It turns out that if $X$ is metrizable then (3) implies (1) too, though we will not show this.
Next, we discuss 2.6 Seminorms

Definition 2.24 (Seminorm). A seminorm on a vector space $X$ is map $p : X \to [0, \infty)$ which has all the axioms of a norm (see Definition C.7) except it is allowed that $p(x) = 0$ for some $x \neq 0$. I.e.,

1. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.
2. $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{C}$ and $x \in X$.

Lemma 2.25. If $p$ is a seminorm on a vector space $X$ then: (1) $p(0) = 0$, (2) $|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$, (3) $\ker(p)$ is a vector subspace of $X$ and (4) $B := \{ x \in X \mid p(x) < 1 \}$ is convex, balanced and absorbing.

Proof. For (1), use $p(\alpha x) = |\alpha| p(x)$ with $\alpha = 0$. From the triangle inequality, we have

$$p(x) = p(x - y + y) \leq p(x - y) + p(y).$$

Since this holds with $x, y$ interchanged and $p(x - y) = p(y - x)$, we have (2) (which actually implies automatically $p \geq 0$ with $y = 0$). Next, to show $\ker(p)$ is a subspace, let $x, y \in \ker(p)$ and $\alpha \in \mathbb{C}$. Then

$$0 \leq p(\alpha x + y) \leq |\alpha| p(x) + p(y) = 0$$

so $\alpha x + y \in \ker(p)$ as well.

Let us show that $B$ is balanced: let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Then

$$\alpha B = \{ \alpha x \mid p(x) < 1 \}$$

$$\quad = \left\{ \alpha x \left| \frac{1}{|\alpha|} p(\alpha x) < 1 \right. \right\}$$

$$\quad = \{ x \mid p(x) < 1 \}.$$ 

If $x, y \in B$ and $t \in (0, 1)$ then

$$p(tx + (1 - t)y) \leq tp(x) + (1 - t)p(y) < t + 1 - t = 1$$

so that $B$ is convex. Finally, to see $B$ is absorbing, let $x \in X$. Pick some $s \in (0, \infty)$ with $s > p(x)$. Then

$$p(s^{-1}x) = s^{-1}p(x) < 1$$

so that $s^{-1}x \in B$ or $x \in sB$. □

Next, we discuss families of seminorms

Definition 2.26 (Separating family). A family $\mathcal{P}$ of seminorms on $X$ is called separating iff $\forall x \in X \setminus \{ 0 \}$ there exists some $p \in \mathcal{P}$ with $p(x) \neq 0$. 

Proof. Assume $\Lambda$ is continuous. We try to show it is bounded. So let $E$ be a bounded set of $X$, and we need to show $\Lambda E$ is a bounded set of $Y$. To that end, let $W \in \text{Nbhd}(0_Y)$. Continuity of $\Lambda$ implies that there is some $V \in \text{Nbhd}(0_X)$ with $\Lambda V - \Lambda 0 = \Lambda V \subseteq W$. Now since $E$ is bounded, $E \subseteq tV$ for large $t$, so

$$\Lambda E \subseteq t\Lambda V = t\Lambda V \subseteq tW.$$ 

Next, we claim convergent sequences are bounded. Indeed, use the fact that for a convergent sequence $x_n \to x$, the set $\{ x_n \} \cup \{ x \}$ is compact (any open cover has a finite sub-cover) and then employ Theorem 2.11. But now since $\Lambda$ is bounded, $\{ \Lambda x_n \} \subseteq E$ is bounded and we get (3) out of (2). □
Definition 2.27 (The Minkowski functional). For any convex absorbing set \( A \subseteq X \), define its Minkowski functional \( \mu_A \) as
\[
\mu_A (x) := \inf \{ t > 0 \mid t x \in tA \} \quad (x \in X).
\]
The fact that \( A \) is absorbing implies \( \mu_A : X \to [0, \infty) \).

Claim 2.28. Note that if \( p \) is a seminorm and \( B \equiv \{ x \in X \mid p(x) < 1 \} \) then as above, \( B \) is convex and absorbing, and \( \mu_B = p \).

Proof. From the above we have seen that for any \( x \in X \) and \( s > p(x) \), \( x \in sB \) which implies \( \mu_B (x) \leq s \) and hence \( \mu_B \leq p \). Conversely, if \( t \in (0, p(x)] \) then \( p \left( \frac{1}{t} x \right) = \frac{1}{t} p(x) \geq 1 \) and hence \( x \notin tB \). Thus \( p(x) \leq \mu_B (x) \).

Theorem 2.29. Assume \( \mathcal{P} \) is a separating family of seminorms on a VS \( X \) (note \( \text{Open} (X) \) is not assumed to exist). To each \( p \in \mathcal{P} \) and \( n \in \mathbb{N} \) define
\[
V_n (p) := \left\{ x \in X \mid p(x) < \frac{1}{n} \right\}.
\]
Let \( \mathcal{B} \) be the collection of all finite intersections of \( \{ V_n (p) \}_{n \in \mathbb{N}, p \in \mathcal{P}} \). Then \( \mathcal{B} \) is a convex, balanced local base for a topology on \( X \) which makes it a locally convex TVS with (1) any \( p \in \mathcal{P} \) is continuous and, (2) \( E \subseteq X \) is bounded iff for every \( p \in \mathcal{P}, \| p \|_E \) is bounded.

Proof. Define \( A \in \text{Open} (X) \) iff \( A \) is the union of translates of members of \( \mathcal{B} \), which automatically defines a translation-invariant topology on \( X \), for which \( \mathcal{B} \) is a local-base.

Let us see that \( \text{Open} (X) \) is \( T_1 \): Since \( \mathcal{P} \) is separating, for \( x \neq 0 \), \( p(x) > 0 \) for some \( p \in \mathcal{P} \). Hence \( p(x) > \frac{1}{n} \) for some \( n \) sufficiently large, whence \( x \notin V_n (p) \), i.e., \( 0 \notin x - V_n (p) \), that is, \( x \) is not in \( \{ 0 \} \). This is equivalent to the fact that \( \{ 0 \} \in \text{Closed} (X) \), and since \( \text{Open} (X) \) is translation-invariant, we are \( T_1 \).

Let us see that addition is continuous. For any \( U \in \text{Nbhd} (0) \), we must have by the definition of the local base,
\[
V_{n_1} (p_1) \cap \cdots \cap V_{n_m} (p_m) \subseteq U
\]
for some finite \( p_1, \ldots, p_m \in \mathcal{P} \) and \( n_1, \ldots, n_m \in \mathbb{N} \). Define
\[
V := V_{2n_1} (p_1) \cap \cdots \cap V_{2n_m} (p_m).
\]
Then
\[
V + V = V_{2n_1} (p_1) \cap \cdots \cap V_{2n_m} (p_m) + V_{2n_1} (p_1) \cap \cdots \cap V_{2n_m} (p_m) \subseteq U.
\]
This shows that addition is continuous.

To see that scalar multiplication is continuous: for \( x \in X \) and \( \alpha \in \mathbb{C} \), let \( U, V \) be as above. Then thanks to \( V \) being absorbing (why is it?), \( x \in sV \) for some \( s > 0 \). Define \( t := \frac{s}{1 + s + \alpha} \). If \( y \in x + tV \) and \( |\alpha - \beta| < \frac{1}{s} \) then
\[
\beta y - \alpha x = \beta (y - x) + (\beta - \alpha) x
\]
which lies in
\[
|\beta| tV + |\alpha - \beta| sV \subseteq V + V \subseteq U
\]
since \( |\beta| t \leq 1 \) (why)? and \( V \) is balanced.

Hence \( X \) is a locally convex TVS. From the definition of \( V_n (p) \), every \( p \in \mathcal{P} \) is continuous at zero, and so on the whole of \( X \) thanks to \( |p(x) - p(y)| \leq p(x - y) \).

Now let \( E \subseteq X \) be bounded. For any \( p \in \mathcal{P}, V_1 (p) \in \text{Nbhd} (0) \) so that \( E \subseteq kV_1 (p) \) for some \( k \) sufficiently large as \( E \) is presumed bounded. Hence \( p(x) < k \) for every \( x \in E \). Hence \( p|_E \) is bounded.
Conversely, if $p|_E$ is bounded for any $p \in \mathcal{P}$, we need to show that $E$ is bounded. Let $U \in \text{Nbhd}(0)$. Thus $p_i|_E \leq M_i$ for all $i = 1, \ldots, m$ where $m$ is the same as the finite intersection (2.2) above. If $n > M_i n_i$ for all $i = 1, \ldots, m$ then $E \subseteq nU$, so that $E$ is indeed bounded.

2.7 Quotient spaces

**Definition 2.30.** For any vector space $X$ and a vector subspace $N \subseteq X$, we define

$$\pi(x) := x + N \quad (x \in X).$$

The space $\{ \pi(x) \}_{x \in X}$ is called $X/N$, the quotient of $X$ modulo $N$, with addition and multiplication defined via

$$\pi(x) + \pi(y) \equiv \pi(x + y), \quad \alpha \pi(x) \equiv \pi(\alpha x).$$

One shows that since $N$ is a vector subspace, $X/N$ is a vector space, and that $\pi: X \to X/N$ is a linear mapping which is surjective and has $N$ has its kernel. Recall also that considered as topological spaces, $\text{Open}(X)$ induces a topology $\text{Open}(X/N)$ (the final topology associated with $\pi: E \in \text{Open}(X/N)$ iff $\pi^{-1}(E) \in \text{Open}(X)$).

**Theorem 2.31.** Let $N$ be a closed subspace of a TVS $X$. Then

1. $X/N$ is also a TVS with respect to the quotient topology, and $\pi: X \to X/N$ is linear, continuous and open.

2. If $\mathcal{B}$ is a local base for $\text{Open}(X)$ then $\pi(\mathcal{B})$ is a local base for $\text{Open}(X/N)$.

3. The following properties of $X$ are inherited by $X/N$: local convexity, local boundedness, metrizability, normability, $F$-space and Banach space.

**Proof.** See Rudin 1.41.

2.7.1 Seminorm quotient

If $p$ is a seminorm on a TVS $X$, set $N := \ker(p)$, which as we’ve seen is a subspace. Define then $\tilde{p}: X/N \to [0, \infty)$ via

$$\tilde{p}(\pi(x)) := p(x) \quad (x \in X).$$

Note that $\tilde{p}$ is well-defined. Indeed, if $\pi(x) = \pi(y)$ then $x + N = y + N$ or $x - y \in N$, i.e., $p(x - y) = 0$. But since $|p(x) - p(y)| \leq p(x - y)$ this implies $p(x) = p(y)$ and hence well-definedness. One may furthermore verify that $\tilde{p}$ is a norm on $X/N$.

2.8 Examples

**Example 2.32.** Let $r \in [1, \infty)$ and define

$$L^r([0,1]) \equiv \{ f: [0,1] \to \mathbb{C} \mid f \text{ is msrbl. and } p_r(f) < \infty \}$$

with

$$p_r(f) \equiv \left( \int_{t \in [0,1]} |f(t)|^r \, dt \right)^{\frac{1}{r}}.$$

We claim that $p_r$ is a seminorm on $L^r([0,1])$ (indeed, $p_r(f) = 0$ if $f = 0$ almost-everywhere, but may not be identically zero). However, $L^r([0,1])/N$ with $N \equiv \ker(p_r)$ yields a Banach space on which $\tilde{p}_r$ is a norm.
Example 2.33. Let \( \Omega \in \text{Open}(\mathbb{R}^n) \setminus \{ \emptyset \} \). Then we know that
\[
\Omega = \bigcup_{n \in \mathbb{N}} K_n
\]
with each \( K_n \neq \emptyset \) compact, which may be chosen such that \( K_n \subseteq \text{interior}(K_{n+1}) \). On \( C(\Omega) \), we define a separating family of seminorms
\[
p_n(f) := \sup \{ |f(x)| \mid x \in K_n \}.
\]
Via Theorem 2.29 this yields a convex topology. Actually this topology is metrizable since the basis is countable, and the metric is given by
\[
d(f, g) = \max_n 2^{-n} p_n(f - g).
\]
One may show that \( d \) is a complete metric. However, \( C(\Omega) \) is not locally bounded, and is hence not normable!

Example 2.34. Let \( \Omega \in \text{Open}(\mathbb{C}) \setminus \{ \emptyset \} \), define \( C(\Omega) \) as above and let \( H(\Omega) \) be the subspace of holomorphic functions. Actually sequences of holomorphic functions that converge uniformly on compact sets have holomorphic limits, so \( H(\Omega) \) is a closed subspace of \( C(\Omega) \).

3 Banach spaces and Banach algebras

Definition 3.1 (Norm). A vector space \( V \) is called a normed vector space iff there is a map
\[
\|\cdot\| : V \to [0, \infty)
\]
which obeys the following axioms:
1. Absolute homogeneity:
\[
\|\alpha \psi\| = |\alpha| \|\psi\| \quad (\alpha \in \mathbb{C}, \psi \in V).
\]
2. Triangle inequality:
\[
\|\psi + \varphi\| \leq \|\psi\| + \|\varphi\| \quad (\psi, \varphi \in V).
\]
3. Injectivity at zero: If \( \|\psi\| = 0 \) for some \( \psi \in V \) then \( \psi = 0 \).

To any norm \( \|\cdot\| \) a metric is associated via
\[
d : V^2 \to [0, \infty),
\]
\[
(\psi, \varphi) \mapsto \|\psi - \varphi\|.
\]
Hence every normed vector space is also a metric space automatically. Recall that a metric space is termed complete if every Cauchy sequence on it converges.

Definition 3.2 (Banach space). If a normed vector space \( (V, \|\cdot\|) \) is complete when regarded as a metric space, then we refer to it as a Banach space.

Definition 3.3 (Dense subsets). If \( (V, \|\cdot\|) \) is a Banach space and \( S \subseteq V \) then we say \( S \) is dense in \( V \) iff for any \( \psi \in V \) and any \( \varepsilon > 0 \) there exists some \( \varphi \in S \) such that
\[
d(\psi, \varphi) < \varepsilon.
\]

Definition 3.4 (Separable spaces). If \( (V, \|\cdot\|) \) is a Banach space which contains a countable, dense subset then \( V \) is called separable.
Definition 3.5 (Banach algebra). A Banach algebra is an associative algebra \( \mathcal{A} \) which is also a Banach space, such that its norm satisfies
\[
\| AB \| \leq \| A \| \| B \| \quad (A, B \in \mathcal{A}) .
\]

If \( \mathcal{A} \) has a unit \( \mathbb{1} \) then it is called unital and commutative iff its multiplication is commutative.

From now on we shall always assume our algebras are unital.

Claim 3.6. In a Banach algebra, multiplication is continuous.

Proof. [TODO ]

Recall that in an algebra \( \mathcal{A} \), an element \( A \in \mathcal{A} \) is called invertible iff \( \exists B \in \mathcal{A} : AB = BA = \mathbb{1} \). The inverse \( B \) is unique should it exist, and is denoted by \( A^{-1} \).

Claim 3.7. On a Banach algebra \( \mathcal{A} \), the set of invertible elements \( \mathcal{G}(\mathcal{A}) \) is open and \( A \mapsto A^{-1} \) is continuous.

Proof. [TODO ]

Definition 3.8 (Entire functional calculus). If \( f : \mathbb{C} \to \mathbb{C} \) is entire then we may define \( f \) via an absolutely convergent power series
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbb{C}) .
\]

Then, for any \( A \in \mathcal{A} \), we define \( f(A) \in \mathcal{A} \) as
\[
f(A)
\]

4 Hilbert spaces

Definition 4.1 (Inner product space). An inner-product space is a vector space \( \mathcal{H} \) along with a sesquilinear map
\[
\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}
\]
such that

1. Conjugate symmetry:
\[
\langle \psi, \varphi \rangle = \overline{\langle \varphi, \psi \rangle} \quad (\varphi, \psi \in \mathcal{H}) .
\]

2. Linearity in second argument:
\[
\langle \psi, \alpha \varphi + \tilde{\varphi} \rangle = \alpha \langle \psi, \varphi \rangle + \langle \psi, \tilde{\varphi} \rangle \quad (\varphi, \tilde{\varphi}, \psi \in \mathcal{H}, \alpha \in \mathbb{C}) .
\]

3. Positive-definite:
\[
\langle \psi, \psi \rangle > 0 \quad (\psi \in \mathcal{H} \setminus \{0\}) .
\]

To every inner product one immediately may associate a norm, via
\[
\| \psi \| := \sqrt{\langle \psi, \psi \rangle} \quad (\psi \in \mathcal{H}) .
\]

The converse, however, hinges on the norm obeying the parallelogram law

Claim 4.2. If a norm satisfies the parallelogram law:
\[
\| \psi + \varphi \|^{2} + \| \psi - \varphi \|^{2} \leq 2 \| \psi \|^{2} + 2 \| \varphi \|^{2} \quad (\varphi, \psi \in \mathcal{H})
\]

then
\[
\langle \psi, \varphi \rangle := \frac{1}{4} \left[ \| \psi + \varphi \|^{2} - \| \psi - \varphi \|^{2} + i \| i \psi - \varphi \|^{2} - i \| i \psi + \varphi \|^{2} \right]
\]
defines an inner product whose associated norm is \( \| \cdot \| \equiv \sqrt{\langle \cdot , \cdot \rangle} \).
**Definition 4.3** (Hilbert space). A Hilbert space $\mathcal{H}$ is an inner-product space with $\langle \cdot, \cdot \rangle$ such that the induced norm $\| \cdot \|$ from this inner product makes $\mathcal{H}$ into a Banach space (i.e., a complete metric space w.r.t. to the metric induced by $\| \cdot \|$).

We denote the space of *linear* maps $\mathcal{H} \to \tilde{\mathcal{H}}$ between two Hilbert spaces by $\mathcal{L} \left( \mathcal{H} \to \tilde{\mathcal{H}} \right)$ and $\mathcal{L} (\mathcal{H}) \equiv \mathcal{L} (\mathcal{H} \to \mathcal{H})$. For finite dimensional Hilbert spaces, $\mathcal{L} (\mathcal{H})$ is simply the space of matrices on that vector space. For infinite-dimensional Hilbert spaces, we call the elements of $\mathcal{L} (\mathcal{H})$ *operators*.

**Definition 4.4** (Operator norm). Given any $A \in \mathcal{L} \left( \mathcal{H} \to \tilde{\mathcal{H}} \right)$ the operator norm is defined as

$$\| A \|_{\text{op}} \equiv \| A \| := \sup \left\{ \| A \psi \|_{\tilde{\mathcal{H}}} \mid \psi \in \mathcal{H} : \| \psi \| = 1 \right\}.$$ 

**Claim 4.5.** For each linear $A \in \mathcal{L} \left( \mathcal{H} \to \tilde{\mathcal{H}} \right)$, if $\| A \| < \infty$ then $A : \mathcal{H} \to \tilde{\mathcal{H}}$ is continuous.

**Proof.** [TODO]

With the identification of $\mathcal{L} (\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*$ it is clear that $\mathcal{L} (\mathcal{H})$ is a vector space.

We define the space of all bounded linear functionals

$$\mathcal{B} (\mathcal{H}) := \{ A \in \mathcal{L} (\mathcal{H}) \mid \| A \| < \infty \}.$$ 

**Claim 4.6.** The operator norm makes $\mathcal{B} (\mathcal{H})$ into a Banach space which is also a Banach algebra.

**Proof.** [TODO]

5 Bounded operators on Hilbert space
6 Linear functionals
7 Spectral theory and functional calculus
8 Fredholm theory
9 Unbounded operators
10 Maybe: C-star algebra techniques
A Useful identities and inequalities

B Glossary of mathematical symbols and acronyms

Sometimes it is helpful to include mathematical symbols which can function as valid grammatical parts of sentences. Here is a glossary of some which might appear in the text:

- The bracket $\langle \cdot, \cdot \rangle_V$ means an inner product on the inner product space $V$. For example,
  \[ \langle u, v \rangle_{\mathbb{R}^2} \equiv u_1 v_1 + u_2 v_2 \quad (u, v \in \mathbb{R}^2) \]
  and
  \[ \langle u, v \rangle_{\mathbb{C}^2} \equiv \overline{u_1} v_1 + \overline{u_2} v_2 \quad (u, v \in \mathbb{C}^2). \]

- Sometimes we denote an integral by writing the integrand without its argument. So if $f : \mathbb{R} \to \mathbb{R}$ is a real function, we sometimes in shorthand write
  \[ \int_a^b f \]
  when we really mean
  \[ \int_{t=a}^b f(t) \, dt. \]
  This type of shorthand notation will actually also apply for contour integrals, in the following sense: if $\gamma : [a, b] \to \mathbb{C}$ is a contour with image set $\Gamma := \text{im} (\gamma)$ and $f : \mathbb{C} \to \mathbb{C}$ is given, then the contour integral of $f$ along $\gamma$ will be denoted equivalently as
  \[ \int_{\gamma} f \equiv \int_{\Gamma} f(z) \, dz \equiv \int_{t=a}^b f(\gamma(t)) \gamma'(t) \, dt \]
  depending on what needs to be emphasized in the context. Sometimes when the contour is clear one simply writes
  \[ \int_{z_0}^{z_1} f(z) \, dz \]
  for an integral along any contour from $z_0$ to $z_1$.

- iff means “if and only if”, which is also denoted by the symbol $\iff$.

- WLOG means “without loss of generality”.

- CCW means “counter-clockwise” and CW means “clockwise”.

- $\exists$ means “there exists” and $\nexists$ means “there does not exist”. $\exists!$ means “there exists a unique”.

- $\forall$ means “for all” or “for any”.

- : (i.e., a colon) may mean “such that”.

- ! means negation, or “not”.

- $\land$ means “and” and $\lor$ means “or”.

- $\implies$ means “and so” or “therefore” or “it follows”.

- $\in$ denotes set inclusion, i.e., $a \in A$ means $a$ is an element of $A$ or $a$ lies in $A$.

- $\ni$ denotes set inclusion when the set appears first, i.e., $A \ni a$ means $A$ includes $a$ or $A$ contains $a$.

- Speaking of set inclusion, $A \subseteq B$ means $A$ is contained within $B$ and $A \supseteq B$ means $B$ is contained within $A$.

- $\emptyset$ is the empty set $\{ \}$.  

- While $=$ means equality, sometimes it is useful to denote types of equality:
  - $a := b$ means “this equation is now the instant when $a$ is defined to equal $b$”.
  - $a \equiv b$ means “at some point above $a$ has been defined to equal $b$”.
  - $a = b$ will then simply mean that the result of some calculation or definition stipulates that $a = b$.
  - Concrete example: if we write $i^2 = -1$ we don’t specify anything about why this equality is true but writing $i^2 \equiv -1$ means this is a matter of definition, not calculation, whereas $i^2 := -1$ is the first time you’ll see this definition. So this distinction is meant to help the reader who wonders why an equality holds.
B.1 Important sets

1. The unit circle

$$S^1 \equiv \{ z \in \mathbb{C} \mid |z| = 1 \}.$$ 

2. The (open) upper half plane

$$\mathbb{H} \equiv \{ z \in \mathbb{C} \mid \text{Im} \{ z \} > 0 \}.$$ 

C Vocabulary from topology

**Definition C.1.** Given a set $S$, a topology on $S$ is a set of subsets $\mathcal{T}$ of $S$ (i.e., it is a subset of the power set $\mathcal{P}(S)$) with the properties that:

1. $S, \emptyset \in \mathcal{T}.$
2. $A \cap B \in \mathcal{T}$ for any $A, B \in \mathcal{T}.$
3. $(\bigcup_{\alpha \in \mathcal{G}} A_\alpha) \in \mathcal{T}$ for any $\{ A_\alpha \}_{\alpha \in \mathcal{G}} \subseteq \mathcal{T}.$ Here $\mathcal{G}$ is any index set, which need not be countable.

If we have a space $S$ which we know is a topological space and we want to refer to its topology, we denote this by $\text{Open}(S)$.

**Definition C.2.** A neighborhood of a point $x \in S$ is any open set $U \in \text{Open}(S)$ that contains $x$: $x \in U \subseteq \text{Open}(S)$.

We denote the set of neighborhoods of a point $x$ as $\text{Nbhd}(x) \subseteq \text{Open}(S)$.

**Definition C.3.** A topological space $S$ is called Hausdorff iff for any $x, y \in S$ such that $x \neq y$, there are $U \in \text{Nbhd}(x), V \in \text{Nbhd}(y)$ such that $U \cap V = \emptyset$.

**Definition C.4.** A subset $B \subseteq \text{Open}(S)$ is called a base or basis for $\text{Open}(S)$ iff any $U \in \text{Open}(S)$ may be written as

$$U = \bigcup_{\alpha \in \mathcal{G}} B_\alpha$$

for some $\{ B_\alpha \}_{\alpha \in \mathcal{G}} \subseteq B$. Here $\mathcal{G}$ is some (not necessarily countable) index set.

**Definition C.5.** A set $T \subseteq S$ is compact iff every open cover of $T$ has a finite sub-cover.

**Definition C.6** (Metric). Given a set $S$, a metric on $S$ is a map $d : S^2 \rightarrow [0, \infty)$ such that

1. $d(x, y) = 0$ implies that $x = y$ for all $x, y \in S$.
2. $d(x, y) = d(y, x)$ for all $x, y \in S$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in S$.

**Definition C.7** (Norm). A vector space $V$ is called a normed vector space iff there is a map $\| \cdot \| : V \rightarrow [0, \infty)$ which obeys the following axioms:

1. Absolute homogeneity:

$$\|\alpha \psi\| = |\alpha| \|\psi\| \quad (\alpha \in \mathbb{C}, \psi \in V).$$

2. Triangle inequality:

$$\|\psi + \varphi\| \leq \|\psi\| + \|\varphi\| \quad (\psi, \varphi \in V).$$

3. Injectivity at zero: If $\|\psi\| = 0$ for some $\psi \in V$ then $\psi = 0$. 

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To any norm $\| \cdot \|$ a metric is associated via

$$d : V^2 \rightarrow [0, \infty)$$

$$(\psi, \varphi) \mapsto \| \psi - \varphi \| .$$

References


