1. Let $P, Q$ be two orthogonal projections onto subspaces $M, N$ in a Hilbert space $H$ such that $[P, Q] = 0$.

   (a) Show $P^\perp \equiv 1 - P$, $Q^\perp$, $PQ$, $P + Q - PQ$ and $P + Q - 2PQ$ are orthogonal projections.

   (b) What is the relation between the projections in the previous item and $M, N$?

2. Let $P, Q$ be two orthogonal projections onto subspaces $M, N$ in a Hilbert space $H$. Show that

   \[ \lim_{n \to \infty} (PQ)^n \]

   exists and is the orthogonal projection onto $M \cap N$.

3. Let $A \in \mathcal{B}(H)$. Show that the set of $\lambda \in \sigma(A)$ such that $\lambda$ is not an eigenvalue of $A$ and $\text{im}(A - \lambda \mathbb{1})$ is closed but not the whole of $H$ is an open subset of $\mathbb{C}$.

4. Define the numerical range $N(A)$ of $A \in \mathcal{B}(H)$ via

   \[ N(A) := \{ \langle \psi, A\psi \rangle \mid \psi \in H \land \|\psi\| = 1 \} \]

   (a) Show that \( \sigma(A) \subseteq N(A) \).

   (b) Find an example where $N(A)$ is not closed and \( \sigma(A) \not\subseteq N(A) \).

   (c) Find an example where \( \sigma(A) \neq N(A) = \overline{N(A)} \).

5. Show that if $A \in \mathcal{B}(H)$ has $A = A^*$ then

   \[ \left\| (A - z\mathbb{1})^{-1} \right\| \leq \frac{1}{|\text{Im} \{ z \}|} \quad (z \in \mathbb{C} : |\text{Im} \{ z \}| > 0) \]

6. Show that if $A \in \mathcal{B}(H)$ is an isometry then $\text{im}(A)$ is closed in $H$.

7. Let $V \in \mathcal{B}(L^2([0, 1] \to \mathbb{C}))$ be given by

   \[ V(\psi) := \int_0^1 \psi \quad (\psi \in L^2) \]

   (a) Show that $V$ is well-defined (it is a bounded linear map) with

   \[ V^*(\psi) = \int_0^1 \psi \quad (\psi \in L^2) \]

   (b) Show that the spectral radius of $V$, $r(V)$, equals zero and that $\sigma(V) = \{ 0 \}$. 

(c) Show that $\|V\| = \frac{2}{\pi}$.

8. Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$ be the Fourier series given by

$$\ell^2(\mathbb{Z}) \ni \psi \mapsto \left( [0, 2\pi] \ni k \mapsto \sum_{n \in \mathbb{Z}} e^{-i kn} \psi_n =: \hat{\psi}(k) \right).$$

Let $A \in B(\ell^2(\mathbb{Z}))$ be the discrete Laplacian:

$$A = R + R^*$$

where $R$ is the bilateral right shift operator

$$R\delta_n := \delta_{n+1} \quad (n \in \mathbb{Z})$$

and $\{\delta_n\}_{n \in \mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$. Calculate

$$\mathcal{F} A \mathcal{F}^* \in B(L^2(\mathbb{S}^1)).$$

9. [extra] Call an operator $A \in B(\ell^2(\mathbb{Z}^d))$ local iff

$$\inf_{x,y \in \mathbb{Z}^d} - \frac{1}{\|x-y\|} \log (|\langle \delta_x, A \delta_y \rangle|) > 0.$$ 

Show that if $A = A^*$ and $A$ is local then $(A - z \mathbb{1})^{-1}$ is local too for any $z \in \mathbb{C} : |\text{Im} \{z\}| > 0$. 
