1 Banach algebras and the spectra of elements in it

In the following, \( \mathcal{A} \) is a \( \mathbb{C} \)-Banach algebra.

1. Prove Fekete’s lemma: If \( \{ a_n \}_{n \in \mathbb{N}} \subseteq \mathbb{R} \) is sub-additive then \( \lim_{n \to \infty} \frac{1}{n} a_n \) exists and equals \( \inf \frac{1}{n} a_n \).

2. Let \( R : \mathbb{C} \to \mathbb{C} \) be a rational function, i.e.,
\[
R(z) = p(z) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l} (z - z_k)^{-l}
\]
where \( p \) is a polynomial, \( n \in \mathbb{N} \), and \( \{ z_k \}_{k=1}^{n}, \{ c_{k,l} \}_{k,l} \subseteq \mathbb{C} \). Let now \( a \in \mathcal{A} \) such that \( \{ z_k \}_{k=1}^{n} \subseteq \rho(a) \).
Assume further that we choose some \( \sigma(a) \subseteq \Omega \in \text{Open (} \mathbb{C} \text{)} \) such that \( R \) is holomorphic on \( \Omega \), and \( \gamma_j : [a, b] \to \Omega, j = 1, \ldots, m \) a collection of \( m \) oriented loops which surround \( \sigma(a) \) within \( \Omega \), such that
\[
\frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} \frac{1}{z - \lambda} \, dz = \begin{cases} 1 & \lambda \in \sigma(a) \\ 0 & \lambda \notin \Omega \end{cases}.
\]
Using Lemma 6.26 in the lecture notes (= Lemma 10.24 in Rudin) show that \( R(a) \) obeys the Cauchy integral formula, in the sense that
\[
p(a) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l} (a - z_k)^{-l} = \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} R(z) (z1 - a)^{-1} \, dz.
\]

3. Let \( \mathcal{A} \) be such that there exists some \( a \in \mathcal{A} \) with \( \sigma(a) \) not connected. Show that then \( \mathcal{A} \) contains some non-trivial idempotent (an element \( b \in \mathcal{A} \) with \( b^2 = b \notin \{ 0, 1 \} \)).

4. Assume that \( \{ a_n \}_{n \in \mathbb{N}} \subseteq \mathcal{A} \) is a sequence such that \( \exists \lim_{n} a_n =: a \in \mathcal{A} \). Let \( \Omega \subseteq \text{Open (} \mathbb{C} \text{)} \) contains a component of \( \sigma(a) \). Show that \( \sigma(a_n) \cap \Omega \neq \emptyset \) for all sufficiently large \( n \). \textit{Hint:} If \( \sigma(a) \subseteq \Omega \cup \bar{\Omega} \) where \( \bar{\Omega} \subseteq \text{Open (} \mathbb{C} \text{)} \) (in particular this means \( \Omega \cap \bar{\Omega} = \emptyset \)), define \( f : \mathbb{C} \to [0, 1] \) such that \( f|_{\Omega} = 1 \) and \( f|_{\bar{\Omega}} = 1 \).

5. Let \( X, Y \) be two Banach spaces and \( A, B \) be two bounded linear operators on \( X, Y \) respectively. Let \( T \in \mathcal{B}(X \to Y) \). Show that the following two assertions are equivalent:

(a) \( TA = BT \).
(b) \( Tf(A) = f(B)T \) for any \( f : \mathbb{C} \to \mathbb{C} \) holomorphic in some open set \( U \) which contains \( \sigma(A) \cup \sigma(B) \).