MAT 520 HW5

3. Since $\sigma(a)$ is disconnected, we can write $\sigma(a) \subset U \cup V$ where U, V are open subsets of $\mathbb{C}$, each of which encloses some components of $\sigma(a)$. Consider the holomorphic function f that is 1 on U and 0 on V. Define $b = f(a)$. Then $b^2 = (f(a))^2 = f(a) = b$ since f squares to itself as a function. In particular b is nontrivial since $\sigma(f(a)) = f(\sigma(a)) = \{0, 1\}$.

4. If $\sigma(a)$ is connected, then the result follows from Theorem 10.20 in Rudin. Otherwise suppose $\sigma(a) \subset \Omega_0 \cup \Omega$ where Ω_0 and Ω are disjoint open subsets and Ω enclosed a component of $\sigma(a)$. For all n large enough, we have $\sigma(a_n) \subset \Omega_0 \cup \Omega$ by Theorem 10.20 in Rudin. Consider the holomorphic function f that is 1 on Ω and 0 on Ω_0. Now $f(a) \neq 0$. Thus

$$\|f(a_n)\| \geq \|f(a)\| - \|f(a_n) - f(a)\| > 0$$

for all n large, since $\|f(a_n) - f(a)\| \to 0$ (to show this, we can use the estimate in Lemma 6.14 and the integral formula (6.7) of Theorem 6.28 in the lecture note, and also the resolvent identity.) Thus $\sigma(a_n) \cap \Omega \neq \varnothing$ for all n large; otherwise $f(a_n) = 0$.

5. That (b) implies (a) is clear. If (a) is true, than $TR(A) = R(B)T$ holds for rational functions R without poles in U. We can approximate a holomorphic function f on U by rational functions $\{R_n\}$ without poles in U, uniformly on compact subsets of U. Thus $R_n(A) \to f(A)$ and $R_n(B) \to f(B)$ in norm (see Theorem 6.28 in the lecture note), and hence $Tf(A) = f(B)T$.