Functional Analysis
Princeton University MAT520
HW4, Due Oct 6th 2023 (auto extension until Oct 8th 2023)

October 1, 2023

1 Weak stuff

Solve at least two of the following six problems.

1. Prove that any norm-closed convex bounded subset of a reflexive Banach space is weakly compact.

2. Define the Banach space $\ell^\infty (\mathbb{N}) := \{ a : \mathbb{N} \to \mathbb{C} \mid \|a\|_\infty < \infty \}$ with the norm $\|a\|_\infty \equiv \sup_{n} |a_n|$. Also define $\ell^1 (\mathbb{N}) := \{ a : \mathbb{N} \to \mathbb{C} \mid \|a\|_1 < \infty \}$ with the norm $\|a\|_1 := \sum_{n \in \mathbb{N}} |a_n|$. Our goal here is to use the Banach-Alaoglu theorem to exhibit an element of $(\ell^\infty)^*$ which is not in $\ell^1$.

(a) Define $\{ \mu_n \}_{n} \subseteq (\ell^\infty)^*$ via 
$$
\mu_n (a) := \frac{1}{n} \sum_{j=1}^{n} a_j \quad (a \in \ell^\infty, n \in \mathbb{N}).
$$
Show that $\mu_n \in (\ell^\infty)^*$ indeed and $\|\mu_n\| \leq 1$.

(b) Show there exists some $\mu \in (\ell^\infty)^*$ that is the limit of $\{ \mu_n \}_n$.

(c) Show that $(\ell^1)^* = \ell^\infty$.

(d) Hence we may think of $J (\ell^1) \subseteq (\ell^\infty)^*$ where $J$ is the natural isometric injection. Show that the limit $\mu$ constructed above does not lie in $J (\ell^1)$. That is, show that for any $x \in \ell^1$, $J(x) \neq \mu$.

3. Let $\{ f_n \}_n$ be given by 
$$
f_n (t) := e^{int} \quad (t \in [-\pi, \pi]).
$$
Show that if $p \in [1, \infty)$ then $f_n \to 0$ weakly in $L^p ([\pi, \pi])$, but not in the norm topology of $L^p ([\pi, \pi])$.

4. Consider $L^\infty ([0,1])$ with its norm topology (the essential supremum norm), and, since $(L^1 ([0,1]))^* = L^\infty ([0,1])$, the weak-star topology on $(L^1 ([0,1]))^*$, which is a topology on $L^\infty ([0,1])$. Show that $C ([0,1])$ (the space of all continuous functions) is dense in $L^\alpha$ but not in $L^\beta$, for either $(\alpha, \beta) = (1, \infty)$ or $(\alpha, \beta) = (\infty, 1)$.

5. Let $X$ be an infinite-dimensional Banach space and define 
$$
S := \{ x \in X \mid \|x\| = 1 \}.
$$
Show that the weak-closure of $S$ is 
$$
B := \{ x \in X \mid \|x\| \leq 1 \}.
$$

6. Let $X$ be a Banach space, and $\{ L_n \}_n \subseteq X^*$ be a sequence which converges to some $L \in X^*$ in the weak-star sense. Assume that $\{ x_n \}_n \subseteq X$ converges to some $x \in X$ in norm. It is true that $L_n (x_n) \to L (x)$ in $\mathbb{C}$?
2 Banach algebras

Solve at least ten of the following dozen problems.

Here \( \mathcal{A} \) is a Banach algebra and \( x, y, \ldots \) are elements in it; \( G_\mathcal{A} \) is the set of invertible elements and \( r : \mathcal{A} \to [0, \infty) \) is the spectral radius.

7. Use \( (xy)^n = x(yx)^{n-1} y \) to show that \( r(xy) = r(yx) \).

8. Show that if \( x, xy \in G_\mathcal{A} \) then \( y \in G_\mathcal{A} \).

9. Show that if \( xy, yx \in G_\mathcal{A} \) then \( x, y \in G_\mathcal{A} \).

10. On the Banach space \( \ell^2 (\mathbb{N} \to \mathbb{C}) \), define the right shift operator \( R \in B (\ell^2 (\mathbb{N} \to \mathbb{C})) \):

\[
(Ra)_n := \begin{cases} a_{n-1} & n \geq 2 \\ 0 & n = 1 \end{cases}
\]

and the right shift operator

\[
(La)_n := a_{n+1} \quad (n \in \mathbb{N}) .
\]

Calculate \( RL \) and \( LR \). Conclude that one may have \( xy = 1 \) but \( yx \neq 1 \) in a Banach algebra.

11. Show that \( xy - 1 \) is invertible iff \( yx - 1 \) is invertible.

12. Show that if \( z \in \mathbb{C} \setminus \{ 0 \} \) then \( z \in \sigma(xy) \) iff \( z \in \sigma(yx) \). I.e.,

\[
\sigma(xy) \cup \{ 0 \} = \sigma(yx) \cup \{ 0 \} .
\]

Find an example where \( \sigma(xy) \neq \sigma(yx) \).

13. Define \( \mathcal{A} := C^2 ([0,1] \to \mathbb{C}) \), the space of functions with continuous second derivative. Define, for \( a, b > 0 \),

\[
\| f \| := \| f \|_\infty + a\| f' \|_\infty + b\| f'' \|_\infty .
\]

Show that \( \mathcal{A} \) is a Banach space. Show that \( \mathcal{A} \) is a Banach algebra (with pointwise multiplication) iff \( a^2 \geq 2b \). You may consider the functions \( x \mapsto x \) and \( x \mapsto x^2 \).

14. Show that if \( z \in \partial \sigma(x) \) then \( x - zI \in \partial G_\mathcal{A} \).

15. Let \( x \in \partial G_\mathcal{A} \). Show there exists some \( \{ y_n \} \subseteq \mathcal{A} \) with \( \| y_n \| = 1 \) and

\[
\lim_{n \to \infty} xy_n = \lim_{n \to \infty} y_n x = 0 .
\]

Try to characterize the type of Banach algebras in which there are such elements \( x \) (which are called topological divisor of zero).

16. On \( \ell^2 (\mathbb{N} \to \mathbb{C}) \) define \( T \in B (\ell^2 (\mathbb{N} \to \mathbb{C})) \) via

\[
T(a_1, a_2, a_3, a_4, \ldots) := (-a_2, a_1, -a_4, a_3, \ldots) .
\]

Calculate \( \sigma(T) \).

17. Show that if \( x \in \mathcal{A} \) is nilpotent (i.e. \( \exists n \in \mathbb{N} \) with \( x^n = 0 \)) then \( \sigma(x) = \{ 0 \} \).

18. Show that \( r \) is upper semicontinuous.