1. Let \( X \) be the position operator on \( L^2(\mathbb{R}) \). Show that
\[
D(X) := \left\{ \psi \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} x^2 |\psi(x)|^2 \, dx < \infty \right\}
\]
is the largest vector space \( V \) such that for each \( \psi \in V \), \( X\psi \in L^2 \).

2. Let
\[ A := \{ \psi : [0, 1] \to \mathbb{C} \mid \psi \text{ is ac and } \psi' \in L^2([0, 1]) \} . \]
Let \( A_1, A_2 \) both be defined as
\[ \psi \mapsto -i\psi' \]
on the respective domains
\[
D(A_1) := A, \\
D(A_2) := \{ \psi \in A \mid \psi(0) = 0 \} .
\]
Show that both domains are dense in \( L^2([0, 1]) \) and that \( A_1, A_2 \) are closed. Finally show that
\[
\sigma(A_1) = \mathbb{C}, \quad \sigma(A_2) = \emptyset .
\]

3. Show that if \( A \) is a symmetric operator on a Hilbert space \( \mathcal{H} \) then the following are equivalent:
   (a) \( A \) is essentially self-adjoint.
   (b) \( \ker(A^* \pm i1) = \{ 0 \} \).
   (c) \( \text{im}(A \pm i1) = \mathcal{H} \).

4. Let \( A := -i\partial \) on
\[
D(A) := \{ \psi \in A \mid \psi(0) = \psi(1) = 0 \}
\]
with \( A \) as above.
   (a) Show that \( A \) is symmetric as an operator \( A : D(A) \to L^2([0, 1]) \).
   (b) Calculate \( A^* \) (along with \( D(A^*) \)) and conclude \( A \) is closed, symmetric but \textit{not} self-adjoint.
   (c) For any \( \alpha \in \mathbb{C}, \, |\alpha| = 1 \), define \( A_\alpha := -i\partial \) on the domain
\[
D(A_\alpha) := \{ \psi \in A \mid \psi(0) = \alpha \psi(1) \} .
\]
Show that \( A_\alpha \) is self-adjoint, and that it is an extension of \( A \), and is extended by \( A^* \):
\[ A \subseteq A_\alpha \subseteq A^* . \]
Conclude that \( A \) has uncountably many self-adjoint extensions.
5. Show that $A$ is closable iff $\Gamma(A) = \Gamma(B)$ for some operator $B$. Show that this operator $B$ is the closure $\overline{A}$ of $A$.

6. Let $\{ \varphi_n \}_{n}$ be an ONB for $\mathcal{H}$ and $\psi \in \mathcal{H}$ any vector which is not a finite linear combination of $\{ \varphi_n \}_{n}$. Let $\mathcal{D}$ be the set of vectors which are finite linear combinations of $\{ \varphi_n \}_{n}$ and of $\psi$. Define $A : \mathcal{D} \rightarrow \mathcal{H}$ via

$$A \left( b\psi + \sum_{i=1}^{N} a_i \varphi_i \right) := b\psi.$$ 

Calculate $\Gamma(A)$ and show that $\Gamma(A)$ is not the graph of a linear operator.

7. [R&S VIII, 2] Let $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ be injective.

(a) Show that if $A$ is closed and has a closed range then $\exists C \in (0, \infty)$ such that

$$\|A\psi\| \geq C\|\psi\| \quad (\psi \in \mathcal{D}(A)). \tag{0.1}$$

(b) Show that if $A$ has dense closed range and obeys (0.1) then $A$ is closed.

(c) Show that if $A$ is closed and obeys (0.1) then it has a closed range.

8. Calculate the adjoint of $-\partial^2 : C^\infty_0(\mathbb{R}) \rightarrow L^2(\mathbb{R})$. Determine if $-\partial^2$ is essentially self-adjoint. Here $C^\infty_0(\mathbb{R})$ is the set of functions $f : \mathbb{R} \rightarrow \mathbb{C}$ smooth of compact support.

9. Let $-i\partial : C^\infty((0, \infty)) \rightarrow L^2([0, \infty))$ where the domain is the set of smooth functions with compact support away from the origin. Is it essentially self-adjoint?