November 21, 2023

1. Provide an example for a non-normal operator \( A \in B(\mathcal{H}) \) and a point in the resolvent set \( z \in \rho(A) \) where
\[
\left\| (A - z1)^{-1} \right\| \leq \frac{1}{\text{dist}(z, \sigma(A))}
\]
does not hold.

2. Let \( A = U |A| \) be the polar decomposition of an operator \( A \in B(\mathcal{H}) \). Let
\[
f_n(x) := \begin{cases} \frac{1}{n} & x \geq \frac{1}{n} \\ x & x < \frac{1}{n} \end{cases} \quad (x \geq 0).
\]
Prove that
\[
U = \mathcal{s-lim}_{n \to \infty} Af_n(|A|).
\]
[extra] Do the same with
\[
f_n(x) := \frac{1}{x + \frac{1}{n}} \quad (x \geq 0).
\]

3. Prove that if \( A \in B(\mathcal{H}) \) is normal then
\[
\|A\| = r(A)
\]
where \( r(A) \) is the spectral radius of an operator.

4. Let \( A \in B(\mathcal{H}) \) be normal. Show there exists some finite measure space \((M, \mu)\) and a unitary
\[
U : \mathcal{H} \to L^2(M, \mu)
\]
such that there exists a bounded Borel function \( f : M \to \mathbb{C} \) such that
\[
(UAU^*\psi)(m) = f(m)\psi(m) \quad (m \in M; \psi \in L^2(M, \mu)).
\]

5. Show that if \( A, B \in B(\mathcal{H}) \) are two self-adjoint operators such that \([A, B] = 0\) then there exists a finite measure space \((M, \mu)\) and a unitary
\[
U : \mathcal{H} \to L^2(M, \mu)
\]
such that there are two bounded Borel functions \( f, g : M \to \mathbb{R} \) which obey
\[
(UAU^*\psi)(m) = f(m)\psi(m)
\]
\[
(UBU^*\psi)(m) = g(m)\psi(m)
\]
for all \( m \in M \) and \( \psi \in L^2(M, \mu) \).
6. Prove that for $A \in \mathcal{B} (\mathcal{H})$ self-adjoint and $\chi_\lambda (A)$ the projection-valued measure of $A$, we have

$$\lambda \in \sigma (A) \iff [\chi_{(\lambda - \varepsilon, \lambda + \varepsilon)} (A) \neq 0 \quad (\varepsilon > 0)] \quad (\lambda \in \mathbb{R}).$$

7. Let $\mathcal{H}$ be a separable Hilbert space. Prove that the only operator-norm-closed star-ideals in $\mathcal{B} (\mathcal{H})$ are \{0\}, $\mathcal{K} (\mathcal{H})$ (the compact operators) and $\mathcal{B} (\mathcal{H})$ itself.

8. Let $\mathcal{H} = \ell^2 (\mathbb{Z})$ and on it define the discrete Laplacian

$$-\Delta = 2\mathbb{1} - R - R^*$$

where $R$ is the bilateral right shift operator.

(a) Recall the definition of a cyclic vector from the lecture notes. Show that for any $x \in \mathbb{Z}$, $\delta_x$ is a cyclic vector for $-\Delta$.

(b) Define $f : \mathbb{C}^+ \to \mathbb{C}$ via

$$f (z) := \left< \delta_0, (\Delta - z \mathbb{1})^{-1} \delta_0 \right>.$$  

Find an explicit expression for $f$ using the Fourier series which was presented in the previous homework.

(c) Calculate the limit

$$\lim_{\varepsilon \to 0^+} \text{Im} \{ f (E + i\varepsilon) \}$$

for the two cases $E \in (0, 4)$ and $E \in \mathbb{R} \setminus (0, 4)$.

(d) Calculate the spectral measure of $-\Delta$ and determine its type (w.r.t. the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

9. Let $\mathcal{H} = \ell^2 (\mathbb{Z})$ and on it define the multiplication operator

$$V (X)$$

via

$$(V (X) \psi) (x) := V (x) \psi (x) \quad (x \in \mathbb{Z}, \psi \in \ell^2 (\mathbb{Z}))$$

where $V : \mathbb{Z} \to \mathbb{R}$ is some bounded sequence.

(a) For $x \in \mathbb{Z}$, is $\delta_x$ a cyclic vector for $V (X)$?

(b) For any $x \in \mathbb{Z}$, define $f_x : \mathbb{C}^+ \to \mathbb{C}$ via

$$f_x (z) := \left< \delta_x, (V (X) - z \mathbb{1})^{-1} \delta_x \right>.$$  

(c) Calculate both

$$\lim_{\varepsilon \to 0^+} \text{Im} \{ f_x (E + i\varepsilon) \}$$

and

$$\lim_{\varepsilon \to 0^+} \varepsilon \text{Im} \{ f_x (E + i\varepsilon) \}$$

for all $E \in \mathbb{R}$ (separate into cases).

(d) Calculate the spectral measure of $V (X)$ and determine its type (w.r.t. the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

10. On $\ell^2 (\mathbb{N})$, let $\hat{R}$ be the unilateral right shift operator. Calculate $\ker \hat{R}$, $\ker \hat{R}^*$ and $\text{im} \hat{R}$ and show that $\hat{R}$ is a Fredholm operator. Calculate its Fredholm index.

11. Show that on $\ell^2 (\mathbb{N})$, $\frac{1}{X}$ where $X$ is the position operator, is not a Fredholm operator by calculating $\text{im} \frac{1}{X}$ and showing that it is not closed.