Claim: In $\mathbb{C}^n$, vector addition and scalar multiplication are continuous.

Proof: We prove $(\mathbb{C}^n)^2 \rightarrow \mathbb{C}^n$ is continuous.

$$(u, v) \mapsto u + v$$

It suffices to take some $B_r(z) \subseteq \text{Open}(\mathbb{C}^n)$ and show $+^{-1}(B_r(z)) \subseteq \text{Open}(\mathbb{C}^n)^2$. Since open sets in the product topology are unions of products of open balls,

$$+^{-1}(B_r(z)) = \bigcap \{ (u, v) \in (\mathbb{C}^n)^2 \mid \|u + v - z\| < r \}$$

Let $(u, v) \in +^{-1}(B_r(z))$, i.e., $u + v \in B_r(z)$.

Since $B_r(z) \subseteq \text{Open}(\mathbb{C}^n)$, there exists $\varepsilon > 0 : B_{\varepsilon}(u + v) \subseteq B_r(z)$.

Claim: $B_{\varepsilon/3}(u) \times B_{\varepsilon/3}(v) \subseteq +^{-1}(B_{\varepsilon}(u + v))$

Proof: If $(\hat{u}, \hat{v}) \in B_{\varepsilon/3}(u) \times B_{\varepsilon/3}(v)$, then

$$\|\hat{u} + \hat{v} - u - v\| \leq \varepsilon/3 < \varepsilon$$
\[ \Rightarrow B_{\epsilon/3}(u) \times B_{\epsilon/3}(w) \subseteq \text{NEbd}(u \cup w) \]

and \[ B_{\epsilon/3}(u) \times B_{\epsilon/3}(w) \subseteq +r(B_{\epsilon}(u \cup w)) \subseteq +r(B_{r}(\tilde{z})). \]

\[ \Rightarrow +r(B_{r}(\tilde{z})) \in \text{Open}(C^n) \times C^n) \text{ and hence } +r \text{ is cont. } \]

Next, W.T.S. \( \bullet : C \times C^n \rightarrow C^n \) is cont.

\( (x, u) \mapsto xu \)

Again W.T.S. \( +r(B_{r}(\tilde{z})) \in \text{Open}(C \times C^n) \).

Let \( (x, u) \in +r(B_{r}(\tilde{z})) \iff \|xu - z\| < r, \)

\[ \iff xu \in B_{r}(\tilde{z}) \]

So \( \exists \epsilon > 0 \) : \( B_{\epsilon}(xu) \subseteq B_{r}(\tilde{z}) \).

Want \( \|xu - xu\| < \epsilon \).

\[ \|xu - xu\| = \|xu - xu + xu - xu\| \]

\[ \leq \|x\| \|u - u\| + \|x - x\| \|u\| \]

\[ \leq (\|x - x\| + \|x\|) \|u - u\| + \|x - x\| \|u\| \]

\[ \leq (1 + \|x\|) \|u - u\| + \|x - x\| \|u\| \|u\| \leq \frac{\|x\|}{2} < \epsilon. \]

So pick \( U := B_{\min\{1, \frac{\|x\|}{2}\}}(x) \times B_{\min\{1, \frac{\|x\|}{2}\}}(u). \)
Claim: \( C \) w/ the French metro metric is NOT homeomorphic to \( C \) w/ Euclidean metric.

**Pf.:** Example 2 in lecture notes.

Since we know \( \exists \) only one TVS (up to homeomorphisms) in \( \dim n < \infty \), the French metro metric's top. cannot make a TVS out of \( C \).

Claim: \( \overline{A + B} \subseteq \overline{A + B} \)

**Proof:** Let \( a \in \overline{A}, b \in \overline{B} \).

W.T.T.S. \( a + b \in \overline{A + B} \).

Let \( \mathcal{W} \in \text{Nhd}(a + b) \). So W.T.T.S. \( \mathcal{W} \cap (A + B) \neq \emptyset \).

Since addition is cont., \( \exists (U, V) \in \text{Nhd}(a) \times \text{Nhd}(b) : U + V \subseteq W \).

Since \( a \in \overline{A}, \exists \tilde{a} \in A \cap U \)

\( \tilde{a} \in \overline{U} \subseteq \overline{A} \), \( b \in \overline{B} \).

Since \( \overline{U} \subseteq \overline{A} \cap \overline{B} \), \( \tilde{a} + b \in \overline{A + B} \).
Claim: If $A \subseteq X$ is a subspace, then so is $\overline{A}$.

Proof: By Rudin pp. 6, 7.

$S \subseteq X$ is a subspace $\iff \forall x \in S \left\{ \begin{array}{l} \alpha S + \beta S \subseteq S \\ \forall \alpha, \beta \in \mathbb{C} \end{array} \right.$

Clearly since $0_x \in A$ and $A \subseteq \overline{A}$, $0_x \in \overline{A}$.

WTS $\alpha \overline{A} + \beta \overline{A} \subseteq \overline{A} \implies \forall \alpha, \beta \in \mathbb{C}.$

Claim: $\alpha \overline{A} = \overline{\alpha A}$ $\forall \alpha \in \mathbb{C}$

Proof: If $\alpha = 0$ true.

Else: Let $\overline{A} = \bigcap F_{\in \text{Closed}(X)} F$

$f(u) := \frac{1}{\alpha}u$

$\alpha \overline{A} = f^{-1}(\overline{A}) = f^{-1}\left( \bigcap_{F \in \text{Closed}(X)} F \right) \implies A
\[ f^{-1}(F) = \bigcap_{F \in \text{Closed}(X) \land F \supseteq A} F \supseteq aA. \]

Hence \( \alpha \bar{A} + \beta \bar{A} = \bar{\alpha A} + \bar{\beta A} \)

\[ \alpha \bar{A} + \beta \bar{A} = \bar{\alpha A} + \bar{\beta A} \]

\[ \bar{\alpha A} + \bar{\beta A} = \bar{\alpha A} + \bar{\beta A} \]

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**Claim:** \( 2A \subseteq A + A \)

**Proof:** Let \( a \in A \). Then \( 2a = a + a \in A + A \).

**Claim:** Unions and intersections of balanced are balanced.

**Proof:** Let \( \exists B \subseteq A \) be balanced, i.e.,
\[ \pm B_\alpha \subseteq B_\alpha \quad \forall \alpha, \mid \alpha \mid \leq 1. \]

Let now \( \exists \in C : \mid \exists \mid \leq 1 \). With \( \exists \),
\[ \pm \bigcup \alpha B_\alpha \subseteq \bigcup \alpha B_\alpha \quad \text{and} \]
\[ \pm \bigcap \alpha B_\alpha \subseteq \bigcap \alpha B_\alpha . \]

If \( \exists = 0 \), \( \pm B_\alpha = \emptyset \) so \( \exists \in B_\alpha \forall \alpha \)
and \( \exists \) anything to prove.

Else, let \( \forall \in \bigcap \alpha B_\alpha \). Then
\[ \frac{1}{2} \forall \in B_\alpha \forall \alpha . \]

Since \( B_\alpha \) is balanced, \( \forall \in B_\alpha \forall \alpha \).

Similarly for the union. \( \square \)

---

**Claim:** If \( A, B \) are balanced, so is \( A + B \).

**Proof:** Let \( \exists \in C : \mid \exists \mid \leq 1 \) and \( \forall \in Z(A + B) \).
Then \( \forall \in Z(A + B) \exists \alpha \in A, B \).
So \( \forall = zA + zB \in A + B \) as \( A, B \) are balanced. \( \square \)
Claim: If $A \subset B$ are odd, then $A \cup B$ is odd.

Pf. Let $N \in \text{Nbd}((X))$. WTS.

$(A \cup B) \subseteq tN$

for all $t > 0$ large enough.

by cont. $\exists M \in \text{Nbd}((X)) : M \cup M \subseteq N$. for $t$

Then $A \subseteq tM$

$b \subseteq tM$

$\Rightarrow A \cup B \subseteq tM + tM \subseteq t(M + M) \subseteq tN$. \[\square\]

Claim: If $A, B$ are opt. then $A \cup B$ is opt.

Pf. $+$: $X^2 \rightarrow X$ is cont.

$A \times B \in \text{Cpt}(X^2)$ by def. of prod. top. $A \cup B \equiv + (A \times B)$ and cont. image of cpt. is cpt. \[\square\]

Claim: $\exists A, B \in \text{Closed}(X) : A \cup B \notin \text{Closed}(X)$. 

[88]
Proof: Let \( A \subseteq C \) be given by
\[
A := N \in \text{Closed}(C)
\]
\[
B := \left\{ -n + \frac{1}{n} \mid n \in \mathbb{N} \right\} \in \text{Closed}(C).
\]
\[
\frac{1}{n} \in A + B \quad \forall \ n \in \mathbb{N} \text{ and } \Omega(\mathbb{E}A+B).
\]

Claim: \( X, Y \) TVS w/ \( \dim(C_{Y}) < \infty \).

\( \Lambda : X \to Y \) lin. & surj.

Then (1) \( \Lambda \) is open and

(2) \( \text{ker}(\Lambda) \in \text{Closed}(X) \Rightarrow \Lambda \) is cont.

Proof: By Rudin Thm. 1.21 (a), \( Y = \mathbb{C}^{n} \) WOT.

Let \( \{e_{j} : j = 1\} \) be the std. basis.

Since \( \Lambda \) is surj., \( \exists f_{j} : e_{j} \in X : \Lambda f_{j} = e_{j} \).

Define \( \Gamma : \mathbb{C}^{n} \to X \) via
\[
\tau_{\Lambda} \to \sum_{j=1}^{n} \tau_{j} f_{j}
\]

By def., \( \Gamma \) is lin.

By Rudin’s Lemma 1.20, \( \Gamma \) is cont.
By L.N. Claim 3.21, suffice to show that if \( N \in \text{Nbd}(0, x) \) then \( \Lambda N \) contains some \( M \in \text{Nbd}(0, y). \)

Study \( \Pi^{-1}N \). Since \( \Lambda \Pi \mathcal{U} = \emptyset \quad \forall \mathcal{U} \in \mathcal{C} \):

\[
\Pi^{-1}N = \Lambda \Pi \mathcal{U}^{-1} N \subseteq \Lambda N.
\]

But \( \Pi \) is cont., so \( \Pi^{-1}N \in \text{Open}(\mathcal{C}^n) \) and by linearity, \( O_{\mathcal{C}^n} \in \Pi^{-1}N \).

Hence \( \Lambda N \) is indeed open. \( \Rightarrow \) (1).

Next, assume \( \ker \Lambda \in \text{Closed}(X) \) and WTS \( \Lambda : X \to \mathcal{C}^n \) is cont.

Unfortunately, the easiest way to do this seems to involve quotient TVS, so no pts. will be deducted for mistakes here.

\( \hat{\Lambda} : X / \ker(\Lambda) \to \mathcal{C}^n \) is a VS isomorphism and hence a TVS isomorphism.
Note $\mathbb{R}/\ker{\alpha}$ only makes sense if $\ker{\alpha}$ is closed, and $V_S \rightarrow TVS$ iso. bcs. of finite dimensions.

## All

$$G := \{ f : [0, 1] \rightarrow \mathbb{C} \mid f \text{ is cont.} \}$$

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx$$

### Claim: $d$ is a metric on $G$.

#### Pf.:

1. If $d(f, g) = 0$,

$$\int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx = 0$$

   Since integrand $\geq 0$ and cont.,

   it must equal $\geq 0$ and cont.,

   it must equal zero $\Rightarrow f = g$. \(\square\)

2. $d$ is symm. \(\checkmark\)

3. Triangle ineq. follows from
The fact
\[ d(a,b) := \frac{|a-b|}{1 + |a-b|} \]
obey \Delta \neq \text{ on } \mathbb{C}^n.

Note \([0,\infty) \ni \alpha \mapsto \frac{\alpha}{1 + \alpha} \] is increasing, and we're trying to show
\[ r(1a-b1) \leq r(1a-c1) + r(1c-b1) \]

Note \[ r(\alpha) + r(\beta) \geq r(\alpha \beta) . \]

Indeed,
\[ \frac{\alpha}{1 + \alpha} + \frac{\beta}{1 + \beta} \geq \frac{\alpha + \beta}{1 + \alpha \beta} \]
\[ \frac{\alpha (1 + \beta) + \beta (1 + \alpha)}{(1 + \alpha) (1 + \beta)} = \frac{\alpha + \beta + 2\alpha \beta}{1 + \alpha \beta} \]
\[ \geq \frac{\alpha + \beta + \alpha \beta}{1 + \alpha \beta + \beta} \]
\[ \geq \frac{\alpha + \beta}{1 + \alpha \beta + \beta} . \]

So this follows from ordinary \( \Delta \neq \) on \( \mathbb{C} \).
Claim: \( C \) is a VS.

\textbf{Pf.} Obvious.

---

Claim: \( (C,d) \) is a TVS.

\textbf{Pf.} (1) All metric spaces are T1. \( \checkmark \)

(4) We note \( d \) is transl.-linear:

\[ d(p,g) = d(p+h, g+h). \]

Hence

\[ d(f+g, \hat{f}+\hat{g}) = d(f-f, g-g) \]

\[ \leq d(f-f, 0) + d(0, g-g) \]

and the rest of the proof follows as in \( \text{[Q1]} \).

(1) We don't have homogeneity, but

\[ d(\lambda f, \lambda g) \leq (1+|\lambda|) d(f, g). \]

Indeed,
\[
\frac{|a|}{1 + |a|^2} = \frac{|a|}{1 + |a|^2} \leq \max\{1, |a|^2\} \frac{|a|}{1 + |a|^2}
\]

But if \( |a| < 1 \),

\[
\frac{1 + |a|^2}{|a|} \geq 1 + |a|^2
\]

So

\[
\frac{|a|}{1 + |a|^2} \leq \max\{1, |a|^2\} \frac{|a|}{1 + |a|^2} \leq (1 + |a|) \frac{|a|}{1 + |a|^2}
\]

The rest of the proof follows similarly to \([\Box]\).

---

**Claim:** Let \( V \in \text{Nbhd}(0) \). Then \( \exists \ f: X \to \mathbb{R} \) cont. \( \forall x \in V \) such that \( f(0) = 0 \) and \( f(x) = 1 \) \( \forall x \in V \).

**Proof:** Let \( \{V_n\} \) be a seq. in \( \text{Nbhd}(0) \) which are all balanced and obey:

\[
V_n + V_n \subseteq V_{n-1}
\]
Define
\[ D := \left\{ q \in \mathbb{Q} \mid q = \sum_{n=1}^{\infty} \alpha_n 2^{-n} \right\} \]
and \( \alpha : \mathbb{N} \to \{0, 1\} \) is s.t. \( |\alpha^{-1}(1)| < \infty \).

For \( q \in D \), let \( \alpha(q) \) be the corresponding finite seq.

Then \( q > 0 \) and \( q \leq 1 \).

Define \( A : D \cup [1, \infty) \to \mathbb{P}(X) \)
\[ q \mapsto \left\{ \sum_{j=1}^{\infty} \alpha_j(q) \mathcal{V}_j \mid q \in D \right\} \]

\[ f : X \to [0, 1] \]
\[ x \mapsto \inf \left( \sum_{n \in \mathbb{N}} 1 \mathcal{V}_n \mid x \in A(r) \right) \].

Since \( 0_x \in \mathcal{V}_n \) \( \forall \ n \), \( 0_x \in A(r) \) \( \forall \ r \),
\[ \Rightarrow f(0_x) = 0. \]
If $x \in V^c$, want $\int_{\Delta} f(x) = 1$.  

But if $x \in V^c$, $x$ cannot lie in any $V_m$, and hence not in any of its sums.

Claim: $f$ is cont.

\[ \begin{array}{l}
\text{Pf: } \text{1) } f \text{ is cont. @ } 0_X : \\
\quad \forall \varepsilon > 0, \text{ let } N : 2^{-N} < \varepsilon . \\
\quad \text{Then } \| f(V_N) \| < 2^N < \varepsilon . \\
\text{2) } |f(x) - f(y)| \leq |f(x-y)| \\
\quad \text{which follows as in the proof of Roulins 1.24.}
\end{array} \]

\[ X := \left\{ f : (0,1) \to \mathbb{C} \mid f \text{ cont. } \right\} \text{ vs. } \]

\[ V(f,r) := \left\{ g \in X \mid |g(x) - f(x)| < r \forall x \in (0,1) \right\} \]
Claim: \( \{ V(f, r), f \in X, r > 0 \} \) is NOT a basis.

**Pf:** Need \( \forall f, g \in X, r, s > 0; \)

\[ V(f, r) \cap V(g, s) \neq \emptyset, \]

some \( V(h, t) \subseteq V(f, r) \cap V(g, s) \).

Take \( V(x, 0) \cap V(-x, 0) \) which intersect at the zero point.

But it is impossible to find \( V(h, r) \) inside this as

area tends to zero.

So this is a sub-basis.
Claim: \( f \) is cont.

**Pf.** Define for \( f, g \in X \): 
\[
R(f,g) := \{ h \in X \mid f < h < g \}.
\]

Then \( V(f,r) = \{ g \in X \mid g - f < r \} = \{ g \in X \mid f - r < g < f + r \} = R(f - r, f + r) \).

Actually \( R(f,g) \) is open \( \Box \).

Then if \( g + h \in V(f,r) \),

\[
g + h \in \bigcap_{j=1}^{n} V(f_j, r_j) \subseteq V(f,r)
\]

\[
R(f_j - r_j, f_j + r_j)
\]

\[
R(\min_{j} f_j, \max_{j} f_j) + r_j
\]

\[
L = H
\]

\[
L_1 := g - \frac{1}{2}(g + h - L)
\]

\[
L_2 := h - \frac{1}{2}(g + h - L)\]
\[ H_1 := y + \frac{1}{2} (H - (y+h)) \]
\[ H_2 := h + \frac{1}{2} (H - (y+h)) \]

Then 
\[ (g, h) \in R(C_1, H_1) \times R(C_2, H_2) \subseteq t^{-1} (R(C, H)). \]

To see scalar mul. is not cont., consider \( (x \mapsto \frac{1}{x}) \in X \) w/ mul. by 0, which yields the zero fn. However, \( \mathcal{N} \) nbhd of \( (0, x \mapsto \frac{1}{x}) \) which will land in an arbitrarily small ball of the zero fn.

\[ \Box \]

To see \( R(f, g) \) are open, write 
\[ R(f, g) = \bigcup_{\alpha} \bigcap_{e=1}^{m} V (f^e, r^e), \]

\[ \mathcal{N} \]