Measure Theory
Princeton University MAT425
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Spring 2025 Written Midterm Exam

Note: the following is the midterm exam held on Mar 5th 2025 from 7:30pm-10pm.

Please PRINT IN CAPITALS your first and last name in the box:

Sample Solutions

state the *Honor Code*:

and sign it

.

Now please wait, without turning the page, until you are told to start the exam, at which point you shall have 150 minutes.

Please write *SLOWLY, legibly and neatly*, clearly indicating the structure of your answer (e.g. dividing into claims, sub-claims etc). In particular, if we can't understand your handwriting you *will* lose points, even if your solution is correct.

This is a closed-book exam. No external aids are allowed.

It has three parts: definitions, short questions (no justification necessary) and long questions (full proofs required).

Conventions

- λ is the Lebesgue measure on $\mathbb{R}^d$ for $d \in \mathbb{N}$ (we use the same symbol for any d).
- Without any further specification, if reference to the measurable structure of a topological space is made, the relevant σ-algebra is the *Borel* one.
- We denote integrals of functions *without* reference to the integration variable as
$$\int_A f \mathrm{d}\mu \, .$$
Sometimes when the function is given explicitly as a formula this notation gets clunky, but still, to remain consistent, we unfortunately use
$$\int_A [x \mapsto f(x)] \, \mathrm{d}\mu \equiv \int_{x \in A} f(x) \mathrm{d}\mu(x) \, .$$
- Unless otherwise specified $\mathbb{N} \equiv \{1, 2, 3, \cdots\}$ does not include the number zero.
- The space L^1 is the space of *absolutely* integrable functions, and non-standardly, we use the notation
$$L^1(X \to Y, \mu) \equiv \left\{ f : X \to Y \mid f \text{ is msrbl. and } \int_X |f| \mathrm{d}\mu < \infty \right\}$$
where $Y \subseteq \mathbb{C}$ to indicate also the range of the function under consideration. If L^1 appears without further specification the meaning is probably that the domain is $\mathbb{R}^d$, the co-domain is $\mathbb{C}$ and the measure is the Lebesgue measure.
- The σ-algebra on which the counting measure acts is the power set.
- A measurable space is the duo $(X, \mathfrak{M})$ of a non-empty set and a σ-algebra in it. A measure space is the triplet $(X, \mathfrak{M}, \mu)$ where $(X, \mathfrak{M})$ is a measurable space and μ is a measure on $\mathfrak{M}$.

Part I: Definitions (8 points)

Each definition is worth 2 points. Be as precise as possible; no partial credit.

1. Provide the definition of a σ-algebra on a nonempty set X.

   Jacob Def. 2.1 in LN

2. Provide the definition of a nonnegative measure on a σ-algebra.

   Jacob Def. 2.28 in LN

3. Provide the definition of a measurable function between two measurable spaces.

   Jacob Def. 2.4 in LN

4. Provide the definition of an outer measure.

   Jacob Def. 2.64 in LN

Part II: Short questions (21 points)

In the following questions, *no* justification is necessary. Simply provide the shortest possible correct response. Each question is worth 3 points.

5. Are there Lebesgue measurable subsets of $\mathbb{R}$ which are not Borel measurable?

   Jacob Yes, see Remark 2.77 in LN.

6. Is the pointwise limit of measurable functions always measurable?

   Jacob Yes, see Corollary 2.23 in LN.

7. Provide an example of a measure on some measurable space which is *not* σ-finite.

   Jacob The counting measure on an uncountable set (with its power set being the sigma algebra). It is impossible to cover the set with a countable collection of finite sets (the only sets having finite measure).

8. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of measure-zero sets. Is $\bigcup_{i\in\mathbb{N}} A_i$ of measure zero?

   Jacob Yes. One can show that measures are always sigma-sub-additive (even on non-disjoint sequences). Then use that sigma-sub-additivity.

9. Does the monotone convergence theorem require the functions to be bounded?

   Jacob No. See Theorem 2.47 in LN.

10. Is the countable union of a collection of σ-algebras again a σ-algebra?

    Jacob No. This is already false for just two σ-algebras $\mathfrak{M}_1, \mathfrak{M}_2$ so we can take a countable sequence which is constant after the sequence element. For the case of two, take e.g. $X = \{1, 2, 3, \}$ and

    $$\mathfrak{M}_1 := \{\{1\}, \{2, 3\}, X, \varnothing\}, \qquad \mathfrak{M}_2 := \{\{2\}, \{1, 3\}, X, \varnothing\}$$

    so that

    $$\mathfrak{M}_1 \cup \mathfrak{M}_2 = \{\{1\}, \{2, 3\}, \{2\}, \{1, 3\}, X, \varnothing\}$$

    is not closed under unions as it does not contain $\{1\} \cup \{2\} = \{1, 2\}$.

11. Is every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{C}$ the pointwise limit of a sequence of continuous functions?

Jacob No. This is true if we relax to "almost-everywhere", see e.g. Lusin's theorem. Otherwise, one may prove that the pointwise limit of continuous functions is continuous on a dense G_δ set. However, $\chi_{\mathbb{Q}}$ for example is measurable yet is discontinuous everywhere, and so may never arise as the pointwise limit of continuous functions.

Part III: Long questions (72 points)

In the following questions, you must justify your work with full proofs and convince us that you not only know what the correct answer is, but also *why* it is so. You may freely invoke any result from any textbook or lecture notes just so long as you properly cite and explain what it is that you're invoking so we can look it up, and of course, that you don't cite the very thing you're asked to prove.

Each question is worth 12 points.

13. (a) Prove that if $f \in L^1(\mathbb{N} \to \mathbb{C}, c)$ with c the counting measure then

$$\lim_{n\to\infty} f(n) = 0\,.$$

Serban Since $f : \mathbb{N} \to \mathbb{C}$ is in $L^1(c)$ we have (**1 point**):

$$\sum_{n\in\mathbb{N}} |f(n)| < \infty$$

In order for the series to converge we must have $\lim_{n\to\infty} f(n) = 0$ (**1 point**).

(b) Prove that there exists a continuous $f \in L^1(\mathbb{R} \to [0, \infty)\,, \lambda)$ such that

$$\limsup_{x\to\infty} f(x) = \infty\,.$$

Hint: construct a continuous version of the function equal to n on the segment $[n, n + \frac{1}{n^3})$ for $n \geq 1$.

Serban We define the function f according to the hint to equal n on $\left[n + \frac{1}{4n^3}, n + \frac{3}{4n^3}\right)$, to equal zero outside intervals of the form $\left[n, n + \frac{1}{n^3}\right]$ for all $n \geq 2$, to be a linear function from 0 to n on $\left[n, n + \frac{1}{4n^3}\right]$, and a linear function from n to 0 on $\left[n + \frac{3}{4n^3}, n + \frac{1}{n^3}\right]$. (**2 points**)
By construction the function f is continuous (**1 point**). Also since $f(n + 1/2n^3) = n$ for all $n \geq 2$ we get that $\limsup_{x\to\infty} f(x) = \infty$ (**1 point**).
Finally, we see that by the construction of f we have that $0 \leq f(x) \leq n$ for all $x \in \left[n, n + \frac{1}{n^3}\right]$ and $n \geq 2$. Thus, the integral of f satisfies (**2 points**):

$$\int_{\mathbb{R}} |f| d\lambda \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

so indeed $f \in L^1$.

(c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is *uniformly* continuous and L^1, then

$$\lim_{|x|\to\infty} f(x) = 0\,.$$

Serban We assume by contradiction that $\lim_{|x|\to\infty} f(x) \neq 0$, so WLOG there exists a sequence $x_n \to \infty$ such that $f(x_n) \to a \neq 0$ (**1 point**). We can also assume that $a > 0$ for simplicity and that $|x_n - x_m| > 1$ for all $n \neq m$. Moreover, for $n \geq N$ large enough we have that $f(x_n) > a/2$. Let $0 < \epsilon < a/4$. By uniform continuity we have that there exists $0 < \delta < 1/2$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ (**1 point**). In particular, we get that $|f(x)| > a/2 - \epsilon > a/4$ for all $|x - x_n| < \delta$ and $n \geq N$ (**1 point**). We then get that:

$$\int |f| d\lambda \geq \sum_{n=N}^{\infty} \int_{x_n-\delta}^{x_n+\delta} f(x)dx > \sum_{n=N}^{\infty} \int_{x_n-\delta}^{x_n+\delta} \frac{a}{4} dx = \sum_{n=N}^{\infty} \frac{a\delta}{2} = \infty,$$

which contradicts $f \in L^1$ (**1 point**).

14. Provide a counter-example (and prove it is so) of a measure space $(X, \mathfrak{M}, \mu)$ and a decreasing sequence of measurable sets $\{A_i\}_{i\in\mathbb{N}}$, i.e., $A_i \supseteq A_{i+1}$ where

$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) \neq \lim_{i\to\infty} \mu(A_i)\,.$$

Chayim Note that by Theorem 2.29(4), we *do* have equality if $\mu(A_n) < \infty$ for some n. So our example *must* consist of sets of infinite measure only. Choose $X = \mathbb{R}$, $M = \mathcal{B}(\mathbb{R})$, $\mu = \lambda$. Define $A_n :=[n, \infty)$ ($n \in \mathbb{N}$). Then for all $n \in \mathbb{N}$ we have $\mu(A_n) = \infty$. On the other hand, $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$. Hence

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = 0 \neq \infty = \lim_{n\to\infty} \mu(A_n)$$

(C.f. Example 1.20 in Rudin's *Real and Complex analysis.*)

(3 pts) Proper description of measure space and decreasing sets (example must either actually work or be close to working)

(4 pts) Show that $\lim_{n\to\infty} \mu(A_n) = \infty$

(4 pts) Show that $\bigcap_{n\in\mathbb{N}} A_n$ has finite measure

(1 pt) Overall clarity

15. Show that for $f \in L^1(\mathbb{R}^d \to \mathbb{R})$, if

$$\int_A f \mathrm{d}\lambda = 0 \qquad (A \text{ Lebesgue msrbl.})$$

then

$$\lambda\left(f^{-1}(\{0\}^c)\right) = 0\,.$$

Hint: You may freely invoke the Tschebyschev inequality.

Serban We write $f = f^+ - f^-$ where f^+, f^- are non-negative functions. For any $\alpha > 0$ we have that $\{f > \alpha\} = \{f^+ > \alpha\}$. We recall Chebyshev's inequality (**2 points**):

$$\lambda\big(\{f^+ > \alpha\}\big) \leq \frac{1}{\alpha}\int f^+ d\lambda$$

Using this inequality we get (**4 points**):

$$\lambda\big(\{f > \alpha\}\big) = \lambda\big(\{f^+ > \alpha\}\big) \leq \frac{1}{\alpha}\int f^+ d\lambda = \frac{1}{\alpha}\int_{\{f\geq 0\}} f d\lambda = 0$$

We also notice that $\{f > 0\} = \bigcup_n \{f > 1/n\}$ (**2 points**), so we proved that $\lambda(\{f > 0\}) = 0$ (**2 points**). We use the same argument for $-f$ to get $\lambda(\{f < 0\}) = 0$ and we conclude (**2 points**).

16. Let $\Gamma \subseteq \mathbb{R}^{d+1}$ be given by

$$\Gamma := \big\{(x, y) \in \mathbb{R}^d \times \mathbb{R} \,|\, y = f(x)\big\}$$

for some $f : \mathbb{R}^d \to \mathbb{R}$ which is Lebesgue measurable. Show that Γ is Lebesgue measurable and its Lebesgue measure is zero.

Serban We first show that Γ is measurable. We use the following result from the homework (see also Corollary 3.7 in Chapter 2 of Stein and Shakarchi Real analysis): Suppose that F is measurable on $\mathbb{R}^{d_1}$, then $\tilde{F}(x, y) = F(x)$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (**2 points**). As a result, we obtain that the function $(x, y) \mapsto f(x) - y$ is measurable on $\mathbb{R}^{d+1}$. Since Γ is the zero set of this function, we get that it is measurable (**2 points**).
To compute the measure of Γ we first express its measure as an integral (**1 point**):

$$\lambda(\Gamma) = \int_{\mathbb{R}^{d+1}} \mathbf{1}_\Gamma(x, y) d\lambda$$

Using the fact that $\mathbf{1}_\Gamma$ is measurable, we get by Tonelli's theorem (**2 points**):

$$\lambda(\Gamma) = \int_{\mathbb{R}^{d+1}} \mathbf{1}_\Gamma(x, y) d\lambda = \int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathbf{1}_\Gamma(x, y) dy dx$$

We notice that for fixed $x \in \mathbb{R}^d$ we have $\mathbf{1}_\Gamma(x, y) \neq 0$ iff $y = f(x)$. Thus, $\mathbf{1}_\Gamma(x, y) = 0$ for almost every $y \in \mathbb{R}$, so $\int_{\mathbb{R}} \mathbf{1}_\Gamma(x, y) dy = 0$ (**4 points**). Using this in the above relation we conclude that $\lambda(\Gamma) = 0$ (**1 point**).

17. Let $f \in L^1(\mathbb{R} \to \mathbb{C}, \lambda)$ and define $g : \mathbb{R} \to \mathbb{C}$ via

$$g(x) := \int_{(-\infty, x)} f \mathrm{d}\lambda \qquad (x \in \mathbb{R})\,.$$

Show that g is uniformly continuous.

Serban We use the continuity of the integral (see Proposition 1.12 in Chapter 2 of Stein and Shakarchi: Real analysis, which was proved in Problem 4 of Homework 2): for any $f \in L^1(\mu)$ and $\epsilon > 0$ there exists some $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ for any measurable set with $\mu(E) < \delta$ (**8 points**).
Let any $\epsilon > 0$ and consider $\delta > 0$ as above. For any $|x - y| < \delta$ we have by triangle inequality (**2 points**):

$$|g(x) - g(y)| \leq \int_{[x,y]} |f| d\lambda$$

Using the continuity of the integral, we get that the above is bounded by ϵ. Thus $|g(x) - g(y)| < \epsilon$ so g is uniformly continuous (**2 points**).

18. Calculate

$$\lim_{n\to\infty} \int_{(0,\infty)} \left[x \mapsto \frac{1}{1+x^n} \right] \mathrm{d}\lambda \, .$$

Chayim For convenience, define $f_n(x) := (1 + x^n)^{-1}$. For fixed $x \in (0, \infty)$, we have

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ +\infty & \text{if } x > 1 \end{cases}$$

It follows that

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Clearly $\int_{(0,\infty)} f \, d\lambda = 1$. To show that $\lim_{n\to\infty} \int_{(0,\infty)} f \, d\lambda = \int_{(0,\infty)} f \, d\lambda$, we apply the dominated convergence theorem (Theorem 2.61) with dominating function $g : (0, \infty) \to \mathbb{R}$ given by

$$g(x) := \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ x^{-2} & \text{if } x > 1 \end{cases}$$

Observe

$$\int_{(0,\infty)} g \, d\lambda = \int_0^1 1 \, dx + \int_1^\infty x^{-2} \, dx = 2 < \infty$$

and for each $n \geq 2$, we have (in addition to $f_n(x) \geq 0$ for all $x \in (0, \infty)$)

- For $x \in (0, 1]$, $f_n(x) = \frac{1}{1+x^n} < 1 = g(x)$.
- For $x \in (1, \infty)$, $f_n(x) = (1 + x^n)^{-1} < x^{-n} \leq x^{-2} = g(x)$.

Thus g dominates f_n for $n \geq 2$.

(3 pts) Correctly compute the pointwise limit of the integrand

(2 pts) Indicate that dominated convergence theorem will be used

(3 pts) Construct a valid dominating function (and show that it dominates, at least for large n)

(3 pts) Show that dominating function has finite integral

(1 pts) Attention to detail and clarity of proof.

Bonus long questions (beyond 101 points)

Each successfully solved bonus long question will grant you a total of 12 points, but your grade will be truncated to 100.

19. Provide an example (and prove it is so) of a topological space X where two measures $\mu_1, \mu_2 : \mathscr{B}(X) \to [0, \infty]$ agree on all open sets and yet $\mu_1 \neq \mu_2$.

Chayim Choose $X = \mathbb{R}$. Let μ_1 be the counting measure on $\mathbb{R}$ and define $\mu_2 := 2 \cdot \mu_1$.[a] Consider any open set $U \subseteq \mathbb{R}$. Then either $U = \varnothing$—in which case

$$\mu_1(U) = 0 = 2 \cdot 0 = \mu_2(U)$$

or else U contains an open interval in which case U is infinite and thus

$$\mu_1(U) = \infty = 2 \cdot \infty = \mu_2(U)$$

However, clearly $\mu_1 \neq \mu_2$ seeing as $\mu_1(\{0\}) = 1 \neq 2 = \mu_2(\{0\})$.

(3 pts) Proper description of topological space and measures on that space (example must work or be close to working)

(4 pts) Show that the two measures agree on all open sets

(4 pts) Show that the two measures disagree on some Borel set

(1 pt) Overall clarity

[a] These are naturally measures on all of $\mathscr{P}(\mathbb{R})$, but here we think of them as Borel measures.

20. Let $(X, \mathfrak{M}, \mu), (Y, \mathfrak{N}, \nu)$ be two measure spaces and let $f : X \to \mathbb{C}, g : Y \to \mathbb{C}$ be two measurable functions. Define

$$F(x, y) := f(x)g(y) \qquad ((x, y) \in X \times Y)\,.$$

(a) Show that F is measurable on the measurable space $(X \times Y, \mathfrak{M} \otimes \mathfrak{N})$.

Chayim Define $\tilde{f}, \tilde{g} : X \times Y \to \mathbb{C}$ by $\tilde{f}(x, y) := f(x)$, $\tilde{g}(x, y) := g(x)$. Then clearly $F = \tilde{f}\tilde{g}$. By Theorem 2.20(3), if we can show that $\tilde{f}, \tilde{g}$ are measurable then it will follow that F is too. By symmetry, it clearly suffices to show that $\tilde{f}$ is measurable. To see this, let $W \in \mathscr{B}(\mathbb{C})$. Then $f^{-1}(W) \in \mathfrak{M}$ by the measurability of f. But then

$$\tilde{f}^{-1}(W) = f^{-1}(W) \times Y \in \mathfrak{M} \otimes \mathfrak{N}$$

by definition of the product measure.

(b) Show that if f, g are both L^1 (in their respective spaces) then

$$F \in L^1(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)\,.$$

Chayim Suppose first that X and Y are σ-finite. Then we may apply Tonelli's theorem (Theorem 5.14) to the functions $|f|, |g|$[a] to get

$$\begin{aligned}
\int_{X\times Y} |F|\, d(\mu\times\nu) &= \int_X \left(x \mapsto \int_Y |F|_x\, d\nu\right) d\mu \\
&= \int_X \left(x \mapsto \int_Y |f(x)|\cdot|g|\, d\nu\right) d\mu \\
&= \int_X \left(x \mapsto |f(x)|\cdot \int_Y |g|\, d\nu\right) d\mu \quad \text{(Theorem 2.57)} \\
&= \int_X \left(|f|\cdot \int_Y |g|\, d\nu\right) d\mu \\
&= \left(\int_Y |g|\, d\nu\right)\cdot \int_X |f|\, d\mu \quad \text{(Theorem 2.57 \textit{again})} \\
&< \infty \quad \text{(by hypothesis)}
\end{aligned}$$

We now drop the hypothesis of σ-finiteness. We will use the following trick.

Lemma. *Let (Z, m) be a measure space and suppose $h \in L^1(Z, m)$. Then the set $S := \{z \in Z \mid h(z) \neq 0\}$ is σ-finite.*[b]

Proof. For each $\epsilon > 0$, define $S_\epsilon := \{z \in Z \mid |h(z)| > \epsilon\}$. Clearly

$$S = \bigcup_{n\in\mathbb{N}} S_{1/n}$$

But by the Tschebyschev inequality, we have $m(S_{1/n}) < n \cdot \int_Z |h|\, dm < \infty$. ☐

[a] These are measurable by part (a) and Theorem 2.20(2).

[b] That is, S is a countable union of finite-measure subsets of Z.

Now let $X' := \{x \in X \mid f(x) \neq 0\}$ and let $Y' := \{y \in Y \mid g(y) \neq 0\}$. Letting $\imath : X' \hookrightarrow X$ and $\jmath : Y' \hookrightarrow Y$ be the inclusion maps, define $\mathfrak{M}' := \sigma(\imath)$ and $\mathfrak{N}' := \sigma(\jmath)$. Define measures $\mu' : \mathfrak{M}' \to [0, \infty]$ and $\nu' : \mathfrak{N}' \to [0, \infty]$ via restriction: $\mu'(A) = \mu(A) \quad (A \in \mathfrak{M}')$ and $\nu'(B) = \nu(B) \quad (B \in \mathfrak{N}')$. Then $F(x, y) = 0$ for $(x, y) \in X \times Y - X' \times Y'$. Using this, one shows[c] that

$$\int_{X\times Y} |F|\, d(\mu\times\nu) = \int_{X'\times Y'} |F|\, d(\mu\times\nu) = \int_{X'\times Y'} \left|F|_{X'\times Y'}\right| d(\mu'\times\nu')$$

But the measure spaces X' and Y' are σ-finite, so by our argument above, the rightmost integral is finite.

(3 pts) (a) Essentially correct proof of measurability

(1 pts) (a) Correct handling of any subtleties

(2 pts) (b) Indicate the intention to use Tonelli's theorem (or equivalent)

(2 pts) (b) Correct manipulation of integrals to show finiteness

(1 pt) (b) Noticing that σ-finiteness is needed

(1 pt) (b) Correct handling of the non-σ-finite case.

(1 pt) Overall clarity and proof structure.

[c] The proof involves combing through definitions, but do not be fooled into thinking that the equality simply holds *by definition*.

21. For $s > -1$, calculate

$$\lim_{n\to\infty} \int_{[0,n]} \left[x \mapsto \left(1 - \frac{x}{n}\right)^n x^s\right] \mathrm{d}\lambda\,.$$

Chayim Define $f_n : [0,\infty) \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} \left(1 - \frac{x}{n}\right)^n x^s & \text{if } x \in [0,n] \\ 0 & \text{otherwise} \end{cases}$$

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := e^{-x}x^s$.[a] Observe that (for fixed $x \in [0,\infty)$)

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n x^s \\ &= x^s \cdot \lim_{n\to\infty} e^{n\log(1-x/n)} \\ &= x^s \cdot \lim_{n\to\infty} e^{n(-x/n+o(1/n))} \\ &= x^s \cdot \lim_{n\to\infty} e^{-x+o(1)} \\ &= x^s e^{-x} \\ &= f(x) \end{aligned}$$

Also, using the inequality $\log(1+t) \le t$ (valid for $t > -1$)[b] we get for $x \in (0,n)$

$$f_n(x) = \left(1 - \frac{x}{n}\right)^n x^s = e^{n\log(1-x/n)}x^s \le e^{n\cdot(-x/n)}x^s = e^{-x}x^s = f(x)$$

It should be clear that $f_n(x) \ge 0$ for all $x \in (0,n)$. Hence f dominates f_n. Also

$$\int_{[0,\infty)} f\,d\lambda = \int_0^1 e^{-x}x^s\,dx + \int_1^\infty e^{-x}x^s\,dx$$

The $\int_0^1$ component is finite by comparison with $\int_0^1 x^s\,dx$. The $\int_1^\infty$ component is finite by comparison with $\int_1^\infty e^{-x/2}\,dx$ because $e^{-x}x^s < e^{-x/2}$ for large x.[c] Hence $\int_{[0,\infty)} f\,d\lambda$ is finite. Applying the dominated convergence theorem,[d] we get

$$\lim_{n\to\infty}\int_{[0,n]} f_n\,d\lambda = \lim_{n\to\infty}\int_{[0,\infty)} f_n\,d\lambda = \int_{[0,\infty)} \lim_n f_n\,d\lambda = \int_{[0,\infty)} f\,d\lambda$$

Remark. [e] *The integral $\int_{[0,\infty)} f\,d\lambda = \int_0^\infty e^{-x}x^s\,dx$ is none other than the famous integral expression for the Gamma function (with argument shifted by one). Thus*

$$\lim_{n\to\infty}\int_{[0,n]} f_n\,d\lambda = \int_0^\infty e^{-x}x^s\,dx = \Gamma(s+1)$$

(3 pts) Correctly compute the pointwise limit of the integrand

(2 pts) Indicate that dominated convergence theorem will be used

(3 pts) Construct a valid dominating function (and show that it dominates)

(3 pts) Show that dominating function has finite integral

(1 pts) Attention to detail and clarity of proof.

[a] Everything here depends *implicitly* on s.

[b] This inequality follows immediately from the concavity of the logarithm function.

[c] We are using the continuity of $x \mapsto e^{-x}x^s$ on $(0,\infty)$ to conclude that the integral is finite on any bounded subinterval of $[1,\infty)$.

[d] A more light-handed approach would use the monotone convergence theorem instead.

[e] This part is not necessary for a complete solution.