Measure Theory
Princeton University MAT425
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Spring 2025 Written Midterm Exam

Note: the following is the midterm exam held on Mar 5th 2025 from 7:30pm-10pm.

Please PRINT IN CAPITALS your first and last name in the box:

state the *Honor Code*:

and sign it

.

Now please wait, without turning the page, until you are told to start the exam, at which point you shall have 150 minutes.

Please write *SLOWLY, legibly and neatly*, clearly indicating the structure of your answer (e.g. dividing into claims, sub-claims etc). In particular, if we can't understand your handwriting you *will* lose points, even if your solution is correct.

This is a closed-book exam. No external aids are allowed.

It has three parts: definitions, short questions (no justification necessary) and long questions (full proofs required).

Conventions

- λ is the Lebesgue measure on $\mathbb{R}^d$ for $d \in \mathbb{N}$ (we use the same symbol for any d).
- Without any further specification, if reference to the measurable structure of a topological space is made, the relevant σ-algebra is the *Borel* one.
- We denote integrals of functions *without* reference to the integration variable as
$$\int_A f \mathrm{d}\mu \, .$$
Sometimes when the function is given explicitly as a formula this notation gets clunky, but still, to remain consistent, we unfortunately use
$$\int_A [x \mapsto f(x)] \, \mathrm{d}\mu \equiv \int_{x \in A} f(x) \mathrm{d}\mu(x) \, .$$
- Unless otherwise specified $\mathbb{N} \equiv \{1, 2, 3, \cdots\}$ does not include the number zero.
- The space L^1 is the space of *absolutely* integrable functions, and non-standardly, we use the notation
$$L^1(X \to Y, \mu) \equiv \left\{ f : X \to Y \mid f \text{ is msrbl. and } \int_X |f| \mathrm{d}\mu < \infty \right\}$$
where $Y \subseteq \mathbb{C}$ to indicate also the range of the function under consideration. If L^1 appears without further specification the meaning is probably that the domain is $\mathbb{R}^d$, the co-domain is $\mathbb{C}$ and the measure is the Lebesgue measure.
- The σ-algebra on which the counting measure acts is the power set.
- A measurable space is the duo $(X, \mathfrak{M})$ of a non-empty set and a σ-algebra in it. A measure space is the triplet $(X, \mathfrak{M}, \mu)$ where $(X, \mathfrak{M})$ is a measurable space and μ is a measure on $\mathfrak{M}$.

Part I: Definitions (8 points)

Each definition is worth 2 points. Be as precise as possible; no partial credit.

1. Provide the definition of a σ-algebra on a nonempty set X.

2. Provide the definition of a nonnegative measure on a σ-algebra.

3. Provide the definition of a measurable function between two measurable spaces.

4. Provide the definition of an outer measure.

Part II: Short questions (21 points)

In the following questions, *no* justification is necessary. Simply provide the shortest possible correct response. Each question is worth 3 points.

5. Are there Lebesgue measurable subsets of $\mathbb{R}$ which are not Borel measurable?

6. Is the pointwise limit of measurable functions always measurable?

7. Provide an example of a measure on some measurable space which is *not* σ-finite.

8. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of measure-zero sets. Is $\bigcup_{i\in\mathbb{N}} A_i$ of measure zero?

9. Does the monotone convergence theorem require the functions to be bounded?

10. Is the countable union of a collection of σ-algebras again a σ-algebra?

11. Is every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{C}$ the pointwise limit of a sequence of continuous functions?

Part III: Long questions (72 points)

In the following questions, you must justify your work with full proofs and convince us that you not only know what the correct answer is, but also *why* it is so. You may freely invoke any result from any textbook or lecture notes just so long as you properly cite and explain what it is that you're invoking so we can look it up, and of course, that you don't cite the very thing you're asked to prove.

Each question is worth 12 points.

13. (a) Prove that if $f \in L^1(\mathbb{N} \to \mathbb{C}, c)$ with c the counting measure then

$$\lim_{n\to\infty} f(n) = 0\,.$$

(b) Prove that there exists a continuous $f \in L^1(\mathbb{R} \to [0, \infty), \lambda)$ such that

$$\limsup_{x \to \infty} f(x) = \infty .$$

Hint: construct a continuous version of the function equal to n on the segment $[n, n + \frac{1}{n^3})$ for $n \geq 1$.

(c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is *uniformly* continuous and L^1, then

$$\lim_{|x| \to \infty} f(x) = 0\,.$$

14. Provide a counter-example (and prove it is so) of a measure space $(X, \mathfrak{M}, \mu)$ and a decreasing sequence of measurable sets $\{A_i\}_{i \in \mathbb{N}}$, i.e., $A_i \supseteq A_{i+1}$ where

$$\mu\left(\bigcap_{i \in \mathbb{N}} A_i\right) \neq \lim_{i \to \infty} \mu(A_i)\,.$$

15. Show that for $f \in L^1(\mathbb{R}^d \to \mathbb{R})$, if

$$\int_A f \mathrm{d}\lambda = 0 \qquad (A \text{ Lebesgue msrbl.})$$

then

$$\lambda\left(f^{-1}(\{0\}^c)\right) = 0\,.$$

Hint: You may freely invoke the Tschebyschev inequality.

16. Let $\Gamma \subseteq \mathbb{R}^{d+1}$ be given by

$$\Gamma := \big\{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid y = f(x)\big\}$$

for some $f : \mathbb{R}^d \to \mathbb{R}$ which is Lebesgue measurable. Show that Γ is Lebesgue measurable and its Lebesgue measure is zero.

17. Let $f \in L^1(\mathbb{R} \to \mathbb{C}, \lambda)$ and define $g : \mathbb{R} \to \mathbb{C}$ via

$$g(x) := \int_{(-\infty,x)} f \mathrm{d}\lambda \qquad (x \in \mathbb{R})\,.$$

Show that g is uniformly continuous.

18. Calculate

$$\lim_{n\to\infty} \int_{(0,\infty)} \left[x \mapsto \frac{1}{1+x^n} \right] \mathrm{d}\lambda \, .$$

Bonus long questions (beyond 101 points)

Each successfully solved bonus long question will grant you a total of 12 points, but your grade will be truncated to 100.

19. Provide an example (and prove it is so) of a topological space X where two measures $\mu_1, \mu_2 : \mathcal{B}(X) \to [0, \infty]$ agree on all open sets and yet $\mu_1 \neq \mu_2$.

20. Let $(X, \mathfrak{M}, \mu), (Y, \mathfrak{N}, \nu)$ be two measure spaces and let $f : X \to \mathbb{C}, g : Y \to \mathbb{C}$ be two measurable functions. Define

$$F(x,y) := f(x)g(y) \qquad ((x,y) \in X \times Y)\,.$$

(a) Show that F is measurable on the measurable space $(X \times Y, \mathfrak{M} \otimes \mathfrak{N})$.

(b) Show that if f, g are both L^1 (in their respective spaces) then

$$F \in L^1(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)\,.$$

21. For $s > -1$, calculate

$$\lim_{n\to\infty} \int_{[0,n]} \left[x \mapsto \left(1 - \frac{x}{n}\right)^n x^s \right] \mathrm{d}\lambda\,.$$