# Measure Theory
# Princeton University MAT425
# Lecture Notes

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**Abstract**

These lecture notes correspond to a course given in the Spring semester of 2025 in the math department of Princeton University.

## Contents



## Syllabus

- Main source of material for the lectures: this very document (to be published and weekly updated on the course website–please do not print before the course is finished and the label "final version" appears at the top).
- Official course textbook: No one, main official text will be used but in preparing these notes; I will probably make heavy use of [Rud86] and [SS05].
- Other books one may consult are [Fol99, FR10, Sim15].
- Two lectures per week: Tue and Thur, 1:30pm–2:50pm in Fine Hall 314.
- People involved:
  - Instructor:
    Jacob Shapiro shapiro@math.princeton.edu
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    Office hours: Tuesdays from 4:30pm - 5:30pm in the common area in front of Jadwin A10.

- HW will be published on a regular basis but is NOT to be submitted: do it for your own good. Sample solutions will be published one week later.
- Grade: 50% midterm (written in-person) scheduled-midterm; 45% final exam (oral, in person), 5% bonus.
- Attendance policy: *some* extra credit to students who attend lectures regularly and ask questions or point out mistakes.
- Anonymous Ed discussion enabled. Use it to ask questions or to raise issues (technical, academic, logistic) with the course.
- If you alert me about typos and mistakes in this manuscript (unrelated to the sections marked [todo]) I'll grant you *some* extra credit. In doing so, please refer to a version of the document by the date of typesetting.
  - Thanks goes to: Akshat Agarwal, Heyu Li (×3), Natalia Khotiaintseva, Vernon Hughes.

# Semester plan

List of (big) theorems and topics aimed at being included:

- Abstract measure theory.
- The Lebesgue integral.
- Radon-Nikodym derivative.
- Fubini, dominated convergence, monotone convergence, Fatou.
- Borel-Cantelli.
- Ergodic theorems.
- Carathéodory's theorem.
- The Lebesgue-Stieltjes integral.
- Tempered distributions.
- Hilbert space theory and applications to Fourier Transforms, and partial differential equations.
- Some probability theory?
- Introduction to fractals? Maybe.

Semester plan by date:

- Jan 28th 2025: introduction and abstract measure theory
- Jan 29th 2025:

# 1 Soft introduction

## 1.1 The Riemann integral and its inadequacies

In a single-value analysis class we are introduced to the rigorous definition of the Riemann integral, which is a $\mathbb{C}$-linear map from functions

$$f : [a, b] \to \mathbb{R}$$

into numbers. In particular, the integral is interpreted in multiple ways as:

1. The average value the function takes:

$$\overline{f} \quad = \quad \frac{1}{b-a}\int_{[a,b]} f\,.$$

2. The (signed) area enclosed between the graph of $f$, the horizontal axis, and the vertical lines $x=a$, $x=b$.
3. The appropriate continuum generalization to the discrete sum
$$\sum_{n=1}^{N} f(n)$$
understood in some appropriate sense.

There are various ways to rigorously define the Riemann integral [Rud76]. Let us proceed somewhat informally. The minimal assumption we make on $f$ is that it is bounded (otherwise we do not even ask whether it is Riemann integrable or not). To avoid the complication of partitions[1], let us always consider regular subdivisions of $[a,b]$. Then the lower / upper Riemann sum at $N$ subdivisions is given by

$$L_N(f) := \frac{b-a}{N}\sum_{n=0}^{N-1}\inf\left(\left\{ f(x) \,\middle|\, x\in\left(a+[n,n+1]\frac{b-a}{N}\right)\right\}\right)$$

and

$$U_N(f) := \frac{b-a}{N}\sum_{n=0}^{N-1}\sup\left(\left\{ f(x) \,\middle|\, x\in\left(a+[n,n+1]\frac{b-a}{N}\right)\right\}\right).$$

**Definition 1.1.** If the limits $\lim_N L_N(f)$ and $\lim_N U_N(f)$ exists and are equal, we say that $f$ is Riemann integrable on $[a,b]$ and define its Riemann integral as equal to the result of these equal limits:
$$\int_{[a,b]} f := \lim_N L_N(f) = \lim_N U_N(f)\,.$$

We remind the reader of Lebesgue's theorem. For it we need the notion of measure zero set:

**Definition 1.2** (Zero measure sets)**.** Let $S\subseteq\mathbb{R}$ be given. We say that $S$ has *zero measure* iff for any $\varepsilon>0$ there exists a countable collection of open intervals $\{ U_n \}_{n\in\mathbb{N}}$ such that both conditions below hold:
$$\begin{aligned}\sum_n |U_n| &< \varepsilon\\ S &\subseteq \bigcup_{n\in\mathbb{N}} U_n\,.\end{aligned}$$

**Theorem 1.3** (Lebesgue's theorem)**.** *The bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable iff its set of discontinuities on $[a,b]$ has measure zero.*

Armed with this theorem, it is easy to come up with some examples and counter-examples of Riemann integrable functions:

1. Any continuous function is Riemann integrable.
2. The indicator function on the cantor Set $C$, $\chi_C:[0,1]\to\mathbb{R}$, is Riemann integrable. Its set of discontinuities is the Cantor set $C$ which has measure zero (though it is uncountable).
3. The indicator function on a fat Cantor set is not Riemann integrable.
4. The indicator function onto the rationals $\chi_{\mathbb{Q}}:[0,1]\to\mathbb{R}$ is not Riemann integrable since it is discontinuous everywhere.

This last example is especially heinous: the set on which $\chi_{\mathbb{Q}}$ is different than zero is countable, it should somehow integrate to zero, since the countable set should not interfere with the uncountablity of the whole interval. Hence, already we see some deficiencies of the Riemann integral: what if the function we are trying to integrate doesn't have zero measure? Couldn't we still say something about its average value? This brings us to the study of just which sets are measurable at all, which we will get to eventually. Another question is what about unbounded functions? The improper Riemann integral addresses this to an extent.

[1] We note in passing that while we are allowed to restrict to regular subdivisions, we are not allowed to restrict to *both* regular subdivisions *and* always sample at the starting / ending point of each sub-interval.

**Example 1.4.** Consider the function $f:(0,1)\to\mathbb{R}$ given by $x\mapsto\frac{1}{\sqrt{x}}$ which is clearly *unbounded*. However, we may make sense of it formally by defining $f_n:\left[\frac{1}{n},1\right]\to\mathbb{R}$ by $x\mapsto\frac{1}{\sqrt{x}}$. For finite $n\in\mathbb{N}$, the function $f_n$ is bounded and Riemann integrable, and

$$\int_{\left[\frac{1}{n},1\right]} f_n = \int_{x\in\left[\frac{1}{n},1\right]}\frac{1}{\sqrt{x}}\mathrm{d}x = 2x^{\frac{1}{2}}\Big|_{x=\frac{1}{n}}^{1} = 2-\frac{2}{\sqrt{n}}\to 2\,.$$

If we had a finite number of integrable blow ups like this we could somehow manage. But this approach can go horribly wrong:

**Example 1.5.** Since $(0,1)\cap\mathbb{Q}$ is countable, let $\eta:\mathbb{N}\to(0,1)\cap\mathbb{Q}$ be the bijection which enumerates this set. Define then a sequence of functions $f_n:[0,1]\to\mathbb{R}$ via

$$f_n(x) \quad := \quad \begin{cases}(x-\eta_n)^{-\frac{1}{2}} & x>\eta_n\\ 0 & x\leq\eta_n\end{cases} \qquad (n\in\mathbb{N}, x\in[0,1])\,.$$

Then define $f:[0,1]\to[0,\infty]$ via

$$f(x) := \sum_{n=1}^{\infty}\frac{f_n(x)}{2^n} \qquad (x\in[0,1])\,.$$

$f$ has the weird property that it is unbounded on every open subinterval of $[0,1]$, since each one contains a rational number. *Hence $f$ is not Riemann integrable on every subinterval of $[0,1]$ which is not a singleton.*

But somehow we still feel like we should be able to assign an area under the graph of $f$, since we can do so for each $f_n$:

$$\begin{aligned}\int_{[0,1]} f_n &= \lim_{\varepsilon\to 0^+}\int_{[0,\eta_n-\varepsilon]} f_n + \int_{[\eta_n+\varepsilon,1]} f_n\\ &= \lim_{\varepsilon\to 0^+}\int_{x\in[\eta_n+\varepsilon,1]}(x-\eta_n)^{-\frac{1}{2}}\mathrm{d}x\\ &= \lim_{\varepsilon\to 0^+} 2(x-\eta_n)^{+\frac{1}{2}}\Big|_{x=\eta_n+\varepsilon}^{1}\\ &= 2\sqrt{1-\eta_n}\,.\end{aligned}$$

and somehow it should equal

$$\int_{[0,1]} f = \sum_{n=1}^{\infty}2^{-n}\int_{[0,1]} f_n = \sum_{n=1}^{\infty}2^{-n+1}\sqrt{1-\eta_n}\leq\sum_{n=1}^{\infty}2^{-n+1}<\infty\,.$$

From the more practical and less theoretical perspective, a much more severe limitation of the Riemann integral is how it behaves with limits. Namely, we have

**Theorem 1.6.** *Let $f_n:[a,b]\to\mathbb{R}$ be a sequence of bounded Riemann integrable functions which converges* uniformly *to the bounded function $\lim_n f_n:[a,b]\to\mathbb{R}$. Then $\lim_n f_n:[a,b]\to\mathbb{R}$ is also Riemann integrable, and*

$$\lim_n\int_{[a,b]} f_n = \int_{[a,b]}\lim_n f_n\,.$$

However, establishing *uniform converges* is notoriously difficult, in fact it is false in many interesting applications. For instance, letting let $\eta:\mathbb{N}\to(0,1)\cap\mathbb{Q}$ again be the bijection which enumerates its codomain, define

$$f_n := \chi_{\{\,\eta_j\,|\,j\in[1,n]\cap\mathbb{Z}\,\}}\,.$$

Clearly each $f_n$ is bounded and Riemann integrable. Also, $\lim_n f_n = \chi_{\mathbb{Q}\cap[0,1]}$ pointwise. But as we saw above, this limit is *not* Riemann integrable. We are looking for a way to exchange integration and limit without uniform convergence. We shall see that to do so we need to invent a new, more robust notion of integration.

## 1.2 Intuitive difference between Riemann and Lebesgue integration

We will see that the conceptually, while the Riemann integral divides the *domain* into small pieces and measures the area of each small rectangle, the Lebesgue integral does things somewhat sophisticatedly. To calculate the Lebesgue integral, we first need the notion of a *measure* which generalizes volume on Euclidean space to arbitrary spaces. Then we divide the *codomain* into small chunks and ask what is the measure of the preimage of that chunk in the domain. This turns out to give a more robust definition of the integral, which is not so susceptible to discontinuities and behaves better with limits. For that reason we now turn to abstract measure theory.

# 2 Abstract measure theory (Rudin RCA Chapter 1)

We now want to define the concept of *measurability* and ultimately assign a *measure* to measurable sets. This will be useful when we define the Lebesgue integral, and furthermore, this has applications in probability theory where measurable sets may be considered as those events for which a probability can be calculated.

## 2.1 Measurable sets and measurable functions

On a set $X$, we now want to define a *system of subsets* much like $\text{Open}(X)$ is a system of subsets with certain axioms.

**Definition 2.1** ($\sigma$-algebra)**.** Let $X$ be a set. A collection $\mathfrak{M} \subseteq \mathcal{P}(X)$ is called *a $\sigma$-algebra in $X$* iff $\mathfrak{M}$ obeys the following conditions:

1. $X \in \mathfrak{M}$ (contains the whole space).
2. $X \setminus A \in \mathfrak{M}$ for each $A \in \mathfrak{M}$ (closed under complements).
3. If $\{ A_n \}_{n\in\mathbb{N}}$ is a sequence of subsets such that $A_n \in \mathfrak{M}$ for each $n \in \mathbb{N}$ then

$$\bigcup_{n\in\mathbb{N}} A_N \in \mathfrak{M}\,.$$

   (closed under countable unions).

The tuple $(X, \mathfrak{M})$ where $\mathfrak{M}$ is a $\sigma$-algebra on $X$, is together called *a measure space.*

Note that this definition automatically implies: (1) closure with respect to countable intersections via De Morgan and (2) $\varnothing \in \mathfrak{M}$.

*Remark* 2.2 (Etymology). The prefix $\sigma$ denotes the closure w.r.t. countable unions. If we had merely closure wr.t. finite unions this would be called an *algebra.*

Contrast this with the notion of a *topology* on a given set $X$:

**Definition 2.3** (Topology)**.** Let $X$ be a set. A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a topology on $X$ iff $\mathcal{T}$ obeys the following conditions:

1. $X, \varnothing \in \mathcal{T}$ (contains the whole space and the empty set).
2. $\bigcap_{j=1}^n U_j \in \mathcal{T}$ if $U_1, \cdots, U_n \in \mathcal{T}$ (closed under finite intersections).
3. $\bigcup_{\alpha\in\mathcal{I}} U_\alpha \in \mathcal{T}$ if $U_\alpha \in \mathcal{T}$ for any $\alpha \in \mathcal{I}$, where $\mathcal{I}$ is an arbitrary set (*not* necessary countable) (closed under arbitrary unions).

The tuple $(X, \mathcal{T})$, if $\mathcal{T}$ is a topology on $X$, is together called *a topological space.*

When dealing with a topological space $X$, it is often convenient to denote its (already defined) topology as $\text{Open}(X)$. Similarly, we given a measure space $X$, we denote by $\text{Measurable}(X)$ the $\sigma$-algebra in it, should it be understood from the context.

**Definition 2.4** (Measurable function)**.** Let $f : X \to Y$ be given where $X, Y$ are two measure spaces. We say that $f$ is *measurable* iff $f^{-1}(A) \in \text{Measurable}(X)$ for each $A \in \text{Measurable}(Y)$.

Note that Rudin [Rud86] defines measurable function slightly differently (his codomains are always topological spaces).

*Claim* 2.5. The composition of two measurable functions is again measurable.

*Proof.* Let $f : X \to Y$ and $g : Y \to Z$ be two measurable functions between measure spaces. Let $A \in \text{Measurable}(Z)$. Then $g^{-1}(A) \in \text{Measurable}(Y)$. But then $f^{-1}\left(g^{-1}(A)\right) \in \text{Measurable}(X)$. But $f^{-1}\left(g^{-1}(A)\right) = (g \circ f)^{-1}(A)$ so we conclude $g \circ f$ is measurable. □

**Example 2.6** (The trivial $\sigma$-algebra)**.** Given a set $X$, we may consider its power set $\mathscr{P}(X)$ as a $\sigma$-algebra on it. It is called *the trivial* or *largest* $\sigma$-algebra on $X$. The smallest one is of course $\{\varnothing, X\}$.

**Example 2.7.** Take $X := \{1, 2, 3, 4\}$. Then a possible $\sigma$-algebra is $\{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$.

**Example 2.8.** Let $A \in \mathscr{P}(X)$. Then $\{\varnothing, A, X \setminus A, X\}$ is the smallest $\sigma$-algebra which contains $A$.

We may consider the category of measure spaces, in which measurable functions are precisely the morphisms.

*Remark* 2.9. A topology *need not* be a $\sigma$-algebra: it could fail to contain complements.

*Claim* 2.10. An arbitrary intersection of $\sigma$-algebras is again a $\sigma$-algebra. Not so for unions.

*Proof. TODO, fix this:* (even though a-priori it lies within an intersection of $\sigma$-algebras). Let then $A_n \in \sigma(\mathscr{F})$ for every $n \in \mathbb{N}$. Let $\mathfrak{M} \in \Omega$. Then $A_n \in \mathfrak{M}$ by definition, so $\bigcup_n A_n \in \mathfrak{M}$, as $\mathfrak{M}$ is itself a $\sigma$-algebra. But since $\mathfrak{M} \in \Omega$ was arbitrary, the union lies in the intersection $\sigma(\mathscr{F})$. The other two properties, complements and the entire space, are verified in the same manner.
*TODO*: provide a counter-example. □

**Definition 2.11** ($\sigma$-algebra generated by a function)**.** Let $f : X \to Y$ with $Y$ a measure space and $X$ a set. Then the $\sigma$-algebra generated by $f$ is a $\sigma$-algebra on $X$, denoted by $\sigma(f)$, given by

$$\sigma(f) := \left\{ f^{-1}(A) \,\middle|\, A \in \text{Measurable}(Y) \right\}.$$

One may then rephrase and say that, if $X$ already had a measure space structure, then $f$ is measurable w.r.t. it iff $\sigma(f) \subseteq \text{Measurable}(X)$. Cf. with initial topology.

**Theorem 2.12** ($\sigma$-algebra generated by a collection of subsets)**.** *Let $\mathscr{F} \subseteq \mathscr{P}(X)$ with $X$ some set. Then, there exists a* smallest *(in the sense of set inclusion) $\sigma$-algebra $\sigma(\mathscr{F})$ in $X$ such that $\mathscr{F} \subseteq \sigma(\mathscr{F})$. We call $\sigma(\mathscr{F})$ the $\sigma$-algebra generated by $\mathscr{F}$.*

*Proof.* (See [Rud86] Theorem 1.10) Let $\Omega$ be the family of all $\sigma$-algebras in $X$ which contain $\mathscr{F}$. Of course $\mathscr{P}(X)$ is in $\Omega$, so it is not empty. Define the *set*

$$\sigma(\mathscr{F}) := \bigcap_{\mathfrak{M} \in \Omega} \mathfrak{M}.$$

Clearly $\mathscr{F} \subseteq \sigma(\mathscr{F})$ by construction. The fact that $\sigma(\mathscr{F})$ is itself a $\sigma$-algebra and not just a set follows via Claim 2.10. □

**Definition 2.13** (Borel sets)**.** Given a topology on $X$, by Theorem 2.12 there is a $\sigma$-algebra generated by $\text{Open}(X)$: $\sigma(\text{Open}(X))$. The elements of $\sigma(\text{Open}(X))$ are called *the Borel sets of X*. In particular:

- Closed sets are also Borel sets, since they are the complements of open sets.
- Countable unions of closed sets are also Borel sets. These are called $F_\sigma$'s ($F$=closed, $\sigma$=union (summe)). For example $[a, b)$ is a $F_\sigma$ set of $\mathbb{R}$ with its standard topology.
- Countable intersections of open sets are also Borel sets. These are called $G_\delta$'s ($G$=open, $\delta$=intersection (durchschnitt)). For example $[a, b)$ is also a $G_\delta$ set of $\mathbb{R}$ with its standard topology.

We denote this special $\sigma$-algebra of Borels sets by $\mathscr{B}(X) := \sigma(\text{Open}(X))$.

Thus, given a topology on $X$ we are automatically provided with the Borel $\sigma$-algebra on it! If we don't specify any other $\sigma$-algebra on a (otherwise topological) space, we shall always mean the Borel $\sigma$-algebra.

*Claim* 2.14. Let $f : X \to Y$ be a mapping between two topological spaces such that $f^{-1}(U) \in \mathcal{B}(X)$ for any $U \in \text{Open}(Y)$. Then $f$ is measurable w.r.t. the Borel $\sigma$-algebras on both of these spaces.

*Proof.* We may consider the set

$$\mathfrak{M} := \left\{ A \in \mathcal{P}(Y) \,\middle|\, f^{-1}(A) \in \mathcal{B}(X) \right\} .$$

Cf. with final topology. We may verify it is stable under complements and countable unions, so it is itself a $\sigma$-algebra in $Y$. By hypothesis, $\text{Open}(Y) \subseteq \mathfrak{M}$ and so actually

$$\text{Open}(Y) \subseteq \mathcal{B}(Y) \subseteq \mathfrak{M}$$

by construction of $\mathcal{B}(Y) \equiv \sigma(\text{Open}(Y))$. But by $\mathcal{B}(Y) \subseteq \mathfrak{M}$ we learn that $f$ is measruable w.r.t. $\mathcal{B}(Y)$. ☐

This then coincides with Rudin's definition of measurable function.

**Theorem 2.15** (Rudin's Theorem 1.8)**.** *Let $u, v : X \to \mathbb{R}$ be two measurable functions ($\mathbb{R}$ is considered a measure space w.r.t. $\mathcal{B}(\mathbb{R})$). Let $\varphi : \mathbb{R}^2 \to Y$ be continuous where $Y$ is some topological space. Let $h : X \to Y$ be given by*

$$X \ni x \mapsto \varphi(u(x), v(x)) \in Y .$$

*Then $h$ is measurable w.r.t.* $\text{Measurable}(X)$ *and* $\mathcal{B}(Y)$.

*Proof.* The function $f : X \to \mathbb{R}^2$ given by $u \times v$. We have $h = \varphi \circ f$, so we only have to show $f$ is measurable. Let $R$ be any open rectangle on the plane with sides parallel to the axes: $R = I_1 \times I_2$ for two open intervals $I_1, I_2$ and so

$$f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2)$$

which is measurable by assumption on $u, v$. Since every open set $V \in \text{Open}(\mathbb{R}^2)$ is the countable union of such rectangles $R_i$, we find

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{\infty} R_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(R_i)$$

and hence $f^{-1}(V)$ is measurable and so is $f$. ☐

**Theorem 2.16** (Rudin's Theorem 1.9)**.** *Let $X$ be a measure space. Then*

1. *If $u, v : X \to \mathbb{R}$ are measurable then $f : X \to \mathbb{C}$ defined by $f := u + \mathrm{i}v$ is measurable.*
2. *If $f : X \to \mathbb{C}$ is measurable then $\mathbb{Re}\{f\}, \mathbb{Im}\{f\}$ and $|f|$ are measurable functions from $X \to \mathbb{R}$.*
3. *If $f, g : X \to \mathbb{C}$ are measurable then $f + g$ and $fg$ are too.*
4. *If $A \in$ Measurable$(X)$ then $\chi_A : X \to \mathbb{R}$ is a measurable function.*
5. *If $f : X \to \mathbb{C}$ is measurable then there exists some $\alpha : X \to \mathbb{C}$ measurable such that $f = \alpha |f|$.*

*Proof.* We only prove the last statement. Set $E := f^{-1}(\{0\})$ (a measurable set) and $Y := \mathbb{C} \setminus \{0\}$. Let

$$\begin{aligned} \varphi : Y &\to \mathbb{C} \\ z &\mapsto \frac{z}{|z|} . \end{aligned}$$

Define

$$\alpha(x) := \varphi(f(x) + \chi_E(x)) \qquad (x \in X) .$$

Show that $\varphi$ is continuous on $Y$ to conclude. $\square$

In what follows, it will be convenient to consider the *extended real line* $[-\infty, \infty]$, see Appendix A. In particular we shall always consider it as a measure space w.r.t. $\mathcal{B}([-\infty, \infty])$ unless otherwise specified.

**Theorem 2.17.** *Let $f : X \to [-\infty, \infty]$ be a map with $X$ a measure space. Here we consider $[-\infty, \infty]$ as the extended real line with its topology, see Appendix A. Then if*

$$f^{-1}((\alpha, \infty]) \in \text{Measurable}(X) \qquad (\alpha \in \mathbb{R})$$

*then $f$ is measurable w.r.t.* $\text{Measurable}(X)$ *and* $\mathcal{B}([-\infty, \infty])$.

*Proof.* The set $(\alpha, \infty]$ is already open in $[-\infty, \infty]$ so our goal is to build *any* of the basis elements of $[-\infty, \infty]$ using this basic open set. To that end, let

$$\Omega := \left\{ E \subseteq [-\infty, \infty] \mid f^{-1}(E) \in \text{Measurable}(X) \right\} .$$

Let $\alpha \in \mathbb{R}$ and $\{ \alpha_n \}_n \to \alpha$ *from below*. Then $(\alpha_n, \infty] \in \Omega$ by hypothesis, and we have

$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} (\alpha_n, \infty]^c$$

so we get the other type of basic open set, $[-\infty, \alpha)$. Next, using

$$(\alpha, \beta) \quad = \quad [-\infty, \beta) \bigcap (\alpha, \infty]$$

we see that since every open set of $[-\infty, \infty]$ is a *countable* union of segments of the above types, so that $\Omega$ contains all open sets of $[-\infty, \infty]$ and hence $f$ is measurable. $\square$

## 2.2 Limits of measurable functions

Recall the definition of the lim inf and lim sup: Let $\{ a_n \}_{n \in \mathbb{N}} \subseteq \overline{\mathbb{R}}$ be a given sequence. Then

$$\liminf_{n \to \infty} a_n \equiv \lim_{n \to \infty} \left( \inf_{m \geq n} a_m \right) = \sup_{n \in \mathbb{N}} \inf_{m \geq n} a_m .$$

Similarly,

$$\limsup_{n \to \infty} a_n \equiv \lim_{n \to \infty} \left( \sup_{m \geq n} a_m \right) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} a_m .$$

Evidently, we always have

$$\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$$

and if the limit of $\{ a_n \}_n$ actually exists then both are equal to that limit.

**Theorem 2.18.** *If $f_n : X \to [-\infty, \infty]$ is a sequence of measurable functions then $\sup_{n \in \mathbb{N}} f_n : X \to [-\infty, \infty]$ defined by*

$$X \ni x \mapsto \sup_{n \in \mathbb{N}} f_n(x)$$

*and $\limsup_{n \to \infty} f_n : X \to [-\infty, \infty]$ defined by*

$$X \ni x \mapsto \limsup_{n \to \infty} (f_n(x))$$

*are both measurable.*

*Proof.* Let us denote $g := \sup_{n\in\mathbb{N}} f_n$ and $h := \limsup_{n\to\infty} f_n$. Then, from the definition of $g$ it follows that

$$g^{-1}((\alpha,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty]) \ .$$

Indeed, let us show this. If $x \in g^{-1}((\alpha,\infty])$ then $g(x) > \alpha$. That means $\sup_{n\in\mathbb{N}} f_n(x) > \alpha$ so in particular there must exist $n \in \mathbb{N}$ so that $f_n(x) > \alpha$. Alternatively, if $x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty])$ then there exists some $n \in \mathbb{N}$ for which $f_n(x) > \alpha$. This in particular implies $g(x) > \alpha$.

We conclude that $g$ is measurable. We write

$$h = \inf_{k\geq 1} \sup_{i\geq k} f_i$$

so that $h$ is also measurable by similar representations. ☐

**Corollary 2.19.** *We have*

1. *The limit of every pointwise convergent sequence of complex measurable functions is measurable.*
2. *If* $f, g : X \to \overline{\mathbb{R}}$ *are measurable then so are* $\max(\{f,g\})$ *and* $\min(\{f,g\})$.
3. *In particular, so are* $f^+ \equiv \max(\{f,0\})$ *and* $f^- = -\min(\{f,0\})$.

We may always decompose any $\overline{\mathbb{R}}$-valued function into its positive and negative parts as follows

$$f = f^+ - f^-$$

with $f^\pm$ the positive and negative parts of $f$, and $|f| = f^+ + f^-$ [2].

## 2.3 Simple functions

We shall build a theory of integration starting from primitive functions and then take limits. This will proceed as follows. Given any function

$$f : X \to \mathbb{C}$$

we write it as

$$f = \mathbb{Re}\, f + \mathrm{i}\, \mathbb{Im}\{f\} \ .$$

Then we write

$$\mathbb{Re}\, f = \mathbb{Re}\, f^+ - \mathbb{Re}\, f^-$$

and similarly for the imaginary part, so that any complex function is the (complex) linear combination of four *nonnegative* functions. Measurability is inherited by all four. Then we want to approximate each nonnegative function with even simpler objects, simple functions.

**Definition 2.20** (Simple function)**.** Let $X$ be a measure space and $s : X \to \mathbb{C}$. If $|\mathrm{im}(s)| < \infty$ then $s$ is called a *simple function*. If in addition, $\mathrm{im}(s) \subseteq [0,\infty)$ then $s$ is called a *nonnegative* simple function. We are *not* including $\pm\infty$ as part of $\mathbb{C}$ so that simple functions, by definition, *cannot* take on the values $\pm\infty$.

Clearly simple functions always take on the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

for some $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ and $A_i \equiv \{ x \in X \mid s(x) = \alpha_i \}$.

*Claim* 2.21. A simple function $X \to \mathbb{C}$ of the form $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is measurable iff $A_i \in \mathrm{Measurable}(X)$ for $i = 1, \ldots, n$.

[2] Note a certain minimal property for these objects: Note that if $f = g - h$ with $g, h \geq 0$ then $f^+ \leq g$ and $f^- \leq h$. This is because $f \leq g$ and $0 \leq g$ clearly implies $\max(\{f,0\}) \leq g$.

*Proof.* (We consider $\mathbb{C}$ w.r.t. the Borel sigma algebra, as usual). By Claim 2.14 we only need to check that the pre-image of closed sets is msrbl. Hence let $F \subseteq \mathbb{C}$ be closed. If $F$ does not contain any of the points $\alpha_i$ then $s^{-1}(F) = \varnothing \in \text{Measurable}(X)$. If $F$ contains $\alpha_{i_1}, \dots, \alpha_{i_k}$ then

$$s^{-1}(F) = \bigcup_{j=1}^{k} A_{i_j}$$

and the union of measurable sets is measurable. Conversely, if $s$ is measurable, take the (closed) singleton $\{\alpha_i\}$ to verify that $A_i \in \text{Measurable}(X)$. □

Now we want to establish that any nonnegative measurable function may be approximated by simple functions *from below.*

**Theorem 2.22** (Approximation by simple functions)**.** *Let $f : X \to [0, \infty]$ be measurable. Then there exist simple measurable functions $s_n : X \to [0, \infty)$ such that*

1. $0 \leq s_1 \leq s_2 \leq \cdots \leq f$.
2. $s_n \to f$ *pointwise.*

*Proof.* For every $n \in \mathbb{N}$, define

$$\begin{aligned} \varphi_n : \mathbb{R} &\to [0, \infty] \\ t &\mapsto \begin{cases} 2^{-n} \lfloor 2^n t \rfloor & 0 \leq t < n \\ n & t \in [n, \infty] \end{cases} \end{aligned}$$

which is depicted, at $n = 3$ in Figure 1. The function $\varphi_n$ converges to $t \mapsto t$ as $n \to \infty$. It is doing that in two ways simultaneously:

1. Th region over which it does *not* resemble the identity function, $[n, \infty]$ keeps shrinking.
2. The region over which it does resemble the identity function, it becomes finer and finer at approximation the identity function there by subdividing $[0, n]$ into $2^n$ sub-intervals and being saw-toothed there.

First, note that at each fixed $n \in \mathbb{N}$, $\varphi_n$ is a Borel function. Indeed, it is a simple function that takes on basically $2^n$ values on intervals and as such it is measurable. Moreover, we have

$$\varphi_n(t) \leq \varphi_{n+1}(t) \qquad (t \in [0, \infty], n \in \mathbb{N}) .$$

Indeed, we observe that

$$t - 2^{-n} < \varphi_n(t) \leq t \qquad (t \in [0, n])$$

which leads to the purported monotonicity. Now we set

$$s_n := \varphi_n \circ f$$

which automatically fulfills both of our constraints, using the fact that the composition of measurable functions is measurable Claim 2.5. □

## 2.4 Measures

We now come to the notion of *measure* which for us is to be understood as a generalization of volume in $\mathbb{R}^n$ to much more exotic sets (yet they still have to be measurable).

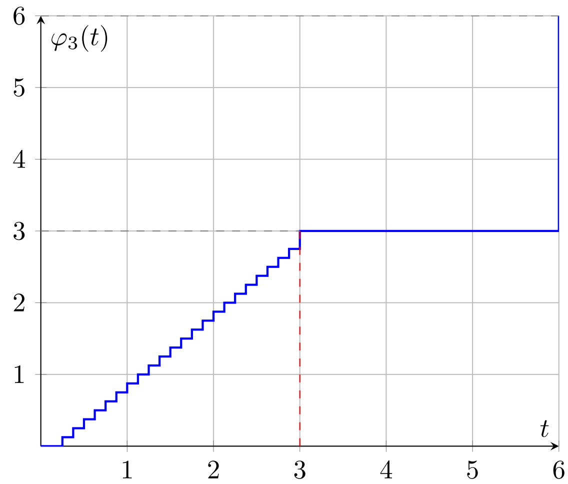


Figure 1: The function $\varphi_3$ approximating the identity.

**Definition 2.23** (Measure)**.** A *complex measure* is a map

$$\mu : \text{Measurable}\,(X) \to \mathbb{C} \cup \{\,\infty\,\}$$

which is countably additive, i.e.,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu\,(A_n) \qquad (A_n \in \text{Measurable}\,(X) : A_n \cap A_m = \varnothing \forall n \neq m) \tag{2.1}$$

and for which $\exists A : \mu\,(A) < \infty$ (otherwise it is not very interesting). If $\text{im}\,(\mu) \subseteq [0, \infty]$ then we say $\mu$ is a *positive measure.*

*Note*: it is customary when using the term *complex measure* to assume $\mu$ never takes on the value $\infty$ (unlike when we use the phrase *positive measure*).

**Theorem 2.24.** *Let* $\mu : \text{Measurable}\,(X) \to [0, \infty]$ *be a positive measure. Then*

1. $\mu\,(\varnothing) = 0$ (so in particular (2.1) holds also for finitely many unions).
2. (*monotonicity*) $A \subseteq B$ implies $\mu\,(A) \leq \mu\,(B)$ for all $A, B \in \text{Measurable}\,(X)$.
3. $\mu$ may be approximated from "inside" as follows:

$$\lim_{n\to\infty} \mu\,(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

   for all increasing sequences $A_n \in \text{Measurable}\,(X)$: $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$.
4. $\mu$ may be approximated from "outside" as follows:

$$\lim_{n\to\infty} \mu\,(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$$

   for all decreasing sequences $A_n \in \text{Measurable}\,(X)$: $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ with $\mu\,(A_1)$ assumed finite.

*Proof.* By assumption, there exists $B \in \text{Measurable}\,(X)$ with $\mu\,(B) < \infty$. Define now a sequence $A_1 := B$, $A_j := \varnothing$ for all $j \geq 2$. This sequence obeys the conditions of (2.1) since it is pairwise disjoint. Hence we find

$$\infty > \mu\,(B) = \mu\,(B) + \sum_{j=2}^{\infty} \mu\,(\varnothing)$$

and the only way this equation could hold is if $\mu(\varnothing) = 0$.

For monotonicity, given $A, B \in \text{Measurable}(X)$ with $A \subseteq B$, let us decompose $B = A \cup (B \setminus A)$ which are now disjoint. Hence additivity implies

$$\mu(B) = \mu(A) + \mu(B \setminus A)$$

and using positivity of the measure, we find this is larger than or equal to $\mu(A)$.

Let us now establish the approximation properties. To do so, given any increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, we decompose it into disjoint parts as follows:

$$\begin{aligned} B_1 &:= A_1 \\ B_n &:= A_n \setminus A_{n-1} \qquad (n \geq 2) \,. \end{aligned}$$

Note that $A_n = \bigcup_{j=1}^n B_j$. So by (2.1) we find

$$\mu(A_n) = \sum_{j=1}^n \mu(B_j)$$

and moreover, since $\bigcup_n A_n = \bigcup_n B_n$, we get

$$\mu\left(\bigcup_n A_n\right) = \sum_{n=1}^\infty \mu(B_n) \,.$$

The result now follows by taking the limit $n \to \infty$ on the penultimate displayed equation.

For approximation from outside, we make the following new variables.

$$C_n := A_1 \setminus A_n \qquad (n \geq 1) \,.$$

This implies $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ and

$$\mu(C_n) = \mu(A_1) - \mu(A_n) \,.$$

Moreover, $A_1 \setminus (\bigcap A_n) = \bigcup_n C_n$, so now we may invoke the previous statement on the sequence $C_n$ to get

$$\begin{aligned} \mu(A_1) - \mu\left(\bigcap_n A_n\right) &= \mu\left(A_1 \setminus \bigcap_n A_n\right) \\ &= \mu\left(\bigcup_n C_n\right) \\ &= \lim_n \mu(C_n) \\ &= \lim_n (\mu(A_1) - \mu(A_n)) \\ &= \mu(A_1) - \lim_n \mu(A_n) \end{aligned}$$

from which our result follows. □

Our main example for a positive measure will be *the Lebesgue measure* on $\mathbb{R}^n$, but it will be a little while before we can define it.

**Example 2.25** (Counting measure)**.** Let $\text{Measurable}(X) = \mathscr{P}(X)$ and define $c : \text{Measurable}(X) \to [0, \infty]$ via

$$S \mapsto |S|$$

(the cardinality of a set, $\infty$ if it is countable or higher). $c$ is called the counting measure.

**Example 2.26** (Unit mass)**.** Let $\text{Measurable}(X) = \{\varnothing, X, \{x_0\}, X \setminus \{x_0\}\}$ be a $\sigma$-algebra and define $\delta_{x_0}$ : $\text{Measurable}(X) \to [0, \infty]$ by

$$S \mapsto \begin{cases} 1 & x_0 \in S \\ 0 & x_0 \notin S \end{cases} .$$

$\delta_{x_0}$ is called the unit mass concentrated at $x_0$. It is closely related to the Dirac delta function.

**Example 2.27.** If we take the counting measure $c$ on $\mathbb{N}$ and set $A_n := \mathbb{N}_{\geq n}$ then $\bigcap_n A_n = \varnothing$ and yet $\mu(A_n) = \infty$. This does not violate the theorem above since the assumption $\mu(A_1) < \infty$ is clearly violated here.

## 2.5 Integrating positive functions

Given a positive measure $\mu : \text{Measurable}(X) \to [0, \infty]$, we now proceed to define the Lebesgue integral associated to $\mu$.

**Definition 2.28** (The Lebesgue integral of positive simple measurable functions)**.** Let $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ be a positive measurable simple function. Then we define the integral of $s$ on a set w.r.t. $\mu$ as

$$\int_E s\mathrm{d}\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) \qquad (E \in \text{Measurable}(X)) .$$

We use the convention $0 \cdot \infty = 0$ in case $\alpha_i = 0$ yet $\mu(S) = \infty$.

**Definition 2.29** (The Lebesgue integral of positive functions)**.** Let $f : X \to [0, \infty]$ be measurable. Then

$$\int_E f\mathrm{d}\mu := \sup_s \int_E s\mathrm{d}\mu$$

where the supremum ranges over all simple measurable functions $s$ which obey $0 \leq s \leq f$. Note if $f$ is simple the two definitions coincide.

# A The extended real line

We shall frequently use the symbol $[-\infty, \infty]$ or $\overline{\mathbb{R}}$ to denote *the extended real line*. As a set it is given by

$$\overline{\mathbb{R}} \quad := \quad \mathbb{R} \cup \{\pm\infty\}$$

and topologically we add the neighborhoods of $\pm\infty$ as those sets which contain the basic open sets

$$(a, \infty]$$

and

$$[-\infty, a)$$

respectively.

**Lemma A.1.** *test lemma*

**Proposition A.2.** *test proposition*

# B Glossary of mathematical symbols and acronyms

Sometimes it is helpful to include mathematical symbols which can function as valid grammatical parts of sentences. Here is a glossary of some which might appear in the text:

- $\text{im}(f)$ is the *range* or *image* of a function: If $f : X \to Y$ then

  $$\text{im}(f) \equiv \{ f(x) \in Y \mid x \in X \} .$$

- The bracket $\langle \cdot, \cdot \rangle_V$ means an inner product on the inner product space $V$. For example,

  $$\langle u, v \rangle_{\mathbb{R}^2} \equiv u_1 v_1 + u_2 v_2 \qquad \left(u, v \in \mathbb{R}^2\right)$$

  and

  $$\langle u, v \rangle_{\mathbb{C}^2} \equiv \overline{u_1} v_1 + \overline{u_2} v_2 \qquad \left(u, v \in \mathbb{C}^2\right) .$$

- Sometimes we denote an integral by writing the integrand without its argument. So if $f : \mathbb{R} \to \mathbb{R}$ is a real function, we sometimes in shorthand write

  $$\int_a^b f$$

  when we really mean

  $$\int_{t=a}^b f(t)\, \mathrm{d}t .$$

  This type of shorthand notation will actually also apply for contour integrals, in the following sense: if $\gamma : [a, b] \to \mathbb{C}$ is a contour with image set $\Gamma := \text{im}(\gamma)$ and $f : \mathbb{C} \to \mathbb{C}$ is given, then the contour integral of $f$ along $\gamma$ will be denoted equivalently as

  $$\int_\Gamma f \equiv \int_\Gamma f(z)\, \mathrm{d}z \equiv \int_{t=a}^b f(\gamma(t))\, \gamma'(t)\, \mathrm{d}t$$

  depending on what needs to be emphasized in the context. Sometimes when the contour is clear one simply writes

  $$\int_{z_0}^{z_1} f(z)\, \mathrm{d}z$$

  for an integral along *any* contour from $z_0$ to $z_1$.

- iff means "if and only if", which is also denoted by the symbol $\Longleftrightarrow$.

- WLOG means "without loss of generality".

- CCW means "counter-clockwise" and CW means "clockwise".

- $\exists$ means "there exists" and $\nexists$ means "there does not exist". $\exists!$ means "there exists *a unique*".

- $\forall$ means "for all" or "for any".
- : (i.e., a colon) may mean "such that".
- ! means negation, or "not".
- $\wedge$ means "and" and $\vee$ means "or".
- $\implies$ means "and so" or "therefore" or "it follows".
- $\in$ denotes set inclusion, i.e., $a \in A$ means $a$ is an element of $A$ or $a$ lies in $A$.
- $\ni$ denotes set inclusion when the set appears first, i.e., $A \ni a$ means $A$ includes $a$ or $A$ contains $a$.
- Speaking of set inclusion, $A \subseteq B$ means $A$ is contained within $B$ and $A \supseteq B$ means $B$ is contained within $A$.
- $\varnothing$ is the empty set $\{\ \}$.
- While $=$ means equality, sometimes it is useful to denote types of equality:
  - $a := b$ means "this equation is now the instant when $a$ is defined to equal $b$".
  - $a \equiv b$ means "at some point above $a$ has been defined to equal $b$".
  - $a = b$ will then simply mean that the result of some calculation *or* definition stipulates that $a = b$.
  - Concrete example: if we write $\mathrm{i}^2 = -1$ we don't specify anything about *why* this equality is true but writing $\mathrm{i}^2 \equiv -1$ means this is a matter of definition, not calculation, whereas $\mathrm{i}^2 := -1$ is the first time you'll see this definition. So this distinction is meant to help the reader who wonders *why* an equality holds.

## B.1 Important sets

1. The unit circle

$$\mathbb{S}^1 \quad \equiv \quad \{\, z \in \mathbb{C} \mid |z| = 1 \,\} \ .$$

2. The (open) upper half plane

$$\mathbb{H} \quad \equiv \quad \{\, z \in \mathbb{C} \mid \mathbb{Im}\{z\} > 0 \,\} \ .$$

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